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TRANSACTIONS.

I. *Account of some Experiments made in Different Parts of Europe, on Terrestrial Magnetic Intensity, particularly with reference to the Effect of Height. By* JAMES D. FORBES, *Esq., F.R.SS. L. & E. &c., Professor of Natural Philosophy in the University of Edinburgh.*

Read 19th December 1836.

1. THE Council of the Royal Society of Edinburgh having, on my application in 1832, entrusted me with HANSTEEN'S Magnetic Intensity Apparatus, in their possession, I feel it to be my duty to communicate to the Society the results then and subsequently obtained with it.

2. The instrument consists of a mahogany box 5 inches long, 4 broad, and 2 deep, with sides and top of glass, having also a wooden tube, screwing into the top, for containing a silk-worm's fibre about 5 inches long, by which the magnetic needle is suspended so as to place itself horizontally, and after being caused to deviate from its point of rest, the time of any given number of oscillations in a horizontal plane is measured,—whilst a graduated circle in the bottom of the box indicates its arc of vibration.

3. The needles which accompanied the instrument, when originally sent from Norway, are two in number, one a cylinder 3 inches long and 0.1 inch in diameter, is marked on its case " No. 1." The other is shorter, thicker, and heavier, and from its form has always been called the " Flat" needle. These were the needles used with this apparatus by Mr DUNLOP, in the experiments made in Scotland at Sir T. M. BRISBANE'S expense, and published in Vol. XII. of the Society's Transactions. A reference to that volume will shew clearly Professor HANSTEEN'S and Mr DUNLOP'S method of observation, which, essentially, I have always followed.

4. If we assume the magnetism of a needle to remain invariable, the inten-

sity of the earth's magnetism at different places, or at the same place at different times, will be (on the principle of the pendulum) inversely as the square of the time required to perform a given number of vibrations in infinitely small arcs, under the different circumstances. But various adjustments have to be attended to, and corrections applied.

5. Nothing more portable or more simple than the instrument in its present form can be desired. These requisites are no doubt obtained at the expense of some accuracy. Mr Snow Harris has shewn * that the influence of the surrounding air upon the needle gives rise to considerable errors, especially when the needle is so small and light, as in Hansteen's apparatus. But greatly to enlarge the needle, and to connect an air-pump with the apparatus, is nearly equivalent to depriving the traveller of its use altogether. Hansteen's instrument was the constant companion of my pedestrian excursions, and had it been in any other form, the present observations would probably never have been made. Besides all this, there are sources of error arising from imperfectly known and irregular variations of the earth's intensity, and equally or more important ones from changes of magnetic intensity in the needle itself, which the improvements in question do not affect. Until by a regular and long continued series of observations, such as those likely to be undertaken at Greenwich, magnetism shall be reduced to more of a science than it is at present, we must beware of pretending to illusory accuracy in a traveller's detached experiments. Those about to be detailed in this paper, will sufficiently indicate the degree of comparability of observations made with Hansteen's instrument, such as it is, and which is by far the best test of their real value. It has certainly rather exceeded than fallen short of my expectations.

§ 1. *Adjustments and Method of observing.*

6. Hansteen's instrument contains no provision for securing the horizontality of the needle, which is of considerable importance. The needles have, indeed, sliding collars of suspension, which may be altered with change of dip, but the box has no adjusting levels. I have always † used a small spirit level for adjusting the bottom of the box, and then, as carefully as I could, made the needle hang parallel to it, but the adjustment was troublesome and unsatisfactory.

7. The needle being levelled and allowed to come to rest, it was drawn out of its position of rest ‡, but always in a horizontal plane, by the approach of a

* Edinburgh Transactions, vol. xii. p. 1.—See also the Observations of Professor Bache; American Phil. Trans. vol. v.

† I cannot answer, however, for two or three of the first observations hereafter to be quoted.

‡ As the torsion of the silk fibre must have some influence, it is not unimportant to remark, that the same thread which was adapted to the instrument in August 1832, has been used ever since.

piece of iron or steel (usually a penknife), and the process repeated until the semi-arc of vibration exceeded 20°, if 300 vibrations were to be observed, or 10° if 100 only were observed, as was more usually the case. This last deviation from Professor HANSTEEN'S practice was not adopted without due consideration. The large commencing arc necessary in order that the vibrations might be distinguishable at the close of 300, increased greatly the errors pointed out by Mr HARRIS. Moreover, there seemed less chance of error in combining several series of 100 vibrations taken in succession, than in using a single series of 300, which, from the time it occupies, is more liable to be interrupted and rendered useless by a gust of wind, or a momentary relaxation of the painful attention required to be exerted by an unassisted observer. Besides, the mere error of the *observed time*, depending on the eye and ear of the observer, will not exceed even in 100 vibrations the uncertainty arising from causes impossible to eliminate,—indeed falls much short of it: yet this is the only error which we diminish by increasing the vibrations in a series to 300.

8. When the semi-arc of vibration had diminished to 20°, or to 10° (as 300, or 100 vibrations were to be observed), the counting of vibrations commenced,—the hour, minute, second, and decimal, of the beginning or 0th vibration being noted, and the second and decimal only for each succeeding 10th vibration, until 360 vibrations (in the first case) or 160 (in the second) were observed. These seconds of time are arranged in columns, so that the times of the 0th, 100th, 200th, 300th vibration, run along the same horizontal line as do the 10th, 110th, 210th, 310th, &c. The time of the 0th being then subtracted from the time of the 300th (or 100th), we have one value of the time of 300 (or 100) vibrations. The 10th, from the 310th (or the 110th) gives a second value, and so on to the 60th and 360th (or 160th), which gives in all seven values of time of 300 (or of 100) vibrations; the mean of which seven values is taken (the minutes being of course supplied), and the hour, minute, and second of conclusion. The thermometer (enclosed in the box) is consulted at the beginning and end, and its indications registered. The rate of diminution of the semi-arc of vibration is also observed, its continued bisection being indicated opposite to the instant at which it occurs in a column parallel to those already named. The rate of the chronometer is likewise to be determined. An example will best illustrate all this.

MAGNETIC INTENSITY.

Place, Greenhill, near Edinburgh.				Ther. Reaum
Date, 7th May 1833.	Beginning,	$5^h\ 3^m\ 41^s.6$		18°.0
Needle, Flat.	End, . .	25 34 .5		17°.3
			Mean,	17.65

Arc.	Sec.	Arc.	Sec.	Arc.	Sec.	Sec.	300 Vibrations.
20°	41.6		47.8		52.0	56.7	$18^m\ 15^s.1$
	18.3		24.0		28.3	32.5	14.2
	54.7	10°	0.4		5.3	9.2	14.5
	31.0		36.5		40.8	45.7	14.7
	8.5		14.0	5°	17.7	22.0	13.5
	45.0		49.8		54.2	58.2	13.2
	21.7		26.0		30.7	34.5	12.8
	58.3		3.3		7.7		
	35.0		39.0		43.6	Mean,	$18^m.14^s.00$
	11.1		15.8		19.8		$= 1094^s.00$
Rate + 3^s							

9. My method of observing was to keep the chronometer at the ear till the instant that the termination of a vibration was observed, then to count five beats of the balance (corresponding to two seconds), which affords time to bring the dial-plate into view, and the seconds entered in the Table are those read off at that time, namely two seconds later than the absolute times. Thus the impracticable attempt to observe two things at once by the eye is avoided. For this and some other suggestions, I am indebted to my friend Captain P. P. King, R. N. The observations are registered in lithographed forms bound into volumes.

10. In the choice of stations I have been extremely particular, often at great personal inconvenience.* Places remote from any trace of habitation have most usually been selected, and in no case have intensity observations been made in a house. The specialties of the sites will be noticed in the following Tables. I have invariably removed all masses of iron from my person; and in my later experiments even took the precaution of carrying thin shoes, in order that the heavily nailed ones which I usually wear, might be removed to a distance. The chronometer, too, has generally been held at some distance from the apparatus. But some direct experiments lead me to believe that the influence of the two last mentioned sources of error is insensible.

* None but those who have been engaged in observations of the very same description, where the eye, the ear, and the memory are all actively employed, can have an idea of the difficulty of always finding sites free from the interruptions of curiosity, or natural obstacles.

§ 2. *Corrections applicable to the Observations.*

11. When the mean of seven values of 100 or of 300 vibrations has been taken, as above explained, a variety of important corrections remain to be applied.

12. I. *Rate of Chronometer.*—The following rule due to Professor HANSTEEN is simple and accurate:—" The logarithm to five decimal places of the observed time is taken, unity is to be *added* to the fifth decimal place for every *two* seconds *per diem* that the watch goes *slow*; and unity *subtracted* for every *two* seconds that the watch goes *fast*." The demonstration is too simple to require notice. The following is a table of corrections:—

TABLE I.

	Log. Additive.		Log. Additive.
Rate + 0^{sec}	0.00000	Rate — 0^{sec}	0.00000
2	0.99999	2	0.00001
4	0.99998	4	0.00002
6	0.99997	6	0.00003
8	0.99996	8	0.00004
10	0.99995	10	0.00005

There is another chronometric correction worth mentioning, arising from the necessarily imperfect division of the seconds' circle of an enamelled dial-plate. In my watch this amounts to a sensible quantity, and has often given an apparent discrepancy to the partial results of a series for which I was not prepared. Upon investigation, I find, however, that the effect upon the *mean* will always be so insignificant as to be hardly worth notice.

13. II. *Arc.*—A correction due to the motion of the magnetic pendulum in circular arcs, cannot be considered as a constant quantity, and therefore not affecting relative results, 1. Because the rate of diminution of arc varies considerably in different experiments, and is directly deduced from the observed law of diminution of arc; and 2. Because we sometimes have to compare observations of 100 vibrations having an initial semi-arc of 10°, with 300 vibrations beginning at 20°. The latter case having alone been considered by HANSTEEN, I re-investigated the theory of the correction, and confirmed his numbers.

14. It is assumed that the arc diminishes geometrically in consequence of resistance, the time increasing arithmetically. The best observations I have made, confirm the truth of this general admission. Again, we have to recollect, that, in consequence of the degradation of the arcs, the reduction to infinitely small arcs for the vibrations between the 0th and the 300th, will be greater than between the 10th and 310th, &c., and that the *mean* of all the corrections (taking this va-

riation into account) must be applied. The law of the diminution of arc, or the factor representing the ratio of the arc of one vibration to that of the immediately preceding one, will be at once deduced from observing after how many vibrations the arc is halved. Let m be that number, then if r be the factor in question, $r^m = \frac{1}{2}$; whence $r = \sqrt[m]{\frac{1}{2}}$, which is known; and therefore m, together with the initial semi-arc of vibration, may be used as the arguments for entering the following table of corrections:—*

* *Investigation.*—Let α be the initial semi-arc of vibration (*taken in parts of radius*), and r the ratio of its diminution by resistance in a single vibration.

Then, for the	1st,	2d,	3d,	4th, ... nth vibration,
The arcs will be,	α,	αr,	αr^2,	αr^3, ... αr^{n-1}

And, by mechanics (POISSON, art. 184.), the times occupied by these vibrations will be (the time of an infinitely small vibration being unity),

$$1+\frac{\alpha^2}{16},\quad 1+\frac{\alpha^2 r^2}{16},\quad 1+\frac{\alpha^2 r^4}{16},\quad 1+\frac{\alpha^2 r^6}{16}\ \dots\ 1+\frac{\alpha^2 r^{2(n-1)}}{16}$$

And the *mean* duration of the n vibrations is,

$$M = 1+\frac{\alpha^2}{16}\cdot\frac{1+r^2+r^4+r^6 \dots r^{2(n-1)}}{n} = 1+\frac{\alpha^2}{16}\cdot\frac{1-r^{2n}}{(1-r^2)n}$$

Hence the mean duration of n vibrations,

From the 0th to the nth, is $1+\frac{\alpha^2}{16}\cdot\frac{1-r^{2n}}{(1-r^2 n)} = 1+\left(\frac{\alpha}{4}\right)^2 . A$

—— — 10th —— ($n+10$th), is . . . . $= 1+\left(\frac{\alpha}{4}\right)^2 r^{10\times 2} . A$

(because the initial arc instead of α is αr^{10})

—— — 20th, —— ($n+20$th), is . . . . $= 1+\left(\frac{\alpha}{4}\right)^2 r^{20\times 2} . A$

—— — 60th, —— ($n+60$th), is . . . . $= 1+\left(\frac{\alpha}{4}\right)^2 r^{60\times 2} . A$

And the mean value of these deviations, is

$$1+\left(\frac{\alpha}{4}\right)^2 . A . \frac{1+r^{20}+r^{40} \dots + r^{120}}{7} = 1+\left(\frac{\alpha}{4}\right)^2 . A . \frac{1-r^{140}}{7(1-r^{20})}$$

The concluding factor may be called B; and substituting the value of r from the text, or $\left(\frac{1}{2}\right)^{\frac{1}{m}}$, we have

$$A = \frac{1-\left(\frac{1}{2}\right)^{\frac{2n}{m}}}{\left(1-\left(\frac{1}{2}\right)^{\frac{2}{m}}\right)n} \qquad B = \frac{1-\left(\frac{1}{2}\right)^{\frac{140}{m}}}{7\left(1-\left(\frac{1}{2}\right)^{\frac{20}{m}}\right)}$$

(the last factor being independent of n), and we have

$$\text{Corrected Time} = \frac{\text{Observed Mean Time}}{1+\left(\frac{\alpha}{4}\right)^2 . A . B.},$$

whence the Tables are computed.

TABLE II.

Initial Semi-Arc $= \alpha = 10°$ No. of Vibrations observed $= n = 100$.		Initial Semi-Arc $= \alpha = 20°$ No. of Observations observed $= n = 300$.	
m	Additive Log.	m	Additive Log.
70	9.99978	70	9.99969
80	9.99975	80	9.99960
90	9.99972	90	9.99953
100	9.99970	100	9.99946
110	9.99967	110	9.99939
120	9.99965	120	9.99933
130	9.99962	130	9.99926

I have every reason to think this correction to be accurate on the whole: the agreement of the two modes of observation being in general very close.

15. III. *Temperature.* This extremely important correction it is very difficult to determine. Without an accurate estimation of it, it would be vain to attempt to decide whether or not the magnetic energy varies with height; because at great elevations the temperature being always diminished, the intensity would appear too great (the magnetic energy in iron being increased by cold, and diminished by heat). I therefore endeavoured to compare the intensity of the needles employed, within the range of temperatures usually observed. The apparatus employed was of this kind. The needle was first allowed to take the temperature of a heated room and vibrated. Then every thing else remaining the same (and of course any local attraction which might exist being unaltered) the apparatus was placed in a cylindrical glass jar, with ice in the bottom, placed in a dish of ice, and covered with a glass plate also covered with ice. A steady temperature, but little above the freezing point, was thus attained, and the oscillation again observed. These experiments were repeated many times. One series was undertaken at Geneva in October 1832, another at Edinburgh on four different days of August 1834. Those for the needle, No. 1, were conducted with the most scrupulous care, nearly 5000 vibrations having been counted for this purpose alone. One set was discarded as differing too much from the others, and the remainder agreed very closely, although made under such different circumstances, and at such different times. The result adopted for Needle No. 1. gave an increase in time of .00045 for a diminution of temperature of 1° Reaumur, and *vice versa;* for the Flat Needle (determined from two concordant series, both observed at Geneva on different days) .00030. From these results the following tables were calculated, giving the reduction in each case to 0° of Reaumur (which being the scale attached to the instrument, was always observed in these experiments.) This seems preferable to referring to any other arbitrary temperature, upon which observers do not generally agree.

TABLE III.

Additive Corrections applicable to *five* Place Logarithms of the *Time*, for the effect of Temperature.

NEEDLE, No. I.						FLAT NEEDLE.					
Temp. Reaumur.	Correction.	Temp. Reaumur.	Correction.	Proportional Parts.		Temp. Reaumur.	Correction.	Temp. Reaumur.	Correction.	Proportional Parts.	
°		°			Corr.	°		°			Corr.
1	9.99980	11	9.99785	°.1	—2	1	9.99987	11	9.99859	°.1	—1
2	9.99961	12	9.99766	.2	4	2	9.99974	12	9.99846	.2	3
3	9.99941	13	9.99746	.3	6	3	9.99962	13	9.99834	.3	4
4	9.99922	14	9.99726	.4	8	4	9.99949	14	9.99821	.4	5
5	9.99902	15	9.99707	.5	10	5	9.99936	15	9.99808	.5	6
6	9.99883	16	9.99687	.6	12	6	9.99923	16	9.99795	.6	8
7	9.99863	17	9.99668	.7	14	7	9.99910	17	9.99782	.7	9
8	9.99844	18	9.99648	.8	16	8	9.99898	18	9.99770	.8	10
9	9.99824	19	9.99629	.9	18	9	9.99885	19	9.99757	.9	12
10	9.99805	20	9.99609			10	9.99872	20	0.99744		

16. I am disposed to think that the correction for temperature is always open to a certain degree of doubt. Perhaps the condition of magnetism in the needle is not necessarily that due to the temperature it possesses at the moment, but rather to a temperature it had formerly. I think I have in some cases perceived indications of this. The Needle No. 1, which is more slender than the "Flat," seemed to be more steady in its indications than the other, and as I have always placed more reliance upon its indications, so the effect of temperature was determined with most care.

17. IV. *Variations in the Earth's Magnetic Intensity.* These variations must affect observations of the relative intensity at two places, if the observations be not simultaneous. These variations are either (1.) *secular*, shewing a progressive change from year to year; (2.) *periodical*, that is subject to short periods of variation and regular, as at different seasons of the year, and at different hours of the day; or, (3.) *accidental*, arising from the aurora borealis, or from unknown causes.* The numerical laws of these three, may be said to be almost equally unknown; the variations of the second class have indeed been studied by HANSTEEN, CHRISTIE, DOVE, and others, but the results are not sufficiently accordant to permit me to apply any of them to my observations. As, however, the

* My friend Professor NECKER of Geneva has pointed out to me one of the first recorded observations of the influence of the aurora upon the magnetic needle, the more interesting because the coincidence was unnoticed (apparently) by the observer himself. In SAUSSURE's *Voyages dans les Alpes*, vol. iv. p. 300, that enterprising traveller notices an auroral appearance, observed from the Col du Geant on the 12th July 1788, and in another part of the same volume (p. 308), records, amongst his magnetical observations, the unsettled state of the needle during the whole of that evening.

epochs are always recorded, this correction may be applied at a future time, and in a more advanced state of science.

18. V. *Variations in the Needles' Magnetism.* In all observations of this kind, this change gives rise to the most troublesome errors. The mode of ensuring an equable magnetic state is unknown, though an approximation may generally be obtained to it. Of the two needles sent to this country in 1827, by Professor HANSTEEN, one (No. 1.) has, after some slight variations, become almost stationary in its magnetism; the other "Flat" has been continually diminishing in intensity. We have seen in the last article that the earth's magnetic action, varying continually and being unknown, we can only properly compare observations made at the same time of the year, and of the day. The progress of change in the needles may be traced by the following tables.*

TABLE IV.

NEEDLE, No. I.				
PLACE.	Date.	Observer.	Log. Time 300 Vibrations.	Log. Ratio of Annual Change.
Makerstoun,	1829, Jan. 20, 3^h	Dunlop.	2.90251	.99997
——	1830, Jan. 23, 3^h	——	2.90248	
Edinburgh,	1829, July 9, 11^h	——	2.90765	.00043
——	1832, June 2, 11^h	Forbes.	2.90890	.99956
——	1833, May 7, 5^h	——	2.90849	
——	1835, May 4, 1^h	——	2.90915	.00066
Paris, . .	1833, June 11, 1^h	——	2.87102	.99995
——	1835, June 13, 4^h	——	2.87092	
The differences since June 1832 are probably imputable to the horary variation alone.				
FLAT NEEDLE.				
PLACE.	Date.	Observer.	Log. Time 300 Vibrations.	Log. Ratio of Annual Change.
Makerstoun,	1828, Dec. 1, 12^h	Dunlop.	3.00582	.00707
——	1830, Feb. 14, 3^h	——	3.01435	
Edinburgh,	1829, July 9, 10^h	——	3.01240	.00546
——	1832, June 2, 12^h	Forbes.	3.02923	.00736
——	1833, May 7, 5^h	——	3.03607	
——	1835, May 4, 2^h	——	3.04579	.00488
Paris, . .	1833, June 11, 2^h	——	2.99871	.00500
——	1835, June 13, 4^h	——	3.00869	

* The mutual action of the needles is a point of importance. Before they came into my possession they were kept in their separate cases, but without farther attention, being packed together in the

19. In the case of No. 1., the magnetism has been considered as stationary throughout the period 1832–1835, with which we are now concerned. In the case of the Flat Needle, this cannot be assumed, nor can we admit the change to have been uniform. It seems probable that much movement, and especially alternations of temperature, accelerate the loss of magnetism, that loss having been greatest in 1832, when most of the following observations were made. This is confirmed by a more minute inspection. Observations were made at Geneva on the 20th August 1832, and again on the 10th November, and between these dates the whole of the alpine series is contained: Now the variation of the logarithms for that period is no less than .00452, or at the rate of .02001 per annum; whilst we have seen that during the period from June 2. 1832 to May 7. 1833, which includes the above, the mean change was only at the rate of .00736 per annum. It is clear then that, in order to render the observations of 1832 comparable with one another, we must assume a much higher rate than the mean, for the months from June to November. Admitting some little doubt as to the Geneva comparisons due to the monthly change of intensity, and the great difference of temperature in the two cases; I think that I shall best satisfy the conditions by assuming the log. time to have increased by .00100 for each month from June to November, leaving .00684 — .00500 = .00184 for the whole of the remaining six months down to May 1833, during which the needle was in a state of almost perfect repose.

20. The mode of allowing for this is the following. All determinations of intensity are *relative*, referring to some intensity as a standard; but I have taken the horizontal intensity at Paris as unity (which is to that at the magnetic equator as 4788 to 10,000 according to HUMBOLDT).* Hence, since the squares of the times of 100 vibrations are inversely as the magnetic forces, the terrestrial horizontal intensity at a station A is to that at Paris, or 1, as the square of the time observed at Paris (which time we may call T_P) is to the square of the time observed at station A (or T_A). Hence,

$$\text{Intensity at A} = \left(\frac{T_P}{T_A}\right)^2$$

If the magnetism of the needle change, we must therefore find by interpolation the time of vibration at Paris for the particular epoch of observation.

same external case in which they came from Norway. This arrangement I have not changed, but in packing them I have taken pains to place the opposite poles nearest one another, an arrangement which seems to have been attended with good effect; and to shew that needles may lie within an inch or two of one another without material injury, when we see the stationary condition of No. 1, and the diminishing rate of variation of the "Flat" Needle.

* Deduced from the measure of *total* intensity 1.3482 at Paris, given in the *Mémoirés d'Arceuil*, tom. i. multiplied by the cosine of the dip (there also given) 69° 12′.

21. In the case of Needle No. 1. the magnetism being stationary, the time of 100 vibrations has been assumed from an observation made (in M. ARAGO'S *Cabinet Magnetique)*, 11th June 1833, as equal to

$247.^{sec}70$; its log. 2.39392

22. In the case of the "Flat" needle, a subsidiary table has been calculated of the times for Paris, corresponding to the epochs when observations were made elsewhere, which appear amongst the details to be given in the sequel. On the 11th June 1833, the log. time of 100 vibrations at Paris was . . . 2.52159

If, for the period from June 1832 to May 1833, we deduct .006184 (by Art. 18.) and for the month of May 1833, .00040 being the rate of change for the current period (Art. 18.), we have a change for one year, subtractive, .00724

T_P, or Time at Paris,	June 11. 1832, . . .	Log. 2.51435
Adding .001 per month, as proposed in Art. 19., we shall have nearly these values (neglecting trifling quantities) as approximations to the value of Log. T_P	July 11, ...	2.51535
	Aug. 11, ...	2.51635
	Sept. 11, ...	2.51735
	Oct. 11, ...	2.51835
	Nov. 11, ...	2.51935

Proceeding similarly with the mean (and very regular) ratios of change in Art. 18, we shall find

Log. T_P	1833, May 7, ...	2.52114
	1835, May 4, ...	2.53100
	... July 20, ...	2.53203
	... July 30, ...	2.53216
	... Aug. 10, ...	2.53230

23. The various corrections now considered being fixed, the application of them, and the deduction of the horizontal intensities related to Paris as unity, becomes easy. I have employed printed forms for this purpose, arranged in pages each containing 5 reductions after the following model, and bound up in books.

MAGNETIC INTENSITY.—Horizontal Needle, No. 1.				
Place,	Geneva. Botanic Garden.		Geneva. Botanic Garden.	
Date,	1832, Nov. 10.		1832, Nov. 10.	
Mean Time,	$11^h\ 33^m$		$11^h\ 42^m$	
No. of Vibrations observed,	100		100	
	Arg.	Log.	Arg.	Log.
Observed Time,	$240\overset{s}{.}14$	2.38046	240.00	2.38021
Rate Chronometer,	$+21^s$	9.99990	$+21^s$	9.99990
Arc, *	$\left\{\begin{matrix}\alpha = 10^\circ \\ m = 110^\circ\end{matrix}\right\}$	9.99967	$\left\{\begin{matrix}\alpha = 10^\circ \\ m = 110^\circ\end{matrix}\right\}$	9.99967
Temperature,	7°.0 R.	9.99863	7°.0 R.	9.99863
Corrected Time (T),	$239\overset{s}{.}14$	2.37866	$239\overset{s}{.}06$	2.37841
Time at Paris (T_P)	....	2.39392	....	2.39392
$\frac{T_P}{T}$	....	0.01526 2	....	0.01551 2
$\left(\frac{T}{T_P}\right)^2$ = Intensity,	1.073	0.03052	1.074	0.03102

* α = Initial semi-arc of vibration; m = number of vibrations which reduce the semi-arc to $\frac{\alpha}{2}$

§ 3. *Observations on Magnetic Intensity.*

24. It now remains to give the observations which have been made, and reduced in the way already detailed. These consist chiefly of two series. One was made in the year 1832, intended to form part of a very general investigation in physical geography, which I meant to pursue throughout a journey of several years. Having been diverted from this by the opening of other prospects, the series remains incomplete as a general investigation, but embraces a connected examination of a great district of the higher and central Alps, calculated to elucidate a question which I had particularly proposed to myself, as to the supposed diminution of magnetism with height. It likewise includes some observations as to the influence of extinct volcanos on the Rhine. The second series was made in the Pyrénées in 1835, with almost an exclusive view to the influence of height, and is confined to one small district. One other important point gained was a very accurate determination of the comparative horizontal intensities at Edinburgh and Paris. The choice of the stations was regulated very much by the views just mentioned: the particular spots of observation, together with the geographical position and elevation of the place, will be given in Table VII. I have thought it better to record in the first place in separate Tables for the two needles, the details of the observations in the order in which they were made, the data

for correction, and the corrected numbers. These are contained in Tables V. and VI. I have thought it needless to incur the labour and expense of printing the individual numbers on which the mean results are founded. They are however preserved in a condition adapted for immediate reference,*

Table V.

NEEDLE, No. I.

Place.	Date.	Mean Time.	No. of Vibrations Observed.	Observed Time.	Rate Chrono-meter.	Arc.† α.	Arc.† m	Temp. Reaum.	Corrected Time 100 Vibrations.	Intensity, Paris =1.
	1832.	h m		s	s				s	
Edinburgh, . . .	June 2.	10 22	300	817.66	+ 13	20	80	16.4	270.26	.840
Brussels,[1] . . .	July 9.	10 29	300	766.67	+ 17	20	70	20.5	252.97	.959
Brussels, . . .	July 9.	10 53	300	766.87	+ 17	20	90	21.6	252.83	.960
Spa, ...	July 17.	5 53	100	252.27	+ 23	10	90	17.35	250.09	.981
Spa, ...	July 17.	6 7	100	252.80	+ 23	10	90	17.0	250.69	.977
Königstuhl, near Heidelberg, .	July 28.	11 4	100	247.60	+ 23	10	90	17.15	245.48	1.018
Königstuhl, . . .	July 28.	11 17	100	247.54	+ 23	10	90	16.75	245.46	1.018
Heidelberg, . . .	July 28.	2 24	100	247·46	+ 23	10	100	15.4	245.51	1.017
Brühl, ...	Aug. 1.	10 55	100	251.54	+ 23	10	100	17.6	249.28	.987
Brühl, ...	Aug. 1.	11 7	100	251.76	+ 23	10	90	17.7	249.51	.985
Brühl, ...	Aug. 1.	11 17	100	251.76	+ 23	10	90	17.55	249.51	.985
Laach, ...	Aug. 4.	1 17	100	252.71	+ 23	10	80	21.7	249.97	.982
Laach, ...	Aug. 4.	1 33	100	252.21	+ 23	10	80	21.0	249.62	.985
Mont Salève, near Geneva, . .	Aug. 17.	1 43	100	240.61	+ 20	10	110	17.0	238.51	1.078
Mont Salève, . .	Aug. 17.	1 53	100	240.79	+ 20	10	110?	17.1	238.71	1.077
Geneva, ...	Aug. 20.	11 21	100	241.17	+ 27	10	100	21.8	238.60	1.078
Geneva, ...	Aug. 20.	11 39	100	241.17	+ 27	10	90	21.9	238.60	1.078
Mont Breven, . .	Aug. 22.	2 12	100	239.27	+ 27	10	90	15.7	237.36	1.089
Chamouni, . . .	Aug. 23.	12 39	100	240.06	+ 27	10	110	18.4	237.82	1.085
Jardin, ...	Aug. 25.	12 29	100	239.40	+ 29	10	120	11.0	237.95	1.084
Chamouni, . . .	Aug. 26.	1 25	100	239.70	+ 27	10	110?	15.0	237.83	1.085
Col des Fours, . .	Aug. 28.	9 28	100	238.19	+ 27	10	110?	4.8	237.43	1.088
Aoste, ...	Aug. 29.	4 8	100	238.71	+ 27	10	130	16.7	236.64	1.096
St Bernard, . . .	Aug. 30.	5 22	100	239.07	+ 27	10	120	6.8	238.08	1.082
Martigny,[2] . . .	Sept. 1.	8 39	100	239.87	+ 27	10	90	14.8	238.05	1.083
Interlaken, . . .	Sept. 10.	4 49	100	241.41	+ 14	90	90	14.3	239.66	1.068
Schmadribach, Valley of Lauterbrunnen, . .	Sept. 12.	12 34	100	240.11	+ 14	10	100	12.3	238.63	1.077
Schmadribach,[3] . .	Sept. 12.	12 44	100	239.71	+ 14	10	110	11.5	238.25	1.081
Grindelwald, . .	Sept. 14.	9 59	100	240.80	+ 14	10	90	14.35	239.08	1.074
Grindelwald, . .	Sept. 14.	10 8	100	240.51	+ 14	10	100	14.35	238.75	1.076
Meyringen, . . .	Sept. 16.	5 9	100	240.70	+ 14	10	120	12.45	239.12	1.073
Meyringen, . . .	Sept. 16.	5 17	100	240.31	+ 14	10	110	11.95	238.80	1.076
Grimsel, ...	Sept. 17.	5 57	100	240.00	+ 14	10	110	7.45	238.96	1.074
Münster[4] (Vallais) .	Sept. 18.	5 26	100	239.96	+ 14	10	110	12.5	238.40	1.080
Münster, . . .	Sept. 18.	5 43	100	240.04	+ 14	10	100	10.4	238.72	1.077
Gemmi, summit, .	Sept. 21.	8 46	100	239.71	+ 14	10	120	9.9	238.41	1.079
Gemmi, ...	Sept. 21.	8 54	100	239.83	+ 14	10	120	9.6	238.60	1.078
Frütigen (Kanderthal)	Sept. 21.	5 2	100	240.56	+ 14	10	110	12.45	239.00	1.074
Frütigen, . . .	Sept. 21.	5 11	100	240.56	+ 14	10	110	11.6	239.09	1.073

† α. indicates the initial semi-arc of vibrations; m. the number of vibrations required to reduce it to half its amount.

[1] Rather windy. [2] Local disturbance suspected. [3] Superior to the last. [4] Suspension not quite free.

* Many of the numerical calculations contained in the remainder of this paper, have been made by two of my pupils Messrs Irvine and Edward, under my own inspection and revision.

Table V.—(*continued.*)

NEEDLE, No. I.—(*continued.*)

Place.	Date.	Mean Time.	No. of Variations Observed.	Observed Time.	Rate Chronometer.	Arc. α.	Arc. m.	Temp. Reaum.	Corrected Time 100 Vibrations.	Intensity Paris = 1.
	1832.	h m		s	s			s	s	
Grindelwald, . . .	Sept. 23.	8 50	100	240.14	+ 14	10	110	6.5	239.21	1.072
Faulhorn, . . .	Sept. 24.	8 13	100	240.57	+ 14	10	110	8.0	249.48	1.070
Faulhorn, . . .	Sept. 24.	8 22	100	240.39	+ 14	10	110	7.25	239.37	1.071
Engelberg, . . .	Sept. 27.	5 22	100	241.13	+ 14	10	110	11.3	239.70	1.068
Engelberg, . . .	Sept. 27.	5 30	100	240.54	+ 14	10	110	10.6	239.20	1.073
Surennes, ...	Sept. 28.	10 49	100	240.81	+ 14	10	120	12.8	239.20	1.072
Surennes, . . .	Sept. 28.	10 59	100	240.99	+ 14	10	110	12.8	239.40	1.071
Klus, near Altorf, .	Sept. 28.	5 24 ?	100	240.81	+ 14	10	110	11.8	239.31	1.071
Klus, ...	Sept. 28.	5 34	100	240.63	+ 14	10	110	10.7	239.25	1.072
St Gothard, . . .	Sept. 30.	8 35	100	240.47	+ 14	10	100	7.1	239.50	1.070
St Gothard, . . .	Sept. 30.	8 50	100	240.09	+ 14	10	110	6.3	239.16	1.072
St Gothard, . . .	Sept. 30.	8 58	100	239.64	+ 14	10	110	6.1	238.76	1.076
Locarno, ...	Oct. 2.	2 34	100	240.20	+ 14	10	110	18.95	238.50	1.084
Orta, ...	Oct. 4.	12 38	100	239.11	+ 14	10	110	18.45	236.91	1.093
Orta, ...	Oct. 4.	12 46	100	239.26	+ 14	10	110	18.85	237.0	1.092
Bellaggio, . . .	Oct. 8.	8 5	100	238.16	+ 14	10	100	12.85	236.60	1.096
Bellaggio, . . .	Oct. 8.	8 16	100	238.41	+ 14	10	90	12.8	236.82	1.094
Reichenau,[1] . .	Oct. 10.	3 32	100	240.43	+ 14	10	110	11.85	238.93	1.075
Reichenau, . .	Oct. 10.	3 41	100	240.07	+ 14	10	100	11.25	238.65	1.077
Wallenstadt, . .	Oct. 12.	9 39	100	241.01	+ 14	10	110	12.05	239.41	1.070
Wallenstadt, . .	Oct. 12.	9 49	100	241.17	+ 14	10	110	12.4	239.62	1.068
Lucerne, ...	Oct. 15.	10 11	100	240.83	+ 14	10	120	8.6	239.66	1.068
Rigi Culm, . . .	Oct. 16.	7 55	100	240.71	+ 14	10	110	3.15	240.36	1.062
Geneva.	Oct. 29.	12 9	100	240.31	+ 21	10	100	10.95	238.97	1.075
Geneva, ...	Nov. 10.	11 33	100	240.14	+ 21	10	110	7.0	239.14	1.073
Geneva, ...	Nov. 10.	11 42	100	240.00	+ 21	10	110	7.0	239.06	1.074
	1833.									
Edinburgh, . . .	May 7.	4 33	300	217.86	+ 3	20	90	18.13	270.10	.841
Edinburgh, . . .	May 7.	4 50	100	272.29	+ 3	10	90	17.9	269.91	.842
Paris,[2]	June 11.	12 38	300	750.94	...	20	110	20.27	247.70	1.000
Paris,[3]	June 11.	12 55	100	250.13	...	10	120	20.2	247.67	1.000
	1835.									
Edinburgh, . . .	May 4.	1 18	100	271.8	...	10	90	9.8	270.42	.839
Edinburgh, . . .	May 4.	1 33	100	271.16	...	10	90	9.2	270.40	.839
Paris, ...	June 13.	3 41	100	249.96	— 40.5	10	100	20.4	247.62	1.001
Paris, ...	June 13.	3 51	100	250.0	— 40.5	10	100	20.15	247.63	1.000
Pic de Bergons, near Luz, Hautes Pyrénées,[4] . . .	July 18.	10 16	100	236.26	+ 20	10	100?	14.75	234.50	1.116
Pic de Bergons,[4] .	July 18.	10 38	100	236.33	+ 20	10	110	14.2	234.60	1.115
Pic de Bergons,[4] .	July 18.	10 51	100	236.56	+ 20	10	100	14.1	234.82	1.113
Luz,[5]	July 20.	11 9	100	235.84	+ 20	10	100	17.65	233.74	1.123
Luz,[6]	July 20.	11 25	100	235.67	+ 20	10	90	18.25	233.54	1.125
Luz,[6]	July 20.	11 34	100	236.03	+ 20	10	80	18.7	233.86	1.122
Luz,[7]	July 20.	11 45	100	235.14	+ 20	10	80	19.05	232.94	1.131
Pic de Bergons, . .	July 21.	10 38	100	236.20	+ 20	10	120?	15.25	234.34	1.117
Pic de Bergons, .	July 21.	10 47	100	236.57	+ 20	10	100?	15.0	234.74	1.113
Pic de Bergons, .	July 21.	10 59	100	236.17	+ 20	10	110	15.1	234.34	1.117
Luz, ...	July 28.	11 10	100	235.69	+ 20	10	80	18.85	233.50	1.125
Luz, ...	July 28.	11 23	100	235.80	+ 20	10	100	18.9	233.60	1.124
Gavarnie, ...	July 29.	11 52	100	235.62	+ 20	10	100	19.2	233.38	1.127
Gavarnie, ...	July 29.	12 4	100	235.60	+ 20	10	100	19.0	233.38	1.126
Ste Marie, valley of Campan, .	Aug. 7.	10 4	100	236.19	+ 20	10	110	19.45	233.91	1.121
Ste Marie, . . .	Aug. 7.	10 16	100	236.30	+ 20	10	100	19.3	234.04	1.120
Pic du Midi, . .	Aug. 8.	10 4	100	235.01	+ 20	10	100	9.45	233.88	1.122
Pic du Midi, . .	Aug. 8.	10 18	100	235.13	+ 20	10	110	9.35	233.90	1.121
Pic du Midi, . .	Aug. 8.	10 38 ?	100	235.19	+ 20	10	110	9.65	233.90	1.121
Brèche de Roland,[8]	Aug. 11.	10 40	100	235.33	+ 20	10	110	12.35	233.80	1.122
Brèche de Roland,[9]	Aug. 11.	10 53	100	235.34	+ 20	10	120	12.6	233.76	1.123

[1] Indifferent observation. [2] Chronometer very near needle. [3] Chronometer three feet from needle.
[4] These observations being not quite unexceptionable, owing to a small compass needle being accidentally retained in the pocket, were repeated three days after. [5] Windy, but observation good. [6] Windy.
[7] The best observations at Luz. [8] A knife in the pocket. [9] Unexceptionable.

Table VI.

FLAT NEEDLE.

Place.	Date.	Mean Time.	No. of Vibrations Observed.	Observed Time.	Rate Chronometer.	Arc.* α.	Arc.* m.	Temp. Reaum.	Corrected Time 100 Vibration.	Intensity Paris = 1.
	1832.	h m		s	s			o	s	
Edinburgh, . . .	June 2.	11 39	300	1075.97	+ 13	20	90	15.95	356.54	.839
Brussels, ...	July 9.	12 55	300	1008.39	+ 17	20	100	23.1	333.38	.965
Königstuhl, near Heidelberg, .	July 28.	11 17	100	327.84	+ 23	10	110	15.1	326.06	1.012
Heidelberg, . . .	July 28.	11 47	100	327.30	+ 23	10	110	15.1	325.52	1.015
Brühl, ...	Aug. 1.	11 34	100	332.80	+ 27	10	140	17.0	330.72	.984
Brühl, ...	Aug. 1.	11 47	100	333.51	+ 27	10	140	17.0	331.43	.980
Salève Summit, . .	Aug. 17.	2 14	100	319.10	+ 20	10	120	17.4	317.14	1.073
Geneva, ...	Aug. 20.	11 55	100	319.23	+ 27	10	120	22.2	316.80	1·076
Mont Breven, . .	Aug. 22.	1 52	100	315.89	+ 27	10	130	16.2	314.01	1.095
Chamouni, . . .	Aug. 23.	12 56	100	318.83	+ 27	10	130	18.6	316.71	1.077
Jardin, ...	Aug. 25.	12 45	100	317.47	+ 27	10	130	11.3	316.04	1.082
Chamouni, . . .	Aug. 26.	1 48	100	318.11	+ 27	10	140	15.3	317.01	1.075
Aoste, ...	Aug. 29.	4 29	100	317.00	+ 27	10	140	16.1	315.12	1.089
St Bernard,[1] . . .	Aug. 31.	8 46	100	318.93	+ 27	10	140?	9.1	317.69	1.072
St Bernard, . . .	Aug. 81.	9 0	100	319.34	+ 27	10	140	9.6	317.84	1.071
St Bernard, . . .	Aug. 31.	10 13	100	318.77	+ 27	10	140	11.3	317.88	1.070
Martigny,[2] . . .	Sept. 1.	8 54	100	318.81	+ 27	10	100	14.5	317.13	1.075
Martigny, . . .	Sept. 1.	9 7	100	318.84	+ 27	10	100	14.6	317.14	1.075
Interlaken, . . .	Sept. 10.	5 4	100	321.20	+ 14	10	140	14.05	319.52	1.061
Interlaken, . . .	Sept. 10.	5 16	100	320.93	+ 14	10	150	14.05	319.18	1.063
Schmadribach, . .	Sept. 12.	12 59	100	319.69	+ 14	10	130?	11.05	318.32	1.069
Grindelwald, . .	Sept. 14.	10 29	100	319.80	+ 14	10	130	14.45	318.12	1.071
Grimsel, ...	Sept. 18.	8 6	100	320.53	+ 14	10	150	6.55	319.54	1.062
Grimsel, . . .	Sept. 18.	8 18	100	319.99	+ 14	10	150	7.0	318.98	1.066
Grimsel.	Sept. 18.	8 29	100	320.27	+ 14	10	150?	7.4	319.25	1.064
Münster, ...	Sept. 18.	5 58	100	319.84	+ 14	10	140	9.5	318.60	1.068
Gemmi Summit, .	Sept. 21.	9 12	100	319.79	+ 14	10	140	9.65	318.54	1.069
Grindelwald, . .	Sept. 23.	7 13	100	320.14	+ 14	10	120	6.7	319.21	1.065
Grindelwald, . .	Sept. 23.	9 30	100	320.46	+ 14	10	140	8.2	319.33	1.064
Faulhorn, . . .	Sept. 24.	8 43	100	320.69	+ 14	10	150	7.5	319.61	1.062
Faulhorn, . . .	Sept. 24.	8 54	100	321.03	+ 14	10	140	7.0	320.02	10.60
Faulhorn, . . .	Sept. 24.	9 6	100	321.09	+ 14	10	140?	6.8	320.11	1.059
Surennes, ...	Sept. 28.	11 20	100	321.39	+ 14	10	140	12.65	319.85	1.062
Klus, near Altorf, .	Sept. 28.	5 49	100	321.13	+ 14	10	130	9.85	319.86	1.061
St Gothard, . . .	Sept. 30.	9 17	100	320.73	+ 14	10	130	6.3	319.81	1.062
Locarno, ...	Oct. 2.	2 52	100	319.50	+ 14	10	150	18.55	317.39	1.079
Bellaggio, . . .	Oct. 8.	8 38	100	317.51	+ 14	10	130	13.4	315.93	1.090
Wallenstadt, . .	Oct. 12.	10 7	100	321.64	+ 14	10	130	13.05	320.03	1.063
Lucerne, ...	Oct. 15.	10 26	100	321.94	+ 14	10	140	9.0	320.74	1.058
Geneva, ...	Nov. 10.	11 58	100	321.10	+ 21	10	130	7.0	320.08	1.067
Geneva, ...	Nov. 10.	12 10	100	321.16	+ 21	10	130	7.0	320.14	1.066
	1833.									
Edinburgh, . . .	May 7.	5 14	300	1094.00	+ 3	20	120	17.65	362.20	.840
Edinburgh, . . .	May 7.	5 36	100	364.34	+ 3	10	120?	17.0	362.2	.840
Paris, ...	June 11.	2 0	300	1005.17	...	20	150	20.8	332.34	1.000
	1835.									
Edinburgh, . . .	May 4.	1 53	100	371.84	...	10	80	9.0	370.65	.840
Edinburgh, . . .	May 4.	2 7	100	371.39	...	10	120	8.7	370.15	.842
Paris, ...	June 13.	4 8	100	342.19	— 40.5	10	130?	19.9	340.04	1.000
Paris, ...	June 13.	4 22	100	342.00	— 40.5	10	130?	19.8	340.10	1.000
Pic de Bergons,[3] .	July 18.	4 23	100	324.63	+ 20	10	150	14.95	322.81	1.112
Pic de Bergons,[4] .	July 21.	11 17	100	324.81	+ 20	10	120	15.0	323.04	1.111
Pic de Bergons,[4] .	July 21.	11 31	100	324.84	+ 20	10	130	15.1	323.04	1.111
Luz, ...	July 28.	11 44	100	323.40	+ 20	10	130	19.0	321.23	1.124
Luz, ...	July 28.	11 58	100	323.48	+ 20	10	130?	19.0	321.31	1.123
Luz, ...	July 28.	12 18	100	323.40	+ 20	10	80	19.0	321.34	1.123
Gavarnie, ...	July 29.	12 32	100	323.43	+ 20	10	140	19.0	321.25	1.124
Ste Marie,[5] . . .	Aug. 7.	10 48	100	323.98	+ 20	10	140	19.55	321.68	1.121
Ste Marie,[6] . . .	Aug. 7.	11 7	100	324.09	+ 20	10	140	19.85	321.82	1.120
Pic de Midi, . . .	Aug. 8.	11 5	100	323.71	+ 20	10	130?	11.25	322.28	1.117
Pic de Midi, . . .	Aug. 8.	11 22	100	324.03	+ 20	10	150	10.4	322.64	1.115
Brèche de Roland,[7]	Aug. 11.	11 17	100	324.37	+ 20	10	140	12.6	322.78	1.114

* *α.* indicates the initial semi-arc of vibration ; *m.* the number of vibrations required to reduce it to half its amount.

[1] The dipping-needle at first near the instrument. [2] Local disturbance suspected. [3] Small compass in pocket.

[4] Unexceptionable. [5] Unsteady. [6] Good. [7] Extremely tremulous. Gusts of wind.

Table VII.

Place.	Particular Situation.	Latitude, N.	Long. from Paris.	Height, Eng. feet.	Observed Intensity; Paris = 1.000. Needle, No. 1.	Mean.	Flat Needle.	Mean.
Edinburgh, . . .	Greenhill, field, June 1832, ...	55° 57′	5° 33′ W.	300	.840	.840	.839	.840
Edinburgh, . . .	——— garden, May 1833, . . .	55 57	5 33	300	.841 .842		.840 .840	
Edinburgh, . . .	——— field, S. of garden, May 1835,	55 57	5 33	300	.839 .839		.840 .842	
Brussels, . . .	Enclosure of Observatory, 20 yards from NW. corner of the building, . .	50 51	2 2 E.	300	.959 .960	.960	.965	.965
Spa, ...	South face of clay-slate hill, close to the town, on the NW. side, . .	...	...	...	.981 .977	.979		
Königstuhl, near Heidelberg, .	Summit of the hill, ...	49 25	6 23	1700	1.018 1.018	1.018	1.012	1.012
Heidelberg, . . .	Prof. Leonhard's garden, on a stone table,	49 25	6 12	300	1.017	1.017	1.015	1.015
Brühl, ...	In a quarry near the bank of the Rhine,	...	...	...	.987 .985 .985	.986	.984 .980	.982
Laach, ...	NW. side of the lake, ...	50 25	4 56	1000	.982 .985	.984		
Mont Salève, near Geneva, . . .	Summit of the Grand Salève, . . .	46 6	3 51	4500	1.078 1.077	1.077	1.073	1.073
Geneva, ...	Botanic Garden, August 1832, . . .	46 12	3 49	1300	1.078 1.078	1.076	1.076	1.071
Geneva, ...	——— ——— November 1832, . .	46 12	3 49	1300	1.075 1.073 1.074		1.067 1.066	
Mont Breven, . .	Summit, ...	45 56	4 30	8400	1.089	1.089	1.095	1.095
Chamouni, . .	On the farther side of the Arve from the village, ...	45 55	4 32	3400	1.085 1.085	1.085	1.077 1.075	1.076
Jardin, ...	At the "Pierre d'Herschel," . . .	45 55	4 39	9000	1.084	1.084	1.082	1.082
Col des Fours, . .	Close to the snow, in a cleft of rock, .	45 45	4 25	8900	1.088	1.088		
Aoste, ...	In a summer-house (built entirely of wood, and without nails) in the garden of the inn, ...	45 44	5 00	1900	1.096	1.096	1.089	1.089
St Bernard, . . .	Between the Hospice and the lake, .	45 52	4 50	8100	1.082	1.082	1.072 1.071 1.070	1.071
Martigny, . . .	Garden of the inn, ...	46 6	4 45	1600	1.083	1.083	1.075 1.075	1.075
Interlaken, . . .	On the bank of the Aar	46 42	5 32	1900	1.068	1.068	1.061 1.063	1.062
Schmadribach, .	Near the upper cascade (at the head of the valley of Lauterbrunnen),	46 31	5 33	5200	1.077 1.081	1.079	1.069	1.069
Grindelwald, . .	Behind the inn; 14th Sept. 1832, . .	46 38	5 42	3700	1.074 1.076	1.075	1.071	1.071
Grindelwald, . .	At the lower glacier; 23d Sept. 1832,	46 38	5 42	3400	1.072	1.072	1.065 1.064	1.064
Meyringen, . . .	Near the church, ...	46 44	5 52	2100	1.073 1.076	1.075		
Grimsel, ...	Near the Hospice, ...	46 34	5 59	6200	1.074	1.074	1.062 1.066 1.064	1.064
Münster (Vallais),	On the bank of the stream, just above the town, ...	46 30	5 57	4200	1.080 1.077	1.078	1.068	1.068
Gemmi, ...	Within shelter-house near the summit,	46 25	5 17	7500	1.079 1.078	1.078	1.069	1.069
Frütigen (Kanderthal),	A little below the town, ...	46 36	5 18	2300	1.074 1.073	1.073		
Faulhorn, . . .	Summit, to the W. of the house, . .	46 40	5 40	8900	1.070 1.071	1.071	1.062 1.060 1.059	1.060
Engelberg, . .	At the side of the road before reaching the town, ...	46 49	6 7	3400	1.068 1.073	1.071		
Surennes, ...	Summit of the Pass, ...	46 49	6 13	7700	1.072 1.071	1.071	1.062	1.062
Klus (near Altorf),	On a muir, ...	46 49	6 19	1600	1.071 1.072	1.072	1.061	1.061
St Gothard, . . .	At the summit-level N. of the Hospice,	46 34	6 14	7100	1.070 1.072	1.071	1.062	1.062
Locarno, ...	In the bed of the torrent below the Convent,	46 10	6 28	700	1.084	1.084	1.079	1.079
Orta (on Lake Orta),	At the side of the road, above the town,	45 47	6 4	1000?	1.093 1.092	1.092		
Bellaggio (Lake Como,)	On the shore N. of the town, . . .	46 00	6 56	700	1.096 1.094	1.095	1.090	1.090
Reichenau, . .	On the bank of the Upper Rhine, a little above the junction of the streams,	46 49	7 4	2000	1.075 1.077	1.076		
Wallenstadt, . .	Near the lake; S. end, ...	47 7	7 00	1400	1.070 1.068	1.069	1.063	1.063
Lucerne, . . .	In a wood-yard on the E. side of the lake, near the bridge, ...	47 03	5 59	1500	1.068	1.068	1.058	1.058
Rigi-Culm, . . .	To the N. of the inn, ...	47 03	6 09	5900	1.062	1.062		
Paris Observatory,	M. Arago's Magnetic Cabinet, June 1833,	48 50	0 00	200	1.000 1.000	1.000	1.000	1.000
Paris Observatory,	——— ——— ——— June 1835,	48 50	0 00	200	1.001 1.000		1.000	
Pic de Bergons, near Luz, Hautes Pyrénées, . . .	Summit, 18th July 1835, ...	42 50	2 18 W.	6900	1.116 1.115 1.113	1.115	1.112	1.111
Pic de Bergons, .	——— 21st July 1835, ...	42 50	2 18	6900	1.117 1.113 1.117		1.111 1.111	

TABLE VII.—(*continued*).

PLACE.	Particular Situation.	Latitude, N.	Long. from Paris.	Height, Eng. feet.	Observed intensity; Paris = 1.600.			
					Needle, No. I.	Mean.	Flat Needle.	Mean.
Luz, Hautes Pyrénées,	Field E. of the village, 20th July, . .	42° 51′	2° 20′ W.	2400	1.123 1.125 1.122 1.131	1.125		
Luz, Hautes Pyrénées,	——— 28th ——— . .	42 51	2 20	2400	1.125 1.124		1.124 1.123 1.123	1.123
Gavarnie, ...	Near the road, to the N. of the inn, .	42 43	2 21	4500	1.127 1.126	1.126	1.124	1.124
Ste Marie, Vallée de Campan, . .	In a field above the village, ...	42 59	2 07	2800	1.121 1.120	1.120	1.121 1.120	1.120
Pic du Midi de Bigorre,	Summit, ...	42 55	2 14	9600	1.122 1.121 1.121	1.121	1.117 1.115	1.116
Brèche de Roland,	W. side; N. aspect of the Breche, . .	42 41	2 20	9300	1.122 1.123	1.123	1.114?	

§ 4. *On the Direction of the Isodynamic Lines (for horizontal Intensity) in the Central Alps, and in the Pyrénées, and on the Influence of Height.*

25. The next question comes to be how to deduce the general results contained in the preceding tables. Where it is merely required to deduce the position of Isodynamic Lines (which may be considered as sensibly straight for a district of moderate extent), projection of the results upon paper would afford quite a sufficient approximation, where the stations are sufficiently multiplied. Thus the variations in latitude and longitude would be determined, and lines of intensity 1.00, 1.01, 1.02, &c. might be drawn with great accuracy upon a geographical map.

26. But the same process will not suffice, if we have a third variable, such as height, and require to extract its influence. The problem, then, is not to draw lines, but planes of equal intensity. For its solution I resolved to use the method of least squares,* which is peculiarly applicable to a question of the kind just stated, and may be made to give, as will immediately be seen, the most probable value of the four following quantities, viz. the variation of intensity for 1′ of latitude; its variation for 1′ of longitude; its variation for 100 feet of elevation; its most probable absolute value at the origin of the co-ordinates, or the station to which the others are referred.

27. I assumed that the intensity of any point whose co-ordinates of latitude, longitude, and height, might be denoted with sufficient accuracy by an expression of the form

$$a x + b y + c z = \mathrm{I} \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

* It would be absurd to claim any merit for the application of a method so universally known. But lest I should be supposed to have borrowed without acknowledgment the method of reduction employed by Professor LLOYD and Captain SABINE in their excellent Magnetic Survey of Ireland (Fifth Report of the British Association), I desire to state, that I had some years ago proposed to myself the present method of reduction as the only one adapted finally to solve (within the present limits of error) the question of the influence of height, which so greatly complicates the problem.

a, b, and c, indicating the position of the point by reference to the three co-ordinates, whilst x, y, and z denote the coefficients of variation of intensity according to each of these, and which are to be discovered. The above expression being the equation to a plane, denotes that the isodynamic lines are not considered as curved, but as straight, which though not absolutely accurate, may be admitted in a country of small extent.

28. Eq. (1) gives the intensity I in terms of a, b, and c, the co-ordinates of the place, a being reckoned in *minutes* of latitude, b in *minutes* of longitude, c in *hundreds of feet* of elevation. It is convenient to assume some station as a point of reference, and write for a, b, and c, the *differences* of the co-ordinates merely, and for I the *difference* of intensities. Let a', b', c', and I' represent these quantities for the fundamental station, and then for any other the expression will be

$$(a - a')\, x + (b - b')\, y + (c - c')\, z = I - I'$$

and by a combination of all the equations of similar form which the observations furnish, we are to deduce the most probable values of x, y, and z, the coefficients of variation in each direction. If, farther, we wish to have the most probable *absolute* value of the horizontal intensity at the fundamental station before mentioned, it must clearly be deduced from the whole mass of the observations, and not from the observation made there alone. Let us suppose, then, that the intensity at the fundamental station requires a small correction, $\delta I'$, we shall write $I' + \delta I'$ instead of I' in the preceding expression, considering $\delta I'$ as another unknown quantity, which will give us a series of equations (for the different points of observation) of the form

$$(a - a')\, x + (b - b')\, y + (c - c')\, z = I - I' - \delta I' \ldots\ldots\ldots (2)$$

or using the letters with subscript numerals instead of $a - a'$, &c. and putting all the unknowns on the left hand, we shall have a series of equations of condition of the form

$$\left.\begin{array}{l} a_1\, x + b_1\, y + c_1\, z + \delta I' = I_1 \\ a\ \ x + b_2\, y + c_2\, z + \delta I' = I_2 \\ \qquad\qquad \text{\&c.} \end{array}\right\} (3)$$

from which the most probable values of x, y, z, and $\delta I'$ are to be deduced by the method of least squares.

29. The observations contained in Table VII. include two groups of observations, to which we mean to apply the method in question. One of these includes the alpine observations made in August, September, and October 1832; the other, a short series in the Pyrénées, made almost entirely with reference to the effect of height in 1835. The remaining observations must be considered for the present as isolated. They are important, however, as fixing the relative horizontal intensities at Paris, Edinburgh, Brussels, Heidelberg, and some points of less note. The admirable coincidence of the Edinburgh observations made in different years gives great confidence in the accuracy of the determination of .8402 for the hori-

zontal intensity, that at Paris being = 1; both needles giving the same mean result to four decimal places. Professor HANSTEEN has given .8428, which must be considered as a close coincidence.*

For Brussels I find by "No. 1,"	. . . . . .	0.960
by "Flat,"	. . . . . .	0.965
Captain SABINE,	. . . .	0.951
M. QUETELET (4 series),	. . . . . .	0.964
M. RUDBERG,	. . . .	0.971

I subjoin a few comparisons of stations common to M. QUETELET's series† and mine.

	Quetelet.	Forbes, No. I.
Castle of Heidelberg,	1.020	1.017
Town of do.	1.024	
Königstuhl (summit),	1.027	1.018
Geneva,	1.080	1.076
Chamouni,	1.093	1.085
St Bernard,	1.097	1.082
Martigny,	1.092	1.083

30. But to return to the calculation of the first group of observations, those including the alpine country of Switzerland, Savoy, and Italy. If we arrange the observations relatively to Geneva as a fundamental station, taking the data from Table VII. and writing the equations of condition in the form (3) art. 28, where a_1 denotes the excess of northern latitude of the given station above that of Geneva in minutes; b_1 the excess of *eastern* longitude in minutes of a degree; c_1 the excess in height, reckoned in hundreds of English feet in round numbers (using of course negative signs to represent the reverse of all this), we shall have the following equations of condition, distinguishing from one another the absolute numbers obtained by the two needles, in order that they may be separately calculated.

TABLE VIII.

Equations of Condition for the Alpine Series.

		By No. I.	By "Flat."
Geneva (Aug. 1832),	$0'x + 0'y + 0z + \delta I' =$	.002	.005
Geneva (Nov. 1832),	$0x + 0y + 0z + \delta I' = -$	.002	—.005
Mont Salève,	$-6x + 2y + 32z + \delta I' =$	.001	.002
Mont Breven,	$-16x + 41y + 71z + \delta I' =$	.013	.024
5. Chamouni,	$-17x + 43y + 21z + \delta I' =$	.009	.005

* Since this paper was read, this result has been still more nearly confirmed by the observations of Professor BACHE of Philadelphia, who, by connecting Edinburgh and Dublin, and taking Professor LLOYD and Captain SABINE's observations for the comparative intensities at Dublin and Paris, has obtained the number .8400.

† See his two papers in the *Mémoires de l'Academie de Bruxelles*, tome iv.; and an abstract in the *Annuaire de l'Observatoire de Bruxelles*, 1834.

TABLE VIII.—(*continued.*)

	Station	Equation	No. I.	By "Flat."
	Jardin,	$-17x + 50y + 77z + \delta I' =$	.007	.009
	Col des Fours,	$-27x + 36y + 76z + \delta I' =$	.012	
	Aoste,	$-28x + 71y + 6z + \delta I' =$	.020	.018
	St Bernard,	$-20x + 61y + 68z + \delta I' =$	.006	.000
10.	Martigny,	$-6x + 56y + 3z + \delta I' =$	.007	.004
	Interlaken,	$30x + 103y + 6z + \delta I' =$	−.008	−.009
	Schmadribach,	$19x + 104y + 39z + \delta I' =$	.003	−.002
	Grindelwald (1.)	$26x + 113y + 24z + \delta I' =$	−.001	.000
	Grindelwald (2.)	$26x + 113y + 21z + \delta I' =$	−.004	−.007
15.	Meyringen,	$32x + 123y + 7z + \delta I' =$	−.001	
	Grimsel,	$22x + 130y + 49z + \delta I' =$	−.002	−.007
	Münster,	$18x + 128y + 29z + \delta I' =$	.002	−.003
	Gemmi,	$13x + 88y + 62z + \delta I' =$	.002	−.002
	Frütigen,	$24x + 89y + 10z + \delta I' =$	−.003	
20.	Faulhorn,	$28x + 111y + 76z + \delta I' =$	−.005	−.011
	Engelberg,	$37x + 138y + 21z + \delta I' =$	−.005	
	Surennes,	$37x + 144y + 64z + \delta I' =$	−.005	−.009
	Klus (near Altorf),	$37x + 150y + 3z + \delta I' =$	−.004	−.010
	St Gothard,	$22x + 145y + 58z + \delta I' =$	−.005	−.009
25.	Locarno,	$-2x + 159y - 6z + \delta I' =$	.008	.008
	Orta,	$-25x + 135y - 3z + \delta I' =$	.016	
	Bellaggio,	$-12x + 187y - 6z + \delta I' =$	.019	.019
	Reichenau,	$37x + 195y + 7z + \delta I' =$	.000	
	Wallenstadt,	$55x + 191y + z + \delta I' =$	−.007	−.008
30.	Lucerne,	$51x + 130y + 2z + \delta I' =$	−.008	−.013
	Rigi-Culm,	$51x + 140y + 46z + \delta I' =$	−.014	

31. From these thirty-one equations of condition for Needle No. I, and twenty-four for the Flat Needle, we obtain by the method of least squares the following values of the four unknown quantities, the calculations having been verified by independent methods.

	No. I.	Flat.
$x =$ Variation of intensity for 1′ of latitude N. increasing,	−.000364	−.000505
$y =$ Variation of intensity for 1′ of longitude E increasing,	+.000055	−.000106
$z =$ Variation of intensity for 100 English feet of height,	−.000033	−.000027
$\delta I' =$ Correction applicable to the registered intensity at Geneva,	+.0016	−.0040

32. To deduce from these numbers the lines of equal horizontal intensity, we must remark that the minute of longitude is shorter than the minute of latitude in the ratio of $7\frac{1}{4}$ to 10 nearly, on an average, in the Alps. The variation of y for a geographical mile or minute would therefore be

For No. I. $= + .000076$. For "Flat" $= + 000146$

And the angle made by the isodynamical lines with the meridian towards the east from north would be

$$\text{Arc whose tang.} = \frac{364}{76}, \text{ and arc whose tang.} = \frac{505}{146} = 78^\circ\ 12' \text{ and } 73^\circ\ 52'$$

33. Of these results I conceive that the former is to be preferred. The discrepancy of the results obtained by No. I and "Flat" are, I presume, attributable to one or both of two causes,—a progressive change in the magnetic state of the needle somewhat different from what has been allowed for,—and a slight error in the correction for temperature, which, during the period of observation (the autumn), was generally diminishing. Now both these points being best ascertained for No. I, I prefer abiding by its indication. In fact, it appears by Table VII. that the intensity of the Flat Needle decreased from August to November (by the Geneva observations) faster than the mean rate of decrease allowed; the consequence of which would necessarily be, that the standard intensity at Geneva for purposes of comparison would be assumed too high, and, as the general order of the observations lay southward and eastward, the apparent increase of intensity in those directions would be smaller than the true, which would give rise to an error of the kind mentioned in Art. 31. The stability of No. I renders its indications the most certain. The agreement as to the effect of height is very satisfactory, considering the minuteness of the quantity. The source of error just alluded to would scarcely affect this result. The most probable intensity for Geneva will be 1.0776 for No. I,* and 1.0670 for the Flat Needle. The results are projected in the Map, Plate I.

34. The observations in the Pyrénées lie within smaller compass, and were chiefly conducted with a view to deduce the influence of height. The sources of local error arising from metalliferous deposits are, however, perhaps greater in this case.

35. Proceeding exactly as before, taking Luz in the valley of Lavedan or Baréges, Hautes Pyrénées, as our point of reference, we obtain the following equations of condition from Table VII., which may be arranged exactly as in Table VIII. incorporating the results of *both* needles. In this case the longitudes being westerly, the variation in longitude must be reckoned the opposite way from that in the former case.

Table IX.

Equations of Condition for the Pyrenean Series.

Luz (1.)	$0'x + 0'y + 0z + \delta I' = .000$
Luz (2.)	$0x + 0y + 0z + \delta I' = -.001$
Ste Marie, ...	$8x - 13y + 4z + \delta I' = -.004$
Pic du Midi, ...	$4x - 6y + 72z + \delta I' = -.005$
Pic de Bergons (1.)	$-x - 2y + 45z + \delta I' = -.010$
——— (2.)	$-x - 2y + 45z + \delta I' = -.010$
Gavarnie, ...	$-8x + y + 21z + \delta I' = +.001$
Brèche de Roland, ...	$-10x + 0y + 69z + \delta I' = -.001$

* The intensity varies .01 for 27′5 of latitude.

36. Combining these by the method of least squares, we obtain the following values:—

$$x = -.000210$$
$$y = +.000100$$
$$z = -.000053$$
$$\delta I' = -.0028$$

Hence it appears, that on the same parallel of latitude the intensity increases in a *westerly* direction, which is the reverse of the result found for the course of the isodynamic lines in the Alps; but, in truth, I do not attach much importance to these observations, unless for the sole consideration of height, on account of the small area of country over which the observations were made. There were probably in the Pyrénées some sources of local disturbance which the observations on the Pic de Bergons particularly indicate, and which, having been repeated with coincident results, could not be owing to an error of observation.* At the same time it is satisfactory to find that the influence due to height is the same in direction, though greater in amount than that obtained in the alpine series. On this subject I proceed to offer some remarks.

37. The first experiments which seem to have had even remotely in view the question of the decrease of magnetic intensity with height are those of SAUSSURE, made during his memorable stay on the Col du Géant in 1788. The observations were too rude, and differ too widely from each other to deserve much confidence; but those made at Chamouni and on the Col du Géant, which were fortunately under almost the same temperature, agree very closely, but give a slightly *greater* intensity to the latter, which is the effect due to the latitude.† The great diminution of intensity in going from Geneva to Chamouni, observed by SAUSSURE, is certainly erroneous, as the reverse has been shewn to take place.

38. In 1804 M. GAY LUSSAC performed his celebrated aerostatic ascent, and from his magnectic observations concluded that no appreciable difference of intensity existed at the surface of the earth and at the height of 23,000 feet. This, however, can only be considered as referring to great and palpable change. The difficulties inseparable from the experiment prevented many oscillations from being observed, or great precision in the times from being attempted, whilst corrections for arc, diurnal variation, and temperature, were not applied. The last of these,

* Since this passage was written, on mentioning to Professor NECKER of Geneva, the anomalous result as to the direction of the isodynamic lines in the Pyrenees (anomalous, because differing from the supposed direction inserted in HANSTEEN's maps, which is deduced from analogy, and not, I believe, from direct observations in that country), he pointed out the curious (though perhaps accidental) coincidence which this result offers to the views he has long entertained as to the general parallelism of the lines of geological elevation, and those of magnetical intensity, which the bearing of the isodynamic lines which I have given for the Alps remarkably confirms.

† SAUSSURE, Voyages aux Alpes, § 2103. Tom iv.

however, could hardly fail to be sensible, the variation of temperature being no less than 36° Cent., and as cold tends to increase the apparent intensity, if no such increase was observed, it might plausibly be argued that the *real* intensity had diminished. It must, however, be observed, that the observations lasted only in general from one to two minutes, and that in so short a period (and depending on a single value of the elapsed time) the acceleration due to the above-mentioned cause would hardly be perceptible. Taking the mean result of the effect of temperature ascertained by myself for No. I. and "Flat" needles, we find the factor — .00037 applicable to the time for a decrease of 1° R. of temperature, which agrees exactly with HANSTEEN's mean correction. If we apply this to GAY LUSSAC's observation we find a correction of — .0108 as a factor applicable to the time, for the effect of — 36° Cent. of temperature. Yet large as this is, amounting to $\frac{1}{100}$th part of the whole, the discrepancies of observation often amounts to double that quantity.* Still we admit with M. KUPFFER that the probability deducible from M. GAY LUSSAC's observations, is in favour of a slow diminution.

39. The next series of observations includes those of HUMBOLDT and GAY LUSSAC, recorded in the *Mémoires d'Arcueil*,† which include observations made in the Alps, though at no great heights; and here no particular influence of height was observed, nor was indeed looked for.‡

40. Since that period the subject seems to have met with little practical attention, until the recent publication of M. KUPFFER's "Voyage au Mont Elbrouz" by the Petersburg Academy. From his observations with a needle by GAMBEY, half a metre long, M. KUPFFER attempts to deduce, not only the fact, but the amount of the diminution with height, and this upon the authority of a single experiment, and at no considerable elevation.§ In fact, all the intricate corrections which this delicate observation requires were little more than guessed at. The difference of geographical position of the two stations (12′ in lat. 38′ in long.) was allowed for by observations made with a different apparatus,—the effect of temperature was deduced from indirect experiments, far from presenting a mutual agreement; and the whole difference of level (4500 French feet) offered a very small basis for so general a conclusion. But, besides this, there is an oversight in M. KUPFFER's deductions (first pointed out to me by Professor NECKER of Geneva), which tends yet farther to diminish the probability of his conclusions. The estimate of the effect of geographical position on the magnetic intensity, M. KUPFFER conceives to be such, that the variation for 12′ in lat. (diminishing from the lower to the upper station) would exactly counterbalance the variation due to 38′ in E.

* See the details of the Observations in the Annales de Chimie. An. xiii. (1805), Tom. lii. p. 75.

† Tom. i. p. 1. ‡ Ibid. p. 10.

§ The observations were not made at the *summit* of Mont Elbrouz, as stated in the Annuaire du Bureau des Longitudes, 1836, p. 288, but near the foot of it, and the difference of height of the stations was less than 5000 English feet. The stations were "Pont de Malka," and "Hauteur de Kharbis."

long. (also diminishing from the lower to the upper station); the one increasing the duration of an oscillation as much as the other diminished it. Now it appears from his own statement on the preceding page (Memoir, p. 87), as well as from the known direction of the isodynamic lines, that these variations conspire with one another, so as to render the anomaly attributed to height *greater* than before. The upper station is S. W. of the lower, the direction of the isodynamic lines is from N. W. to S. E., consequently the variation of position is such as would diminish the time of vibration of the needle, whilst in effect it was found to be increased. From M. KUPFFER's data, I find that the time of one vibration of his great needle by GAMBEY ($24^s.05$ nearly) would be *diminished* about $0^s.104$ for the change of latitude and longitude, whilst it was observed to be *increased* by $0^s.063$. The anomaly, then, instead of being $0^s.063$, as M. KUPFFER states it (and which he attributes to the effect of 4500 French feet of elevation), would be $0^s.167$, or nearly three times as great. M. KUPFFER's *law* of an increase of .000583 of the whole time of vibration, for a rise of 1000 French feet, will therefore, when corrected, amount to .00155, and the factor for the diminution of intensity to twice as much, or .0031, which is just *ten times* as great as my observations indicate, and is so considerable, that, were the conclusion just, it could not fail to be detected by the most ordinary instruments at the most ordinary elevations.

41. But if the anomaly be admitted to exist in M. KUPFFER's observations, whence does it arise? I have no difficulty in answering the question. I shall not dwell upon the incomplete data from which the corrections due to temperature, latitude, &c. are derived; nor upon the entire incompetency of a single observation which unknown causes (for instance, an iron mine, or the occurrence of an aurora borealis) may affect. I take M. KUPFFER's own statement in the geological section of his work, which pronounces the whole country surrounding Mont Elbrouz to afford one continued evidence of ancient volcanic eruption,* to abound in hot ferruginous springs,† to be so intersected by Trachytes,‡ Lavas,§ and Diorites,|| that there is distinct evidence of this tract being nothing else than a crater of elevation, raised by the upheaving force of the trachyte of Mont Elbrouz itself,¶ which he states to be undistinguishable from the rock of Pichincha, the great South American volcano.** Any one who has the slightest acquaintance with the connection between magnetism and volcanic rocks will be at no loss to explain anomalies even greater than those which M. KUPFFER has observed.

42. It was from a persuasion of the entire inconclusiveness of M. KUPFFER's results, as well as of all preceding ones, that I undertook the experiments already detailed, in the hope of compensating for the imperfections of the apparatus by the number and extent of the experiments. I own that until I came to calculate

* Voyage, p. 39. † P. 39, p. 44, p. 55. ‡ P. 44, p. 61, p. 65.
§ P. 60, p. 66. || P. 63. ¶ P. 65. ** P. 35.

the results by the method of least squares, I had little confidence in having obtained any positive result. A careful examination of the station marked on the map, will shew that they were almost invariably chosen so that an elevated station lay between two others at a lower lead, by which the effect of change in latitude and longitude might be eliminated. When we criticise these groups of three series, we find for the most part an agreement greater than I had myself anticipated that the instrument could insure; yet the combination of all with two independent needles, and likewise in two series in different countries and different years, unite in giving a negative coefficient to the height, which I believe to be true and not accidental, though it could not safely be inferred from one or two insulated observations. I should be disposed to deduce its probable value thus, taking the circumstances of the observations into consideration:—

	Weight.	Coefficient of Varied Intensity for 100 feet of Elevation.
Alps, Needle No. I, .	4	.000033
Alps, Flat Needle, .	2	.000027
Pyrénées, both, . .	1	.000053
	Probable mean,	.000034

Hence to produce a variation of .001, an elevation of 3000 feet is necessary. At the height to which GAY LUSSAC ascended, the change of intensity would be nearly .008 of the whole; but the variation in the time of an oscillation would be only half as much.

43. The smallness of the variation fully explains the difficulty of ascertaining its existence from a very limited number of observations. It is hoped that, notwithstanding the imperfection of the instruments, the extent of the induction will entitle the result to some confidence. By adding together the elevations of the distinct stations contained in Table VII, it will be found that the aggregate of the heights to which I have ascended amounts to above 160,000 feet, or *more than thirty vertical miles.*

§ 5. *On the Magnetic Dip.*

44. Although the horizontal magnetic force be only a sort of mathematical abstraction, and bears no direct relation to the earth's action until the effect of dip is considered, we do not therefore think it improper to be made a separate subject of inquiry. From the projected lines of equal horizontal intensity and of equal dip, the lines of equal total intensity are deducible. The two elements may therefore be made the subject of distinct inquiry; and though these elements are probably in a condition of continual change, yet, considering the present errors of observation, any moderate lapse of time between the formation of these curves will not be productive of serious anomalies. By deducing the total intensity

curves from the two partial sets of curves, we also increase the probability of accuracy, since intensity is likely to be so much oftener observed than dip; that the lines of equal horizontal intensity will be better determined than if those points alone were used where the dip was also observed; and thus the whole acquires additional consistency.

45. The general relations of dip and horizontal intensity have been pointed out in the excellent charts of HANSTEEN. Though it is very probable that mountain chains may cause inflections in the general course of the curves, and local attractions produce occasional anomalies, yet the general variation of one or other quantity is always graduated; and though an insulated observation may be spoiled by an abrupt change in either element, the conclusion from a series of experiments cannot be so affected.

46. I state this chiefly to meet two objections to conclusions from experiments of the kind I have detailed, which have at different times been urged. The first is, that the influence of height (for example) upon the horizontal intensity may not be due to a change in the total intensity, but only of the dip. To this we would reply, that no reason can be assigned why the dip should more naturally vary than the intensity; and that it is contrary to all probability that the variation in the latter should be wholly due to the indirect influence of the former. We admit that the change may be due to *both* causes conjointly; but farther, if we adopt HUMBOLDT's estimate (I quote from a reference which I have been unable to verify), which assigns a *diminution* of 2′.5 of dip for 1000 feet of extent, we should have an apparent *increase* of horizontal intensity, if the total intensity remained constant. The *second* objection to which I alluded, I believe no one accustomed to treat such problems will apply to my observations after due examination, namely, that though three stations be in one straight line and equidistant, the elevated station being in the centre, we can draw no conclusion as to the variation of the intensity by comparing the extreme observations with the middle one, because the dip may have altered in the interval. We may indeed have, by a strange accident, a solution of continuity which might produce this effect in a single instance, but its capability of affecting a whole series of observations cannot for a moment be sustained.

47. Observations of dip I have not, however, neglected. My instrument was a very small one (three inches diameter), constructed by Mr ROBINSON for the late Captain KATER, and incapable of indicating such small variations as are required to fix with great accuracy the lines of equal dip. Nor can I hope that the small number of observations which I have accumulated can throw any light upon the influence of height on the dip. Still these observations may fix the dip at several stations with considerable accuracy, and the collation of them shew that tolerable precision may be attained even with an instrument of very small dimensions. Had the observations been as much multiplied as those of intensity,

the isoclinal lines would, even by this instrument, have been determined with very considerable exactness. The following Table contains observations made with only one needle (marked No. II.), the other (from having too thick an axis) having been found to give much more anomalous results.

TABLE X.

Observations of Magnetic Dip with a Three Inch Circle.

Date.	Station.	Marked End N. Pole.	Marked End S. Pole.	Marked End Mean.	Remarks.
1832. April 23.	Edinburgh. Greenhill House, .	71°.0′	71°.46′	71°.23′	Mean 71° 33′.2. But the two last observations are the best. These observations are the only ones made in a house.
April 23.	Edinburgh, ...	71.26	71.42	71.34	
May 23.	Edinburgh, ...	71.31	71.46	71.38	
May 28.	Edinburgh	71.29	71.45	71.37	
July 11.	Brussels Observatory, (same place as intensity)	68.55	69.1	68.58	68° 49′ according to M. Quetelet, May 1832.
July 17.	Spa (same place as intensity, ...	68.20	68.24	68.22	
	Heidelberg,				Mean 66° 37′.1. I have changed the leadings at the first station from 67° to 66°, considering the former as an undoubted error. This is confirmed by observations made at the same three stations by Needle No. I., which gave 66° 48′, 66° 18′, 66° 57′—Mean 66° 41′.
July 27.	*a.* On the bank of the Neckar, two miles above the town, .	66.23	66.45	66.34	
Aug. 7.	*b.* Prof. Leonhard's garden, . . .	66.20	66.44	66.32	
Aug. 7.	*c.* Terrace of the Castle,	66.33	66.59	66.46	
Aug. 4.	Laach; S.E. side of Lake,	67.58	68.22	68.10	Good observation.
Aug. 5.	Andernach; foot of the hill west of the town,	67.33	67.45	67.39	Good. The difference between this and the last probably due to volcanic influence.
Aug. 18.	Cologny, near Geneva, Prof. Necker's garden,	64.45	65.10	64.57	
Aug. 20.	Geneva Observatory, .	64.56	65.14	65.5	65° 48.′5 in 1825 (Arago). The dip diminished at Paris about 27′ between 1825 and 1832.
Aug. 22.	Mount Breven, summit,	64.38	65.11	64.54	
Aug. 23.	Chamouni (same place as intensity), . .	64.45	65.16	65.0	
Aug. 25.	Jardin	64.45	65.7	64.58	
Aug. 29.	Aoste (same place as intensity, ...	64.36	64.58	64.47	
Aug. 31.	St Bernard, near the Lake,	64.43	65.7	64.55	
Sept. 3.	Martigny, ...	64.35	64.43	64.39	Difficult observation. Local influence being suspected, the operation was repeated at Bex.
Sept. 3.	Bex; in an orchard S.W. of the town, . . .	64.47	65.14	65.0	Very good observation.
Sept. 10.	Interlaken; side of the Aar,	65.13	65.30	65.22	
Sept. 22.	... .	65.10	65.39	65.25	
Sept. 29.	Hospital; St. Gothard near the old castle,	65.22	65.31	65.26	Indifferent observation.
Sept. 30.	St Gothard Hospice .	64.50	65.30	65.10	
Oct. 2.	Locarno (Lago Magiore) below the convent,	64.48	65.13	65.0	
Oct. 11.	Pfeffers; near the Bath-house, ...	64.55	65.23	65.9	
1835. June 13.	Paris Observatory; M. Arago's cabinet, .	67.8	67.26	67.17	67° 24′ by M. Arago. July 1835.

Remark.—The dip at Edinburgh is undoubtedly affected by being made within a house. Some observations roughly made at the time in the open air confirm this; and more recently, I have found the dip by the same instrument to be 71° 44′.5 (2d February 1837) and 71° 50′.5 (best, 4th February)—although the dip has been diminishing every year.

48. A careful review of these Observations, compared to those of the usual dipping needles, gives, I think, a favourable impression of the powers of a small instrument. The observations were put in the form of equations of condition for the alpine series, exactly as in the case of intensity; x representing the variation of dip in minutes for 1′ of latitude N. increasing; y the variation for 1′ of longitude E. increasing; z the variation for 100 feet of height. Geneva is taken for the standard of comparison as before; $\delta\Delta'$ representing the correction of the dip at that place.

Table XI.

Equations of Condition for Dip.

Geneva,	$0'x$	$+\ 0'y$	$+\ 0\,z$	$+\ \delta\Delta' =$	$0'$
Cologny,	$0\,x$	$+\ 2\,y$	$+\ 3\,z$	$+\ \delta\Delta' =$	$-\ 8$
Breven,	$-16\,x$	$+\ 41\,y$	$+\ 71\,z$	$+\ \delta\Delta' =$	$-\ 11$
Chamouni,	$-17\,x$	$+\ 43\,y$	$+\ 21\,z$	$+\ \delta\Delta' =$	$-\ 5$
Jardin,	$-17\,x$	$+\ 50\,y$	$+\ 77\,z$	$+\ \delta\Delta' =$	$-\ 7$
Aoste,	$-26\,x$*	$+\ 71\,y$	$+\ 6\,z$	$+\ \delta\Delta' =$	$-\ 18$
St Bernard,	$-20\,x$	$+\ 61\,y$	$+\ 68\,z$	$+\ \delta\Delta' =$	$-\ 10$
Martigny,	$-\ 6\,x$	$+\ 56\,y$	$+\ 3\,z$	$+\ \delta\Delta' =$	$-\ 26$†
Bex,	$3\,x$	$+\ 52\,y$	$+\ z$	$+\ \delta\Delta' =$	$-\ 5$
Interlaken,	$30\,x$	$+\ 103\,y$	$+\ 6\,z$	$+\ \delta\Delta' =$	17
Interlaken,	$30\,x$	$+\ 103\,y$	$+\ 6\,z$	$+\ \delta\Delta' =$	20
Hospital,	$24\,x$	$+\ 135\,y$	$+\ 36\,z$	$+\ \delta\Delta' =$	21
St Gothard,	$22\,x$	$+\ 145\,y$	$+\ 58\,z$	$+\ \delta\Delta' =$	5
Locarno,	$-\ 2\,x$	$+\ 159\,y$	$-\ 6\,z$	$+\ \delta\Delta' =$	$-\ 5$
Pfeffers,	$47\,x$	$+\ 200\,y$	$+\ 17\,z$	$+\ \delta\Delta' =$	4

49. The method of least squares gives us from these equations the following values of the unknown quantities:—

$$x = 0'.543 \quad y = -\ 0'.028 \quad z = 0'.080\delta \quad \Delta' = -\ 3.'4.$$

As already stated, I consider these numbers (particularly z, which gives an *increase* of dip of 1′ for 1250 feet of ascent) as considerably uncertain.

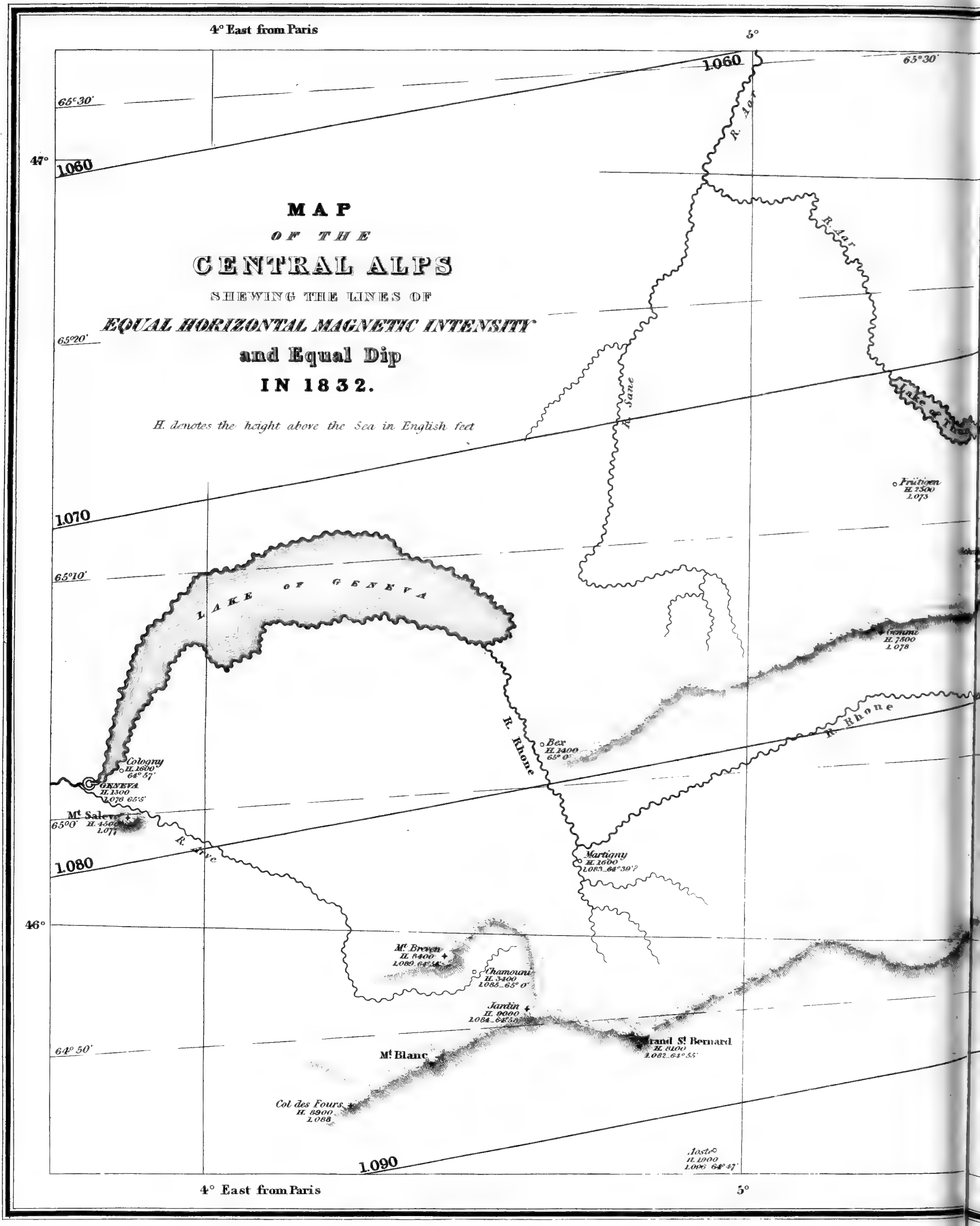
48. If the variation of y for 1′ of longitude, be increased in the ratio of the length of 1′ of latitude to 1′ of longitude (as in Art. 32), it will become $= -\ 0'.039$, and the direction of the isoclinal line to the East of North will be

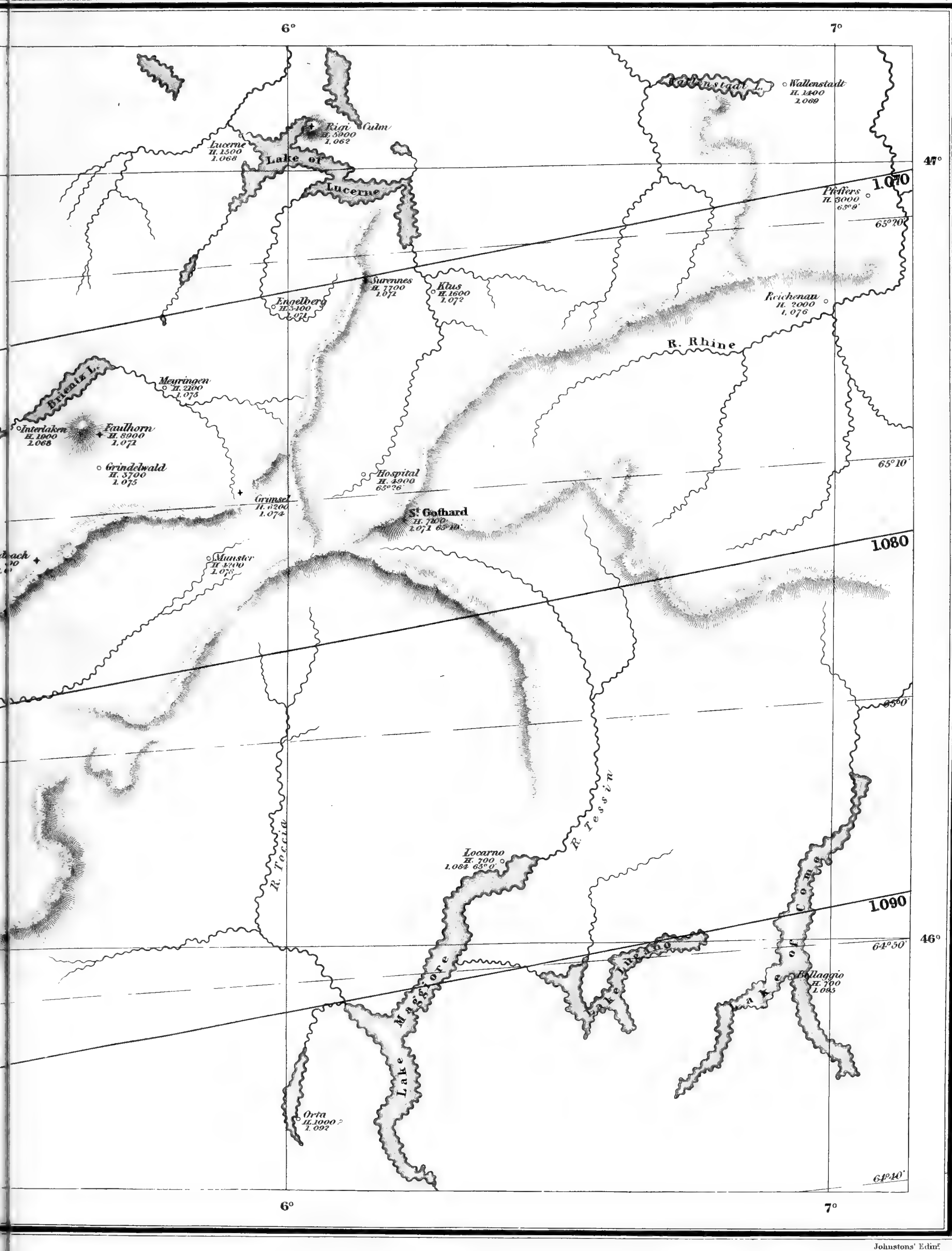
$$\text{Arc whose tang.} = \frac{543}{39} = 85^\circ\ 53'$$

Hence the lines of equal dip would appear to approach nearer to the parallels of latitude than the lines of equal horizontal intensity (Art. 32). The corrected dip at Geneva would be 65° 1′.6, and the dip would increase 10′ for an increase of 18′.4 of latitude.—See Plate I.

* The coefficient ought to have been 28.

† This observation is certainly erroneous, and should have been discarded.


4° East from Paris
5°
1.060
65°30'
47°
MAP
OF THE
CENTRAL ALPS
SHEWING THE LINES OF
EQUAL HORIZONTAL MAGNETIC INTENSITY
and Equal Dip
IN 1832.
H. denotes the height above the Sea in English feet
65°20'
R. Aar
R. Sane
Lake of Thun
Frütigen
H. 2500
1.073
1.070
65°10'
LAKE OF GENEVA
Gemmi
H. 7500
1.078
R. Rhone
Bex
H. 1400
65° 0'
Cologny
H. 1600
64° 57'
GENEVA
H. 1300
1.076 65°5'
Mt Saleve
H. 4500
1.077
65°0'
R. Arve
1.080
Martigny
H. 1600
1.083 64°39'?
46°
Mt Breven
H. 8400
1.089 64°34'
Chamouni
H. 3400
1.085 65° 0'
Jardin
H. 9000
1.084 64°53'
Grand St Bernard
H. 8100
1.082 64°55'
64° 50'
Mt Blanc
Col des Fours
H. 8900
1.088
1.090
Aosta
H. 1900
1.096 64°47'
4° East from Paris
5°



Johnstons' Edinr

49. The lines of equal dip and equal horizontal intensity being known, the direction of the lines of equal total intensity may be deduced geometrically. I am, however, too well aware of the great uncertainty which a small error in the elements produces, to attempt to assign a result which might prove very erroneous indeed.

Postscript.

Since this paper was written, and the results made public, a suggestion has been made in a quarter entitled to attention, as to how far the apparent diminution with height may be due to the hour of the day at which observations at great heights have usually been made. I have already stated, that I have attempted no correction for the hour of the day, owing to the want of accurate data, but I thought it worth while to inquire how far there was any general ground for such an explanation of the observed difference. I accordingly divided my observations into 18 series above 4000 feet, and 22 below that height. I found that the mean hour at which the former were made was $11^h\ 12^m$, the latter at $12^h\ 42^m$. According to the best observations, the intensity would be somewhat less at the former period than the latter, and would so far give a false indication of diminished intensity with height. But the variation for $1^h\ 40^m$ would undoubtedly be trifling, compared even with the small variation which the preceding paper assigns for 5110 feet, which corresponds to the mean difference of height for the two series, the mean height for the first being 7160 feet, and for the second 2050.

II.—*On Paracyanogen and the Paracyanic Acid.* By James F. W. Johnston, A.M., F. R. S. E., Professor of Chemistry and Mineralogy in the University of Durham.

Read 4th April 1836.

INTRODUCTION.

The history of the newer sciences presents many instructive examples of the progress of the human mind in developing the germs of natural knowledge, and building on a single observation entire departments of science. Few pursuits indeed are more interesting, even to the student of immaterial nature, than in the perusal of such a history to trace the footsteps of the several investigators, and to mark how far, and by what means,—whether by new methods or by greater patience of research,—each successive observer advanced the inquiry. We see the human mind, as it were, set free from the trammels of time, and developing its resources on a large and continuous scale, not limited by the powers of one intellect, the length of one life, or the means of one station. We see, at the same time, what varied gifts and opportunities are necessary for the elucidation of a single subject; how these gifts, though not all imparted to one man or to one generation, are yet wisely and beneficently bestowed on the entire species; and how all are thus enabled to co-operate, either in unfolding abstract truth, or in drawing forth those practical results which directly conduce to the amelioration and comfort of all.

These reflections are particularly suggested by the history of that branch of chemical science, to which the facts contained in the following paper are intended to form a small addition.

Early in the last century, about 1710, a solution of potash which had been employed by Dippel in the purification of his animal oil, and afterwards calcined, was accidentally mixed with a solution of sulphate of iron. A beautiful blue precipitate was the result, and by the repetition of the experiment a pigment was obtained, since known in commerce by the name of the Prussian blue. This single observation gave rise to a multitude of researches. Woodward, Macquer, Morveau, Lavoisier, and Bergman, successively experimented on the blue substance with little success. Upwards of seventy years elapsed before any light was thrown upon its true nature;—when in 1782 Scheele obtained from it the hydrocyanic or prussic acid. Five years later, his results were verified and extended by BERTHOLLET. In 1806, the characters and compounds of the new acid were more fully detailed by PROUST; and in 1815, its composition rigorously determined by GAY LUSSAC, in his admirable researches into the properties of cyanogen. Still it was not till 1819 that the exact constitution of the original pigment, Prussian blue, was established by BERZELIUS with any degree of certainty; and not till after the discovery of the red prussiates or ferro-cyanides by LEOPOLD GMELIN in 1822 that the last doubts were removed. Thus, from the time of its accidental discovery, a period of 112 years intervened, before our knowledge became so far extended that we could give a satisfactory reason for the various steps necessary to its production.

Yet the many more or less unsuccessful labours which this long period saw were not spent in vain. Almost every experimenter has recorded observations fitted to be the germ of new researches, and so many branches have already shot forth from the main trunk of investigation, as to render the department which includes cyanogen and its kindred compounds, by far the most complicated and difficult of the chemistry of the present day.

Among the more important memoirs to which we are indebted for the recent development of this branch of the science, may be enumerated those of PORRETT on the ferro and sulpho cyanic acids; of LEOPOLD GMELIN on the suites of compounds formed on replacing the iron of the ferro-cyanides by other metals; of WÖHLER on the cyanic acid; of HOWARD, LIEBIG, and EDMUND DAVY on the fulminic acid; of SERULLAS on the cyanic acid, and many other interesting compounds of cyanogen; and of MOSANDER and others on the complicated combinations which the double cyanides are capable of forming with each other. To these must be added also the beautiful memoirs of LIEBIG and WÖHLER on the cyanic and cyanuric acids; of LIEBIG alone on mellon, and the curious compounds to which, by the action of acids and alkalies, it gives rise; and, most recently, of LEOPOLD GMELIN on the mellonic and hydro-mellonic acids, the investigation of which still remains to a considerable degree imperfect.

While, therefore, the history of this one department shews what lengthened and laborious research the illustration of some branches of nature requires, it shews, at the same time, how little the failures of even a succession of experimenters affects the ultimate advancement of knowledge,—how men may grope on in darkness year after year, perhaps age after age, despairing of success,—and yet may be all the while unconsciously laying the foundations and storing up the materials of future buildings, till at length accident or genius guides some philosopher into a new path, or puts into his hand a new instrument before which all obstacles give way.

I.—*Preparation of Paracyanogen.*

1. When pure dry bicyanide of mercury is heated in close vessels, it gives off metallic mercury and a gas, which is wholly absorbed by solution of caustic potash. This gas is pure cyanogen. The pure dry salt gives off along with it no appreciable quantity of any other gas.

2. In all cases, however, when the pure bicyanide is wholly decomposed, there remains in the retort a greater or less quantity of a black matter resembling charcoal, sometimes in the form of powder, light, porous, and void of lustre: at others more dense and coherent, and when it has been in contact with the sides of the retort exhibiting a shining metallic lustre.

3. Since pure bicyanide of mercury consists wholly of cyanogen and mercury, and since, during the decomposition by heat, pure cyanogen and pure mercury are alone given off, the black residue, when freed from metallic mercury, can contain only carbon and nitrogen, in the same proportion in which they enter into the constitution of cyanogen. It must either be a new body having the same elementary constitution as cyanogen, or it must be a mixture of two or more substances which taken together have such a constitution.

4. This black substance did not escape the notice of Gay Lussac in his researches upon cyanogen. He recognised in it the presence of nitrogen, as many succeeding chemists have done, but he supposed the greater part to be carbon derived from the decomposition of a portion of the cyanogen. This opinion is still generally entertained.

5. Seven or eight years have now elapsed since, in a paper which I had the honour of reading before this Society, and which was afterwards published in Brewster's Edinburgh Journal of Science for 1829, p. 75, I endeavoured to shew that this black matter was not a mere mixture of two or more substances, but was in reality a new body, which, though differing so remarkably in physical and chemical properties, yet contained the same elements as cyanogen, and combined together in precisely the same proportion. Chemists, however, were not prepared at that time for the reception of so extraordinary an opinion. The only case of isomerism then clearly made out, was that of the cyanic and fulminic acids analyzed by Liebig, and even over that case the researches of Edmund Davy still threw some doubt. It was not surprising, therefore, that the result at which I had arrived should be regarded with a suspicion, which the many striking examples of isomerism since discovered has not yet wholly removed.

6. Dr Thomson, in his System of Chemistry, has objected that I have not shewn the absence of hydrogen in the black matter; but, as the dry bicyanide contains no hydrogen, it is obvious that none can be present in any residue it may leave. Liebig having prepared and analyzed a portion of this substance, concluded that

it was not a definite compound, and that the quantity of carbon was variable. The presence of some impurity had probably interfered with the accuracy of his results.

7. While on a visit to that eminent chemist at Giessen in the month of October 1835, he did me the favour to repeat his analysis in presence of Professor Poggendorf and Dr Gregory, and with a result on this occasion perfectly according with my own. Thus, burned with oxide of copper, and the gases separated by caustic potash—

99 vols. left	32.6 vols. of nitrogen.		
92.9 —	— 30.1	—	—
293 —	— 98	—	—

Another portion burned with bichromate of potash, a method lately suggested by Liebig, gave the gases in a like proportion. Thus—

94.5 vols. left	33.5 nitrogen.	
129 —	— 43	—
108 —	— 36	—
175 —	— 58.5	—

8. My attention being thus recalled to the subject, I have since my return prepared this substance by a variety of processes; and have found, that not only has it, when pure, the same composition as cyanogen, but that, like it, it also exhibits with other substances many interesting reactions, and is capable of combining with oxygen to form a new cyanic acid.

9. Pure bicyanide, prepared by saturating prussic acid with peroxide of mercury, reduced to powder, carefully dried and decomposed in a retort gradually heated to redness, left a light bulky powder, which, when burned with oxide of copper, gave a gas, of which, treated with caustic potash,

102 vols. left	35 of nitrogen	
86 —	— 29	—
87.5 —	— 29.5	—

This substance is remarkably difficult to burn. The quantity of nitrogen present is so great that I have only once or twice succeeded in burning it without the formation of a sensible quantity of nitric oxide.

10. When strong prussic acid is set aside, it speedily decomposes, and deposits a black powder in considerable quantity. Dried in vacuo over sulphuric acid or at 212° F., this substance still gives off, when heated in close vessels over a lamp, water, carbonic, and hydrocyanic acids, and ammonia. The black matter that remains, burned with bichromate of potash in large excess, gave a mixture of carbonic acid and nitrogen in the proportion of 2 to 1. Thus,

97.5 vols. left 32 of nitrogen.		92.5 left 30	
95 —	— 31 —	90.5 —	29.5

11. When a solution of cyanogen gas in water is allowed to stand for a considerable time, a similar black matter is deposited in smaller quantity. A portion of this substance, for which I was indebted to the kindness of Professor Wöhler, after heating to redness, to free it from water, ammonia, &c., burned with oxide of copper, and the gases made to pass over a large surface of red hot metallic copper, gave still distinct traces of nitric oxide, and

263	vols. of the gas	left	89	of nitrogen.
110	—	—	36.5	—
96	—	—	32	—
69.5	—	—	22.5	—

12. A strong solution of caustic potash absorbs cyanogen gas very rapidly, and when excess of the gas is present, speedily becomes coloured, forming a dark reddish-brown solution, from which a brown powder is precipitated on neutralizing with an acid. Solution of caustic ammonia has a similar action, but I have not yet prepared the matter in sufficient quantity by these processes to enable me to analyze it.

13. Ether absorbs cyanogen slowly, but in considerable quantity. The solution in close vessels remains colourless. Left in an atmosphere of the gas for thirty-six hours, it had absorbed sixteen times its volume, and began to be slightly coloured. After several days the absorption appeared to cease at twenty-eight volumes, it was of a brown colour, and had deposited a portion of a brown sediment. If water be added to the colourless solution, a black film gradually forms at the common surface of the two fluids.—Liquid ammonia and caustic potash cause a speedy deposition of the black matter.

14. Alcohol absorbs cyanogen rapidly, and in large quantity. When perfectly saturated and set aside, the smell of cyanogen disappears, a sweet penetrating ethereal odour takes its place, and the solution becomes coloured. It is now capable of taking up a second and larger doze of the gas, the smell of which again disappears, on standing for twenty-four hours—the ethereal odour, meanwhile, becoming stronger and the colour darker. The addition of gas may now be repeated, and by alternate changing and setting aside, an English pint of common alcohol may, in the course of a week or ten days, be made to absorb the whole of the gas given off by four or five pounds of bicyanide of mercury. By this time, also, a large deposit will be formed at the bottom of the bottle, and the entire alcohol will have become dark and thick like treacle.

15. If this thick fluid be distilled, a colourless product passes over, having an ethereal odour, but from which water separates no ether, and which, on standing, gradually becomes dark coloured, and gives a further portion of the dark brown or black sediment. After the lapse of some weeks or months, according to circumstances, no farther evidence of decomposition shews itself, the deposi-

tion of black matter ceases, and the thick treacly fluid separates into a black powder, and a transparent and nearly colourless supernatant liquid.

16. It is unnecessary at present to inquire minutely into the nature of the complicated changes which take place during the mutual reaction of cyanogen and alcohol on each other. To this subject I shall have occasion to revert, in a future memoir, when describing certain new substances produced, partially at least, by this reaction, but of which the investigation is still incomplete. Ammonia and hydrocyanic acid are formed in considerable quantity, and probably more than one ethereal substance. The greater number of these appear to be decomposed in their turn, and the black powder is the ultimate and chief result.

17. When the thick alcoholic solution is thrown on the filter, and the substance dried without washing, it gives a mass, having a deep black colour and shining fracture; washed with hot water, it is of a brown, more or less dark, and if boiled in repeated portions of water, as long as any thing is dissolved and dried at 212° F. it is of a dark olive-brown colour. Heated over a lamp in a close tube, it gives off, like the substance deposited in prussic acid, water, carbonic and hydrocyanic acids and ammonia, leaving a black residue, which, at a dull red heat, gives off cyanogen, and slowly disappears.

The black residue burned with oxide of copper, gave a mixture of carbonic and nitrogen gases, of which

88.5 vols.	left	30	of nitrogen.
85.5	—	29	—

This is as near the ratio of $N : C_2$ as can generally be expected.

18. When dry cyanate of silver is heated in a close tube, it is decomposed, giving off nitrogen and carbonic acid in the proportion of 1 : 2 by volume, and there remains behind a black matter mixed with metallic silver. This black substance has the same composition as that left by bicyanide of mercury, and its formation is accounted for by the following formula—

$$2\,(N\,C_2\,O + A\,g\,O) = NC_2 + 2\,A\,g + N + 2\,C\,O_2$$

19. Rectified wood-spirit (bihydrate of methylene) also absorbs cyanogen rapidly and in large quantity. When it has absorbed upwards of thirty times its volume of the gas, it begins to be coloured, and the absorption goes on till it becomes dark brown and opaque, and begins to deposit a dark brown powder. In the course of an hour the quantity of gas taken up amounts to thirty or forty volumes, after the lapse of two days it has increased to between fifty and sixty volumes, and a portion of the deposit was observed. If the wood-spirit be not saturated with the gas, it exhibits no change of colour for some days, but ultimately assumes a reddish-brown colour, more or less dark, and gives a brown deposit.

20. When the ferro-cyanides, Prussian blue and prussiate of potash, are de-

composed by heat in close vessels, a black residue remains, which is generally supposed to be a carburet of iron. This, however, is not necessarily the case.

A portion of the best Prussian blue of the shops was gradually heated to redness in a retort for half an hour, water, hydro-cyanate and carbonate of ammonia, carbonic oxide and nitrogen, were given off. The black matter dissolved in sulphuric acid, with evolution of hydrogen gas, and the solution diluted with water, gave a precipitate of Prussian blue. Burned with bichromate of potash, it gave the carbon and nitrogen in the proportion of 2 to 1. Heated again to redness for half an hour, the carbon was found to be to the nitrogen as 4 to 1, and this proportion was not changed by a third heating for the same length of time.

It appears, therefore, that while water is present, the decomposition goes on in such a way as to remove *all the elements* of each atom of cyanogen as it is decomposed, without affecting the composition of what remains; and that, when all the water is drawn off, the affinity of the iron for carbon comes into play, and nitrogen is given off while a carburet of iron is formed.

Dry prussiate of potash, heated to redness in close vessels, as in the preparation of cyanide of potassium, undergoes a similar decomposition. A portion of the black matter remaining after the separation of the cyanide in this process, gave me the carbon to the nitrogen as 8 to 1. I have not satisfied myself, however, that the black matter thus formed contains paracyanogen at any stage of the process.

II.—*Properties of Paracyanogen.*

1. It is of a black or dark brown colour, and occasionally has considerable lustre. It has no taste or smell, is insoluble in water and alcohol, has a density of about 2, and is a non-conductor of electricity.

2. Heated to redness in close vessels, it is slowly resolved into cyanogen. In the air or in oxygen, at a red heat, it burns away without residue, forming carbonic acid, and, according to the hygrometric state of the atmosphere, more or less ammonia and hydrocyanic acid. Heated to redness in a close vessel, and suddenly exposed to the air, it burns like tinder. Heated in a moist state in close vessels, ammonia, and carbonic and hydrocyanic acids, are first given off, and afterwards cyanogen.

3. Fused with sulphur or phosphorus, or heated to redness in the vapours of these substances, no apparent change takes place. With alkaline sulphurets containing an excess of sulphur, an action takes place at a red heat, but without the formation of any sulpho-cyanide. Heated to redness in an atmosphere of dry chlorine or iodine vapour, a very slight action takes place. Chlorine, if moist, disengages vapours which affect the eyes (chloride of cyanogen?), and among other products muriate of ammonia.

4. Heated with potassium in close vessels, a violent combustion takes place, and a compound is formed, which, when dissolved in water, gives with nitrate of silver a white precipitate. Dried and heated to redness, this precipitate gives off cyanogen and leaves metallic silver. The compound formed therefore is cyanide of potassium.

5. Prepared by heating bicyanide of mercury, it is exceedingly little soluble in a solution of caustic alkali, and is very slowly decomposed when boiled with it. During the boiling ammonia is evolved, and oxalic acid should remain in the solution for $NC_2 + 3HO = NH_3 + C_2O_3$. Prepared by heating to redness the deposit from cyanogen in alcohol, it is slightly soluble in a boiling solution of caustic potash, giving it a dark brown colour. Acids precipitate it apparently unchanged.

6. Fused with dry hydrate of potash, ammonia is evolved. Intimately mixed with caustic or carbonate of potash, and heated to incipient redness in the open air for some time, a salt is formed, possessing the properties of *cyanate of potash.* With nitrate of silver it gives a white flocky precipitate. With nitric acid, violent effervescence from the evolution of carbonic acid, while the filtered solution, evaporated to dryness, gives a mixture of nitrates of potash and ammonia.

7. Concentrated muriatic acid acts upon it very slightly, becoming pale yellow.

III.—*Action of Sulphuric Acid on Paracyanogen.*

Concentrated sulphuric acid ($SO_3 + HO$), dropped upon paracyanogen in the state of fine powder and gently heated, is absorbed by it, forming a dry mass. A larger quantity of acid, aided by heat, dissolves the paracyanogen without any sensible evolution of gas. The solution is of a deep brownish red colour, more or less thick. Water decomposes it, throwing down the paracyanogen *apparently* unchanged. It is difficult, by washing, to free the precipitate entirely from sulphuric acid. A portion of it boiled in water, dried and burned with oxide of copper, gave a gas of which 61.6 vols. left 20, and 82.5 left 26.5, so that the ratio of the carbon to the nitrogen is not sensibly altered.

If the temperature be raised too high during solution, or if the solution be heated to 400° or 500° F., the sulphuric acid and paracyanogen are mutually decomposed, sulphurous and carbonic acids are given off, and sulphate of ammonia is sublimed. The gases are evolved in the proportion *nearly* of one volume sulphurous to two of carbonic acid. This agrees with the formula,

$$NC_2 + 2SO + 3HO = \overline{NH_3 + SO_3} + 2CO_2 + SO_2.$$

During this decomposition, a portion of the paracyanogen becomes oxidised, paracyanic acid being formed, hence the volume of sulphurous acid obtained is generally in slight excess. That paracyanic acid is formed is proved by precipitating the brown solution with water, filtering the acid liquid, and adding a solution of nitrate of mercury, when a yellow precipitate falls more or less copiously of paracyanate of mercury.

If the solution in sulphuric acid be boiled for a sufficient length of time, it becomes colourless, and water gives no longer any precipitate; the paracyanogen, therefore, is wholly decomposed.

The black residue obtained by decomposing the bicyanide can rarely be freed entirely from the minute globules of mercury which adhere to it. If mercury be present in sufficient quantity when the paracyanogen is treated with sulphuric acid, the entire solution on cooling congeals into a mass of minute transparent crystalline needles of a brown colour, which are *possibly* a compound of the sulphates of oxide of mercury and of paracyanogen. They are decomposed by water, and have hitherto baffled my attempts to obtain them free from the great excess of concentrated sulphuric acid.

The action of this acid on paracyanogen seems to indicate, that the black substance is possessed of basic properties, and that it may be in some measure analogous to the carbo-hydrogens.

IV.—*Action of Nitric Acid on Paracyanogen—Paracyanic Acid.*

1. Paracyanogen, after heating to redness, as it is obtained from bicyanide of mercury, is very slightly acted upon by nitric acid. Boiled in this acid, a small portion is dissolved, and if the boiling be long continued, the insoluble matter gradually assumes a lighter colour, and ultimately becomes of an orange yellow. The acid solution, diluted largely with water, becomes turbid, and deposits a yellow precipitate in small quantity.

2. When the black deposit formed in an alcoholic solution of cyanogen, in strong prussic acid, &c. is treated with nitric acid in the cold, it dissolves slowly without any sensible evolution of gas, giving a dark reddish brown solution, from which water again precipitates it apparently unchanged. If, however, heat be applied to the solution, if a current of sulphuretted hydrogen be passed through it, or more rapidly and certainly, if the black matter be gradually added to the hot acid, as each successive portion dissolves, and the dark colour caused by it disappears, copious red fumes are evolved, and a transparent reddish yellow solution is obtained. From this solution water precipitates a bulky bright yellow powder, a further portion falls on saturation with an alkali, and a third, though much smaller quantity, by evaporating to dryness the supernatant yellow solution and washing the residue.

3. If the dark brown solution of paracyanogen in sulphuric acid be heated, and nitric acid gradually added, red fumes are evolved, and a yellow precipitate

gradually falls. If a sufficient quantity of nitric acid have been added to oxidize the whole, or if the solution be added to hot nitric acid slowly, and as the colour disappears, water, as in the former case, throws down a copious yellow precipitate.

4. When a solution of cyanogen in diluted alcohol, after becoming dark coloured, is acted upon by a current of chlorine gas, among other compounds formed is a portion of a yellow substance having the appearance and properties of that obtained by means of nitric acid.

5. The yellow matter obtained by these different processes, well washed and dried at 212° F., is without taste or smell, insoluble in water and alcohol, sparingly soluble in cold, more largely and with slight decomposition in hot nitric and sulphuric acids, and soluble to a certain extent in caustic alkalies.

Heated above 212°, it gradually becomes darker in colour; about 400° F. it begins to blacken and evolve ammonia; over the lamp, in a close tube, it gives off ammonia, hydrocyanic acid, and water, and a mixture of carbonic and nitrogen gases, in the proportion of two vols. of the former to one of the latter. It, therefore, contains water,—either hygrometric or in combination,—and oxygen. A residual black matter remains, which is paracyanogen, and which, by further heating, gives cyanogen gas.

6. Burned with bichromate of potash, it gave a mixture of carbonic acid and nitrogen gases, of which

76.75	left 26.5	of nitrogen,	100.5	left 33.1	
83.5	— 28.4	—	189	— 63	
103	— 34.5				

The ratio of the carbon to the nitrogen is therefore 2 : 1, as in cyanogen, and since it contains oxygen and combines with bases, I have given it the name of the paracyanic acid.

Like paracyanogen, it is exceedingly difficult to burn, either with oxide of copper or with bicyanide of mercury, without the formation of binoxide of nitrogen, but the presence of this gas in the above experiments does not produce any serious error, as the nitrogen in combining with the oxygen undergoes no change of volume, and the caustic potash employed to remove the carbonic acid absorbs very little of the binoxide, until it has been a considerable time in contact with the gaseous mixture.

7. The small quantities I have yet obtained of this acid, have prevented me from submitting it in the uncombined state to a sufficient number of analyses, to establish its composition beyond dispute. 4.22 grs. dried carefully at 212° Fahr., and burned with Bichromate of Potash, gave

Carbonic acid, ...	1.3466 = 31.91	per cent.
Water, ...	1.13 = 26.777	per cent.

As this mode of analysis gives the water with great accuracy, we may deduct 1.13 from 4.22, and we have 3.09 of dry acid, containing 1.3466 of carbon = 43.481 per cent. And as the C : N : : 2 : 1, we have for the constitution of the dry acid,

		Calculation.		Experiment.
C_8 = 611.496	=	43.074	=	43.481
N_4 = 708.144	=	49.882	=	50.035
O_1 = 100.000	=	7.044	=	6.484
Atom. = 1419.640		100		100

The quantity of carbon obtained is a little too high, which may be attributed to the absorption by the caustic potash of a small quantity of binoxide, which was recognised by the smell during the process. This excess of carbon necessarily causes an excess also in the nitrogen calculated from it, and a corresponding deficiency in the per-centage of oxygen.

If the water in the acid be 5 atoms, we have

	Theory.		Experiment.
Carbon, ...	= 30.85	=	31.91
Nitrogen and oxygen,	= 40.78	=	41.303
Water, ...	= 28.37	=	26.777
	100		100

As the acid, however, was dried only at 212° Fahr., I do not place much confidence in the *number of atoms* of water. It should probably retain, when fully dry, at this temperature, only four atoms of water, as the salt of silver appears to do, in which case the formula for the hydrated acid would be $C_8 \, N_4 \, O + 4 \, HO$.

8. Being insoluble in water, or nearly so, this acid does not affect litmus. In hot concentrated solutions of the caustic alkalies, it dissolves with partial decomposition and evolution of ammonia; in weaker solutions it is very sparingly soluble, and very little also is taken up by solution of caustic ammonia. It has a powerful affinity, however, for certain metallic oxides, especially for those of mercury and silver, with both of which it forms compounds very slightly acted upon by water, and with difficulty decomposable even by the mineral acids.

This acid, indeed, is remarkably distinguished from the cyanic (Cy O), and fulminic NC_2O acids by its great stability. It is decomposed very slowly by hot concentrated nitric and sulphuric acids, protracted boiling being necessary to produce complete decomposition. Its affinity for the oxide of mercury is so great, that a weak solution of nitrate of mercury throws down the paracyanate from a solution of the yellow acid in the nitric or sulphuric acid.

V.—*Properties of the Paracyanates.*

1. In powder they are all of a pale yellow or orange yellow colour, the salt of oxide of silver crystallizes in bright red crystals.

2. The salts of the metallic oxides are insoluble or sparingly so in water. They dissolve readily when heated in concentrated acids, and are in great part precipitated without apparent change by cooling, or by copious dilution with water. Acidulated solutions dissolve them more sparingly, and, in all cases, they dissolve more largely in hot than in cold solutions.

3. The alkaline salts are soluble in water, and the acid is precipitated in yellow flocks on the addition of a stronger acid. The metallic salts, if decomposed, are so with difficulty, and only by protracted boiling in solutions of the hydrates and carbonates of the fixed alkalies, or in liquid ammonia. They are partially soluble in such menstrua, giving yellow solutions from which they are in great measure deposited on cooling. By such action the salt of mercury becomes of a dark grey or chocolate colour.

4. Heated in close vessels they give off carbonic acid and nitrogen gases, leaving a black mass, which, at a full red heat, is *generally* resolved more or less completely into cyanogen gas. If the salt is moist, instead of carbonic acid and nitrogen, it gives carbonate of ammonia.

5. Heated in the air they become brown between 300° and 400° Fahr. Above that temperature they blacken and give off white fumes of carbonate of ammonia, which are more or less dense according to the hygrometric state of the atmosphere, At a dull red heat they burn like tinder, leaving only the metal, or if easily oxidizable its oxide.

VI.—*Paracyanate of Mercury.*

1. The paracyanic acid, as already stated, has a strong affinity for the oxides of mercury. A solution of nitrate of protoxide of mercury, though largely diluted, causes an immediate precipitate in solutions of the paracyanic acid, or of any of its soluble salts. It forms, in fact, an excellent test for this acid, where there is reason to suspect its presence.

2. The precipitate is of a pale, rarely of a bright yellow colour, and nearly insoluble in water. When thrown down from solutions of the acid in strong nitric or sulphuric acids, the yellow colour is generally more decided.

3. There appear to be several paracyanates of the oxides of mercury, not to be distinguished by their external characters, yet differing in composition as the basic and neutral salts of these oxides with the nitric acid do. Hence it requires great precautions and the use of carefully prepared nitrates, to insure precipitates of uniform composition.

4. These paracyanates are soluble in hot dilute nitric acid, from which they are again partially separated on cooling, or on the addition of water. By prolonged boiling in the acid, not only the salt, but the paracyanic acid itself, is in a great measure decomposed.

5. Concentrated sulphuric acid dissolves them slowly in the cold, giving a pale yellow solution; when heated, they dissolve more rapidly, in larger quantity, and with evolution of carbonic and sulphurous acids and nitrogen. On cooling, crystals are deposited, which are a mixture of bisulphate and paracyanate. From the liquid portion water separates a brownish precipitate, and the diluted supernatant liquid gives a white with a nitrate of mercury. Sulphuric acid, therefore, partly decomposes the salt, and retains the acid in solution, which it resolves into carbonic acid, nitrogen, and perhaps ammonia, when raised to the boiling temperature.

6. Muriatic acid dissolves them by the aid of heat, but if diluted with water a portion is again thrown down.

7. Solutions of caustic potash or ammonia immediately blacken the protoxide salts, especially if newly precipitated, forming a yellow solution, and leaving protoxide of mercury. Neutralized or rendered acid by nitric acid, the solution again gives a copious precipitate with nitrate of mercury.

8. The newly precipitated salt suspended in water is also rendered dark grey by binoxide of nitrogen and sulphurous acid. The latter forms a yellow solution containing a little mercury, and a considerable quantity of the paracyanic acid, which, when the excess of sulphurous acid is driven off by heat, again falls as a yellow paracyanate on the addition of a salt of protoxide of mercury.

9. Diffused through water, the salts of mercury are blackened by hydrosulphuret of ammonia, or by hydrosulphuric acid, without, however, being wholly decomposed, the black sulphuret, when sublimed, still leaving a residue of paracyanogen. Even when the salt is dissolved in muriatic, and precipitated by excess of hydrosulphuric acid, the sulphuret of mercury is apt to carry down with it a portion of the paracyanic acid, so powerfully do these compounds resist the decomposing action even of the mineral acids.

10. A current of chlorine gas causes a very interesting reaction when passed through water, holding the salt in suspension. It decomposes both the salt and the acid, causing the evolution of carbonic acid, and forming bichloride of mercury, sal ammoniac, and carbonate of ammonia, which remain in solution. The action in this case is represented by the following formula:—

$$(C_8N_4O + HgO) + 4Cl + 14HO = HgCl_2 + 2(NH_4Cl) + 2(NH_3 + CO_2) + 6CO_2$$

11. When a solution of iodide of potassium is digested on a protoxide salt, the latter becomes dark green, grey, and finally black, from the separation of me-

tallic mercury. The solution is yellow or orange coloured, and contains biniodide of mercury, and *probably* paracyanate of potash. Such a reaction is thus explained.

$$2\,(C_8N_4O + HgO) + 3KI = (HgI_2 + KI) + Hg + 2\,(C_8N_4O + KO)$$

and it is probable that the haloid salts in general would act in a similar manner.

11. Heated to 230° F., these paracyanates are decomposed, and, from the sudden evolution of gas, appear to decrepitate. Mercury is sublimed, and carbonic acid, nitrogen, and cyanogen gases may be collected. 67.6 vols. of the mixture collected over mercury gave

7.1	vols. of	*cyanogen*	absorbed by alcohol,
33.5	—	carbonic acid —	— caustic potash,
17.0	—	nitrogen, remaining.	
57.6			

A black residue remains behind, which, by further heating, gives cyanogen only.

12. Burned with bichromate of potash, it gives also carbonic acid and nitrogen gases in the proportion of two to one.

Thus 97.5 vols. left 32.5 of nitrogen, and 95 vols. left 31.5. A portion of a beautifully yellow ($HgO + C_8N_4O$) gave a gas, of which 88.5 left 30 vols.

13. It was not till I came to subject this salt to analysis that I was made fully aware of the necessity of attending minutely to the circumstances under which it was prepared, and to the nature of the mercurial solutions employed. I have analyzed only two portions. The first precipitated from a neutral solution of the acid, dried at 212° F., and burned with bichromate of potash, gave from 24.08 grs. 7.74 grs. of carbonic acid, and 1.32 of water = 5.48 per cent. This gives for the composition of the anhydrous salt

$$C = 9.402 \qquad N = 10.888 \qquad O = 1.529 \qquad HgO = 78.181$$

16.57 grs. of the same salt dissolved in muriatic acid, and precipitated by hydro-sulphuric acid, gave 13.83 grs. of bi-sulphuret = 11.933 of metallic mercury, or 72.016 per cent. This gives for the anhydrous salt 79.200 of protoxide of mercury. This, therefore, was a basic salt, and its constitution

		Calculation.	Experiment. (1.)	(2.)
C_8	= 611.496	= 9.150	9.402	......
N_4	= 708.144	= 10.599	10.888	......
O_1	= 100.000	= 1.499	1.529	......
$2HgO$	= 5263.288	= 78.752	78.181	79.200
	6682.928	100	100	

The water amounts only to about 1.7 atoms.

14. The second portion analyzed was thrown down from an acid solution. 10.88 grs. burned with bi-chromate of potash, and the gases made to pass over red hot copper, gave 4.62 of carbonic acid and 1.09 grs. of water.

10.87 grs. dissolved and boiled in aqua regia, and precipitated as before, gave 7.56 grs. of bi-sulphuret of mercury = 60.003 per cent. of metallic mercury.

These give for the constitution of the *anhydrous* salt,

	(1.)	(2.)
Carbon, ...	= 13.094 . .	......
Nitrogen, ...	= 15.163 . .	......
Oxygen, ...	= 2.141 . .	......
Protoxide of Mercury,	= 69.602 . .	69.3738

Boiled in diluted nitric acid, and dried *in vacuo*, the salt gave me 61.13 per cent. of mercury: it had therefore undergone little change.

A neutral compound ($C_8 N_4 O + HgO$) would contain 15.088 of carbon and 64.958 of protoxide of mercury per cent.

The difference, amounting to 2 per cent. in the quantity of carbon obtained by analysis, may be accounted for by supposing it to contain a portion of the basic salt

VII. *Paracyanate of Silver.*

1. When metallic silver is introduced into a solution of paracyanic in nitric acid, a yellowish-brown powder falls down as the silver dissolves. A similar precipitate is obtained by dissolving solid nitrate or carbonate of silver in such a solution; and the more neutral the solution is rendered, the more completely is the paracyanate thrown down. If allowed to stand in this acid solution, the precipitate sometimes forms minute prismatic or acicular crystals of a deep red colour, grouped into small radiated globules. The supernatant liquid still retains much of the salt in solution, a large portion of which is thrown down as a beautiful bulky yellow powder, on the addition of water. After it ceases to be troubled by further dilution, the nitrate of protoxide of mercury throws down a copious precipitate of paracyanate of mercury, shewing how much less soluble it is than the salt of silver.

2. The paracyanate of silver is precipitated by caustic ammonia from its solutions in acids; but the precipitate does not dissolve in an excess of the alkali. Even when boiled in it, no sensible change is produced. The solution, indeed, acquires a yellow colour, but contains no silver.

3. It undergoes no sensible alteration in the sun's rays.

4. It is soluble in hot dilute nitric acid, from which it partly separates again on cooling.

5. In the air, it may be heated to 300° F., without much change of colour.

At incipient redness, it blackens, and at a red heat burns like tinder, leaving metallic silver.

6. When heated in close vessels, it gives off water, carbonate of ammonia, carbonic acid, and nitrogen, and leaves a black residue, which, at a red heat, yields cyanogen. The carbonic acid and nitrogen are in the ratio of 2 to 1, as they are also when the salt is burned with oxide of copper or bi-chromate of potash.

7. *a.* 3.33 grs. dried at 212° F. in LIEBIG's tube, left of metallic silver 1.34 = 40.24 per cent.

b. 7.16 grs. burned with bi-chromate of potash gave 0.96 grs. of water = 13.40 per cent.

c. 9.08 grs. burned with oxide of copper gave 6.49 grs. of carbonic acid = 1.7945 of carbon, or 19.702 per cent.

These give for the constitution of the salt,

		Calculation.	Experiment.
8C . .	= 611.496	= 18.412	. . . . 19.702 *c*
4N . .	= 708.144	= 21.322	
O . .	= 100.000	= 3.017	
AgO .	= 1451.607	= 43.705	. . . 43.217 *a*
4HO .	= 449.918	= 13.544	. . . 13.400 *b*
	3321.165	100	

Here, as in the analysis of the uncombined acid, the carbon obtained is in excess, and probably from the same cause,—the formation of oxides of nitrogen,—which in burning paracyanogen and its compounds, it seems almost impossible wholly to prevent.

Paracyanate of Iron.—When a solution of sulphate of protoxide of iron is poured into one of paracyanic in nitric acid, a yellow precipitate falls, not very copiously, which is soluble in hot dilute nitric acid, and falls in great part on cooling. Heated in the air it becomes brown, blackens, then burns like tinder, leaving metallic iron, which rapidly oxidizes.

Paracyanate of Lead may be formed by mixing together a solution of nitrate of lead with one of the paracyanic, evaporating to dryness, and washing the mass with cold water.

The washings of both these salts give copious precipitates with nitrate of protoxide of mercury.

Alkaline Paracyanates.—The affinity of this acid for the alkalies seems to be singularly small. In large excess they partially decompose the metallic salts, and dissolve small quantities of the uncombined acid forming yellow solutions; but they exhibit little inclination to form neutral salts. Perhaps it is, that the neutral paracyanates of the alkaline bases are also but sparingly soluble in pure water, and hence the difficulty of obtaining a neutral solution. But this is a matter for future investigation.

Rendered neutral by an acid, the solutions of the alkaline paracyanates are precipitated by the salts of oxide of silver and of protoxide of mercury; an excess of acid decomposes them, and throws down the paracyanic acid.

In my paper published in 1829, I suggested that the nitrogen known to exist in certain varieties of coal, might be present in them in the state of paracyanogen. I find that fine caking coal from Killingworth colliery near Newcastle, which contains nitrogen equal to about one-ninth of the weight of carbon, and leaves only 0.95 per cent. of ash, dissolves completely in hot nitric acid, with copious evolution of red fumes, giving a dark brown solution. Water throws down a reddish-brown powder, and the yellow supernatant liquid gives a reddish-brown precipitate with nitrate of mercury. The substance thrown down by water when burned with bichromate of potash, gave a mixture of gases containing about eight per cent. of nitrogen. The subject, therefore, is worthy of future investigation.

In concluding this paper, I would draw the attention of chemists to the very remarkable discordance between the properties of cyanogen and paracyanogen and their several compounds; a discordance certainly more striking than any other with which we are yet acquainted among isomeric bodies. I have already adverted to the analogy which this black substance seems to exhibit in some of its chemical relations to the hydrocarbons. It also agrees with them in the multiple ratio of the elementary atoms existing in its equivalent as compared with that of cyanogen; and in the less stability and more difficult formation of its compounds, as is the case with the hydrocarbons of greater density. For though in its relation to the acids, the paracyanic is much more stable than the cyanic acid, yet in relation to heat the latter is by far the more permanent.

We have therefore three compounds of carbon with nitrogen in the proportion of two atoms to one.

Cy . . . Cyanogen.
C_2N . . . The radical of the fulminic acid.
C_8N_4 . . Paracyanogen

forming with oxygen the acids CyO, C_2NO, and C_8N_4O respectively.

If we allow ourselves to reason from the obscure analogy of paracyanogen to the hydrocarbons, we may expect a compound $C\ N_4$ to fill up the gap between it and cyanogen, and which may possibly be a liquid.

PORTOBELLO, *January* 1836.

III.—*Experimental Researches into the Laws of Certain Hydrodynamical Phenomena that accompany the Motion of Floating Bodies, and have not previously been reduced into conformity with the known Laws of the Resistance of Fluids.* By JOHN SCOTT RUSSELL, Esq. M. A., F. R. S. Ed.

Read 3d April 1837.

INTRODUCTION.

IN the summer of 1834, I was led to examine with considerable interest some of the phenomena of fluids, from the circumstancc of having been consulted upon the means of improving a system of navigation to be conducted at unusually high velocities. Being well aware, however, of the very imperfect state of that part of Theoretical Hydrodynamics which relates to the Resistance of Fluids to the Motion of Floating Bodies, and that there had been found in its application to the solution of practical questions, discrepancies so wide between the predicted results and the observed phenomena, as to render the principles of the theory exceedingly false guides, when followed as maxims of art, I felt it impossible to recommend conscientiously any mode of procedure founded on defective principles, and I therefore determined to undertake a series of investigations concerning the laws of the resistance of fluids, and the means of applying them to the formation of rules for the arts of practical navigation and naval architecture. In this investigation, I have now been engaged during the leisure of two summers, and I am still continuing to prosecute the investigation.

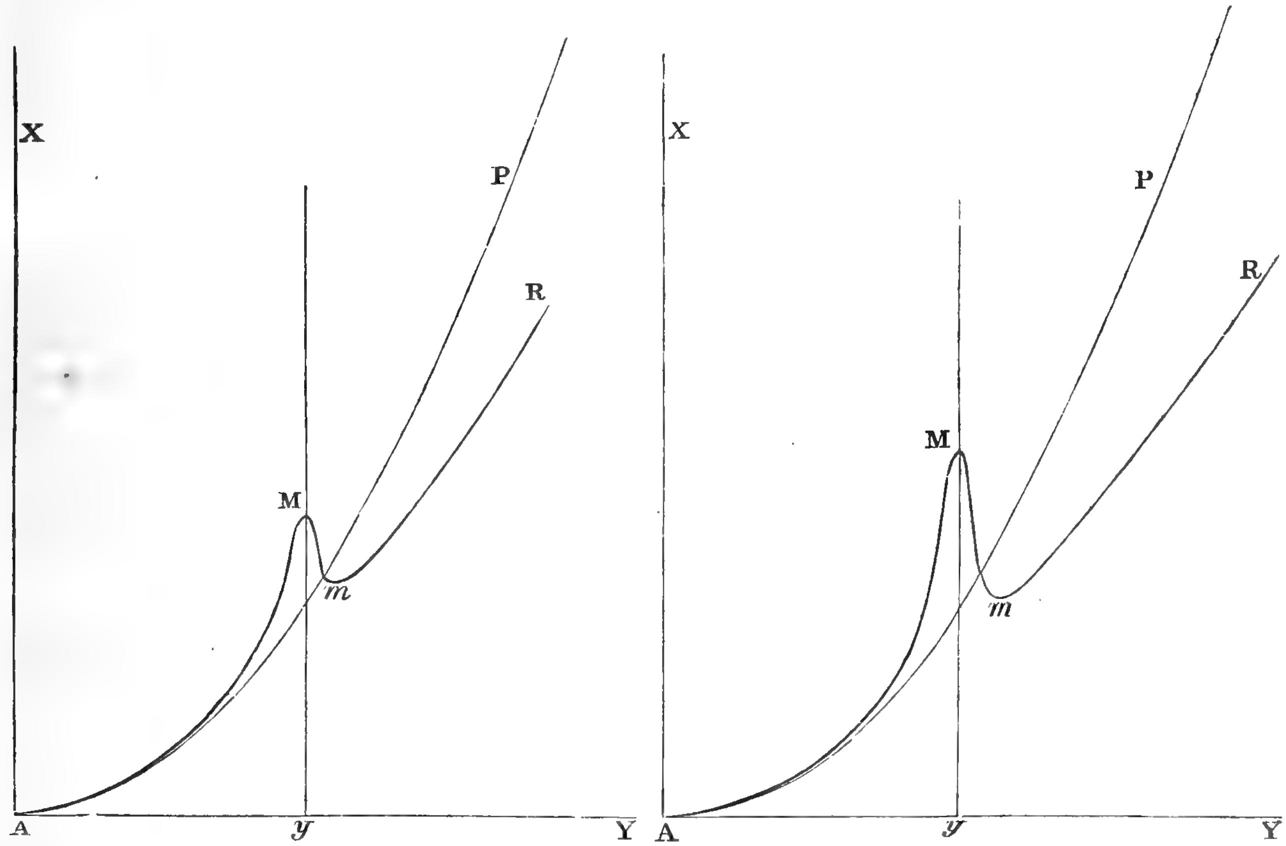
The following papers contain the experiments of the two summers 1834 and 1835, with the resolution of certain anomalous phenomena, and the illustration and application of certain laws that have been developed. The experiments were conducted on a very large scale, and the forms given to the floating bodies were analogous to those which are most highly approved in the practical construction of ships, as well as those of certain theoretical solids. The vessels used were from 31 to 75 feet in length. Accurate Chronometers and Dynamometers of various descriptions used by a number of highly educated and scientific assistant observers, render the experiments worthy of great confidence. In 1834 the power used to overcome the resistance was the force of horses directly applied to the vessels; but although out of a multitude of experiments, some were obtained that were distinguished by uniformity in the application of the force, yet in general the action of that species of moving force was found too desultory and discontinuous to furnish a measure of resistance sufficiently accurate to be used in comparisons of a delicate description, and therefore in 1835, means were provided for rendering the action of the moving force more nearly continuous by using a peculiar apparatus. With this apparatus experiments were made on the resistance of four vessels of about 70 feet in length, at different depths of immersion, so as to give comparative measures of resistance in reference to sixteen forms, at velocities from three to fifteen miles an hour.

The results of the investigations directed to the determination of the law which connects the resistance of the fluid with the velocity of the motion of the floating body, appear to establish the following conclusions,—that the resistance does not follow the ratio of the squares of the velocities, excepting in those cases where the velocity of the body is low, and the depth of the fluid considerable—that the increments of resistance are greater than those due to the squares of the velocities, as the velocity approaches a certain quantity, which is determined by the depth of the fluid—that at this point the resistances attain a first maximum, and that here, by certain elements of the form of the body, and of the dimensions of the fluid, they may become infinite—that immediately after this, there occurs a point of minimum where the resistance becomes much less than that due to the square of the velocity, and after which it continues to receive increments, of which the ratio is less than that due to the increment of the square of the velocity—that according to the law of progression which has been established, the resistance will reach a second point of maximum, when a velocity shall be attained of about 29 miles an hour, after which it will be rapidly diminished with every increase of velocity.

Extracts from the Experiments, shewing the connection between Resistance and Velocity.

Example I.		Example II.	
Velocity.	Resistance.	Resistance.	Squares of Velocities.
3.7	28.0		14.3
4.0	33.75	39.	16.
5.0	51.0		25.
6.1	91.0	111.	38.4
7.1	217	255.	51.5
7.5	265.	330.	57.3
	Point of First Maximum and Minimum.		
8.5	215.	210.	72.6
9.0	234.	235.	81.8
11.3	246.		129.
12.3		352.	153.6
15.1		444.	229.5

The curve of resistance derived from these examples instead of being a parabola, will be of the following species. Fig. 1 and 2.



AX and AY rectangular co-ordinates.
Velocity measured on AY, and resistance on AX.
AP the parabola resulting from the squares of the velocities.
AM*m* R the line of resistance, M the point of first maximum, and *m* the succeeding point of minimum.

The causes of these deviations from the law of the squares of the velocities, are fully investigated in the course of observations forming Part I. of this paper, Part II. being filled with the details of the experiments of 1834, and Part III. with those of 1835.

The first element of deviation which presented itself, was a phenomenon of *Emersion* of the solid from the fluid, due to the velocity of the motion, and by which the dynamical immersion of the floating body is rendered less than its statical immersion in the fluid. The law connecting this emersion with the velocity of the solid is deduced in Sect. (1.) from elementary considerations, and coincides with the experiments.

Having determined the effect of motion upon the floating body itself in relation to the fluid, I have next examined the effect produced on the particles of the fluid itself by the motion of the floating body. At this part of the inquiry I discovered phenomena of a most singular character, by means of which the resolution of the anomalies in resistance has been most successfully effected, and which gives to many of the facts of practical experience a satisfactory explanation, and points the way to many important improvements in the construction of vessels, the navigation of rivers and shallows, inland navigation, and other departments of hydrodynamical engineering. These phenomena arise from the *generation and propagation of Waves* of the fluid by the motion of the floating body.

It has appeared, in the course of these investigations, that the restoration of equilibrium among the particles of the fluid when it has been deranged by the motion of a floating body, is effected, not so much by means of the generation of currents in the fluid, as has hitherto been generally assumed, as by means of the generation of waves of the fluid, in which form the elevations of the fluid raised on the front of the moving body are propagated with an ascertained velocity in the direction of the motion of the disturbing body. It appears that these waves move with a velocity that is nearly uniform,—that they travel to very great distances,—that their velocity is not in any degree connected with the form of the vessel,—that their velocity is not at all dependent on the velocity of the body which generates them,—that their velocity is due alone to the depth of the fluid, being equal to the velocity acquired by a body falling in vacuo through a space equal to half the depth of the given fluid,—and that the height of the wave itself above the fluid will only increase its velocity by so much as it increases the depth of the fluid at that point, reckoning from the summit of the wave.

I next proceeded to examine the nature of the interference of such waves, so as to determine their effect in modifying the resistance of the fluid to the motion of the body giving rise to them. I found immediately that the point of first maximum of resistance coincides accurately with the point at which the velocity of the motion of the floating body becomes *equal* to the velocity of the motion of the propagated waves. It appeared further, that the effect of the formation of these waves, when the velocity of the solid was *less* than the velocity of the waves, was to send forward towards the anterior part of the solid an accumulation of successive waves (to which accumulation I have given the name of the anterior wave), and to create a posterior depression in that part of the fluid from which these waves had been sent out, and thus to change the form of the surface of the fluid in such a manner that the axis of the floating body, formerly horizontal, no longer remained so, but was elevated anteriorly and depressed posteriorly, so as to form a considerable angle of inclination, and greatly increase the anterior section of displacement of the solid, whereby, at velocities less than the velocity of the wave, a very rapid increase of resistance was experienced in approximating to that velocity. It appeared, on the other hand, that at velocities *greater* than that of the waves, the effect of their generation by the floating body was to diminish the resistance given by the fluid, because the elevations of those waves falling behind those points of the body by which they had been raised, constituted an accumulated wave towards the middle of the body, upon the crest of which wave, poised in a position of stable equilibrium, it was borne along in a horizontal position, with a diminished section of immersion at the stem and stern, and consequently with a diminished section of resistance.

Besides the diminution of the resistance to the floating body experienced at velocities greater than that of the wave, it is also rendered apparent why there should be likewise experienced a great diminution of that commotion which takes place in the fluid at velocities less than that of the wave, and how it is that the phenomenon of stern surge, so destructive to the banks of the channel, and so dangerous in the practical navigation of shallow water, is found to disappear entirely at velocities greater than that of the wave in the given depth of fluid.

The effect of the motion of the floating body in changing the form of the fluid is least when the velocity is least and greatest; its effect is greatest when its velocity most nearly approximates to the velocity of the wave.

From the investigations of 1834, a form of great resistance suggested itself to me. A vessel, named in the tables "The Wave," was constructed of this form. This vessel was made the subject of experiment in 1835. It appears from the tables, that the resistance of this vessel was much less than that of other beautifully shaped vessels with which it was compared; and one phenomenon which I observed seems to me to establish as true, that the form of this vessel does not deviate widely from the form of least resistance. This tentative phenomenon seems to me to demonstrate, that the motion communicated to the particles of the fluid is the smallest that is consistent with the translation of the moving body, and it is this,—that at all velocities extending up to seventeen miles an hour, no spray, no heaping up of water at the stem, no lateral currents extending beyond the precincts occupied by the body itself, were ever sensible, but the body entering the water having a smooth and glassy surface, left it unchanged and unruffled. In the motion of all the other forms it was observed, that the water was thrown aside at the stem of the vessel in the form of a "head and feather" of spray, and that "broken water" extended to a distance even among the particles considerably removed from the line of the vessel's motion. The equation to the curve of least resistance was found by supposing the lateral motion given to a particle of the fluid, to receive equal increments in equal times from zero to a given maximum of velocity, after which, by equal decrements in equal times, it should again be brought to rest at the required distance from its original position in the place necessary to permit the transit of the greatest diameter of the immersed body. The curve thus obtained is concave outwards at the stem, and becomes convex towards the maximum breadth, having an intermediate point of contrary flexure. Delineations of this form, and tables of comparative resistances, are given in Part III.

I have given, at the end of the first part of this paper, some illustrations of the subject, drawn from facts and observations in practical experience, which have either been communicated to me or recorded by myself. The navigation of shallow rivers, lakes, seas, and canals, affords many illustrations of the principles I have developed. The canals of Holland and the rivers of America, as well as those of our own country, are navigated on a practical system, which is fully explained by the interference of the wave,—and the improvements of which those species of navigation may be capable, can only be effected in conformity with the knowledge of these laws that has now been obtained. By the propagation of waves, and propelling vessels upon those waves, there is a prospect opened up of attaining velocities upon the surface of the water, that have been hitherto held to be impracticable. The length, however, to which this communication had previously extended compelled me to shorten this part of the paper, and I have been contented rather to shew the applicability of the principles to practical improvement than to carry out this application.

Such are the results of that portion of the investigations I have undertaken, which has been completed; and I feel it to be my duty thus publicly to acknowledge, that if any benefit shall be conferred, either upon theoretical hydrodynamics or the practical arts connected with it, by these inquiries, it is not to myself alone that they owe their value. To so large and extensive a series of experiments my own exertions and my own pecuniary resources would have proved inadequate, had I not been placed in circumstances peculiarly fortunate. Two scientific friends, Alexander Gordon and Andrew Crawford, Esqrs. afforded me invaluable and long continued assistance, and to their labours and those of Dr George Glover, Mr Wilkinson, and Mr Muir, with about a dozen of hired assistants, the experiments owe much of their extent and accuracy. To the Committees of Management of the Edinburgh and Glasgow Canal Companies, who have permitted the use of their public works, and of their servants, and of their moving power, and of their vessels, and defrayed a large expenditure of money, and to Robert Ellis, Esq. W. S., through whose influence principally these privileges were obtained, under the enlightened conviction, that, from the improvement of that science with the applications of which they are so nearly connected, these mercantile companies would be the first to derive important benefit; to them and to him, for his devotion to the interests of science, and for the unwearied kindness, and judgment and skill with which he has assisted me in this undertaking, I consider it my privilege to offer my thanks, as the means of accomplishing a task which might otherwise have proved to be impracticable. To J. W. Smith, Esq. of Phila-

delphia, I am indebted for much valuable information regarding the state of practical navigation in America, of part of which I have availed myself in illustrating the subject. Whatever is still wanting to complete this investigation, I hope, in the course of a few years, if I enjoy life so long, to be able to accomplish; but, in the mean time, one series of the observations has appeared sufficiently entire to be presented by itself, and I have been induced to give them publicity, by the kind recommendation of Professor WHEWELL, who has taken a generous interest in their progress, by which I have been encouraged to pursue the investigation through the many annoyances and disappointments and dangers that necessarily accompany an undertaking of this nature.

PART I.

General Observations on the Phenomena that accompany the Motion of a Floating Body on the surface of a Quiescent Fluid.

Every instance of the want of perfect agreement between the predictions of theoretical mechanics and the results of practical experience, may be traced almost invariably to the existence of certain latent conditions that have been omitted from the fundamental hypotheses. Discrepancies of this nature are sufficiently numerous in the subject of hydrodynamics; so much so, indeed, that in relation to it, the appellations " practical" and " theoretical" are continually applied as terms of antithesis. The various hypothetical constitutions assumed for fluids by NEWTON, BERNOULLI, EULER, D'ALEMBERT, and their followers, have enabled them to obtain one law regarding the resistance of fluids to the motion of solids, which accords very closely with the phenomena of certain solids in certain circumstances and at certain velocities, but that law has not been found adequate to the solution of the case of a solid partly immersed, as when a floating body moves along the surface of a quiescent fluid. This law, which connects the resistance of the fluid with the second power of the velocity, is in very close accordance with the motion of bodies that are wholly immersed, and with the motion of floating bodies that have certain velocities and are placed in certain circumstances; but it has proved widely erroneous in its direct application to the motion of floating bodies in different circumstances and at higher velocities. So far, indeed, does the resistance actually obtained in these cases differ from the theoretical resistance, that examples may be found in every large collection of experiments, and are to be met with in almost every page of those which I have given at the end of this paper, where the resistance, instead of following the law of the squares of the velocities directly, has been found to vary, not only with every different power of the velocities from the first to the fourth power, but also in the inverse ratio of some of those powers. In addition to the examples at the end of this paper, I may here adduce two very obvious illustrations, the one shewing an increase of resistance corresponding to a very high power of the velocities, and the other exhibiting a diminution of resistance with an increase of velocity greater than the former. The experiments were made on the 18th of October 1834 with the floating body, whose form is given in Plate III. Fig. 4, having a mass of 12,579 lbs. All the circumstances attending the performance of the two experiments were alike, and the last column shews the comparative resistances as obtained by a dynamometer.

Example I.

	Space Described.	Time.	Velocity in Feet.	Resistance in lbs.
Experiment I.	1000 feet	$117\overset{s}{.}5$	8.51	233.
Experiment II.	1000 feet	93.5	10.69	425.

Example II.

	Space Described.	Time.	Velocity in Feet.	Resistance in lbs.
Experiment III.	2640 feet	$302\overset{s}{.}$	8.76	261.
Experiment IV.	500 feet	35.	14.28	251.

In the first of these examples, the velocities being in the ratio nearly of 85. to 106., the resistances are nearly as the third powers of the velocities; and in the second case, the velocity being increased from about 5.9 miles an hour to 9.6 miles an hour, the resistance is found to diminish in a ratio of 26.1 to 25.1.

To the imperfection of this branch of science, I may also adduce the testimony of two eminent individuals, to whose exertions we owe much that is now being accomplished for its improvement. In the Essay towards an approximation to a Map of Cotidal Lines, in the Philosophical Transactions of 1833, Professor WHEWELL remarks, that " the phenomena of waves, the motion of water in tubes and canals, in rivers, the motion of winds, and the resistance of fluids to bodies in motion, are all cases in which we are yet far from having drawn our analytical mechanics into a coincidence with experiment, or even a close approximation to it;" and Mr CHALLIS has made the same admission at the end of his Report on the state of the Theory of Hydrodynamics, made for the British Association, where he says, that his " review may serve to shew that this department of science is in an extremely imperfect state, and that possibly it may on that account be the more likely to receive improvement;" and he adds, that " a singular fact relating to the resistance of bodies partly immersed in water has been observed, viz. that a boat drawn on a canal with a velocity of more than four or five miles an hour, rises perceptibly out of the water, making the resistance less than if no such effect took place;" and he further observes, that "theory, although it has never predicted any thing of this nature, now that the fact is proposed for solution, will probably soon be able to account for it on known mechanical principles."

To observe with accuracy the conditions of some of these discrepant phenomena, and reduce to the dominion of known laws certain anomalous facts, so as to obtain a closer approximation to a correct system of theoretical and practical hydrodynamics in those points in which they have hitherto been furthest apart, has been the object of this series of investigations. If the reasoning I have used in the sequel follow accurately from the experiments I have adduced, it will be shewn that there have hitherto been neglected in the calculation from theory of the resistance of fluids to the motion of floating bodies, two important elements of that resistance which affect low velocities by very small quantities, and have therefore escaped observation until certain practical results gave their effects more prominent importance; and that these two elements are, (1.) An emersion of the floating body developing itself as a function of the velocity of the motion, and of the measure of gravitation; and, (2.) The generation of waves by the motion of the floating body which are propagated in the fluid, and which affect the form of the surface of the fluid, the position of the floating body, and the resistance.

It appears to me probable that I shall most readily and simply communicate to others the information I have acquired on this subject, by following the order in which I was myself led to the acquisition of it. My examination was first of all directed to the effects produced by motion upon the floating body itself, and afterwards to the motions of the particles of the fluid in which the body is moved.

SECTION I.—*The Effect produced by Motion on the Immersion of a Floating Body.*

It has been suggested as an explanation of the cases in which the motion of a floating body is observed to be facilitated at high velocities, that the moving power by drawing a vessel partially out of the water, so as to diminish its immersion, may lessen the sectional area of resistance of the solid; and further, that if the moving force be supposed to be applied to the anterior part of a vessel, so as to elevate the prow above the surface of the fluid, the diminished immersion of that part would sufficiently account for the diminished resistance. These suggestions are not confirmed by observation. The amount of force required to produce the said effect by either of these methods, is found to be more than equivalent to the diminution of resistance produced by such force, and it has been observed on the contrary, as will be apparent in the sequel, that great and marked facilitation of the motion is observed when the line of effect of the moving force has a downward, instead of an upward, direction; and that any elevation of the prow or anterior part of a vessel, instead of facilitating its motion, increases the resistance to it.

To determine the real condition of the immersion of the floating body at various velocities, and trace the phenomena to some known mechanical principle, was the object of the first series of my experiments in 1834. For this purpose there was constructed an experimental skiff, a very light vessel of a very small draft of water, and furnished with apparatus for determining resistance and immersion. The skiff and its apparatus are described and delineated in that part of this paper which contains the details of the experiments of 1834. Chronometers, dynamometers, and two modifications of PITOT's tube were observed. Twelve openings in the bottom of the vessel allowed the water to rise in glass-tubes carefully graduated, to the level of the fluid without, and furnished measures of the statical and dynamical immersion of the floating body. The vessel thus furnished, was made the subject of careful experiment at velocities of from 3 to 20 miles an hour.

These experiments give a decided and consistent result. It was found that in every case the statical immersion of the floating body was less than its dynamical immersion. The following are taken from the experiments of 1834, given in Part II. The statical immersion being 2.7 inches, the dynamical immersions observed at given velocities in miles an hour were as follows—

Velocity,	0. ,	3.016,	4.00,	5.165,	6.431,	7.253,	8.11,	9.164,	10.237,	20. +
Immer.	2.7,	2.6 ,	2.5 ,	2.2 ,	1.9 ,	1.8 ,	2.2 ,	2.3 ,	2.0 ,	1.5

After having determined the existence of a dynamical emersion, I endeavoured to discover the law of connection between the diminished immersion and the velocity of the motion. A singular change in the immersion at the velocity 8.11, and those immediately following it, gave me much trouble in my attempts to do this. I at first imagined the experiments might have been erroneous, but obtained the same results on each repetition. It afterwards turned out that these very anomalies shewed the continuity of the law; for it will soon be apparent in following out the subject, that at that very velocity of 8.11, the fluid undergoes a very extraordinary change in its form, which increases the immersion at the middle of the floating body when these immersions had been observed, and diminishes it at other parts of the body. Leaving, therefore, indications 8.11, and those which succeed it, to have the reductions made upon them, which subsequent investigations render necessary, and taking those below that point, and far above it, we may now proceed to examine whether any known principle will lead us to assign a law accordant with these phenomena.

Extensive series of experiments with the tube of PITOT, conducted by the most eminent experimentalists, and confirmed by the accordance of collateral phenomena, have established the doctrine as an axiom in hydrodynamics, That the resistance of a small unit of surface to a fluid, when either the fluid is in motion, or the surface itself, is equal to the statical pressure of a column of fluid having for its height the height due by gravity to that velocity. Had not this been satisfactorily established by previous experiments, and universally received as an unquestionable truth, my own experiments with the tube of PITOT would have been sufficient to shew the truth of the doctrine, which is merely the converse of the theorem, That the statical pressure of a column of fluid generates a velocity in the effluent jet equal to that which is required by a heavy body falling freely by gravity through a height equal to the depth of the fluid. This statical quantity being the measure of the pressure of the fluid upon the anterior surface of the immersed solid, will also be the measure of the quâquaversus pressure of the fluid in every direction, and therefore will measure the pressure of the water upon the vessel causing its emersion. Opposed to this we have the downward pressure arising from the gravity of the solid. Now, the measure of this pressure is the weight of the column of water displaced by the body, the depth of which is equal to the depth of the statical immersion of the solid, and each of these pressures is at every velocity equal to the other, and in the opposite direction to it. Whence,

Let $s =$ Transverse section of Statical Immersion.
$v =$ Velocity of Motion.
$g =$ Measure of Gravitation.
$s' =$ Section of Dynamical Immersion.
$\therefore vs =$ Volume of Fluid displaced by Statical Section; and
$vs' =$ Volume of Fluid displaced by Dynamical Section; and
$\frac{v^2}{2g} =$ Height due to the velocity v.

If ρ be the density of the fluid,

$$s'v\rho = sv\rho - \frac{v^2\rho}{2g}$$

$$\therefore s'v = s\left(v - \frac{v^2}{2g}\right) \text{ and}$$

$$s' = s\left\{1 - \frac{v}{2g}\right\}$$

Proceeding from this equation of the dynamical section, to determine the variation of total resistance, on the condition of proportionality to the law of the squares of the velocities, as regards that portion of the section of the solid which remains immersed, from the general equation

$$R = s v^2 \frac{\rho}{2g}$$

we deduce in this case of diminished section by substitution,

$$R' = s v^2 \left\{ 1 - \frac{v}{2g} \right\} \frac{\rho}{2g}$$

of which the successive differential equations in regard to v are,

$$\frac{d\,R'}{d\,v} = \left\{ 2v - \frac{3v^2}{2g} \right\} \frac{\rho}{2g} \quad \ldots\ldots \quad (1.)$$

$$\frac{d^2\,R'}{d\,v^2} = \left\{ 2 - \frac{3v}{g} \right\} \frac{\rho}{2g} \quad \ldots\ldots \quad (2.)$$

$$\frac{d^3\,R'}{d\,v^2} = \left\{ \quad - \frac{3}{g} \right\} \frac{\rho}{2g} \quad \ldots\ldots \quad (3.)$$

From eq. (1.) if we make

$$\frac{d\,R'}{d\,v} = \left\{ 2v - \frac{3v^2}{2g} \right\} \frac{\rho}{2g} = 0,$$

we obtain, in the case of a maximum or minimum,

$$2 - \frac{3v}{2g} = 0 \quad \text{and} \quad v = \frac{4g}{3}.$$

By substituting this value in eq. (2.) we get

$$\frac{d\,R}{d\,v} = \left\{ 2 - \frac{8}{2} \right\} \frac{\rho}{2g},$$

being a negative quantity, whence it follows, that

$$R' \text{ is a maximum, when } v = \frac{4g}{3};$$

$$s' = 0, \text{ ——— when } v = 2g.$$

These expressions may be converted into the following laws.

Laws of Dynamical Emersion and Diminished Resistance.

1. If a floating body be put in motion with a given velocity, the pressure which it exerts downwards upon the fluid in virtue of gravity, is diminished by a quantity equal to the pressure of a column of the fluid having the height due to the velocity of the motion.
2. The Section of Dynamical Immersion is less than the Section of Dynamical Emersion, in the same proportion in which the difference between the velocity of the motion and the height due to it is less than the velocity of the floating body.
3. The Resistance being taken in the ratio of the square of the velocity upon that part of the section only which remains immersed, the aggregate resistance will increase in the ratio of the squares of the velocities, very nearly

only at low velocities, and at higher velocities it will increase very slowly, and will even diminish as the velocity is increased.

4. The Resistance increases very slowly from about 25 to 29 miles an hour, at which point the velocity being $\frac{4}{3}$ of that which is the measure of the force of gravity for a given point of the earth's surface, or about 43 feet per second, and 29 miles an hour; the resistance has attained a maximum, and rapidly decreases, and continues to do so.
5. At 43.8 miles an hour (when $v = 2g$), the floating body emerges wholly from the fluid, and skims its surface.

It should be observed, that the phenomena corresponding with these results will be modified when the depth of the fluid is small, by the wave and other elements of resistance, upon the consideration of which we are to enter in another part of this paper.

It is also to be observed, that the form of the floating body is no element in the formula of emersion—that the law is a general one. This caution is the more necessary, because Mr CHALLIS has given a formula of emersion for a sphere, derived from the summation of all the elementary forces acting upwardly upon the sphere, which are obtained from resolving the oblique forces on each point of the sphere into co-ordinates of vertical and horizontal action. The particular case treated by Mr CHALLIS, although true for a sphere, does not apply to an elongated body, so as to diminish its emersion, but merely changes its position, and in such a manner as to increase, instead of diminishing, the resistance of the fluid. The effect he refers to is a great evil incident to a certain form of vessel, which otherwise possesses considerable advantages. The law of Diminished Resistance and Immersion which I have developed, is perfectly general in its application, and wholly independent of casual form. It has for its foundation merely the simple principle, That gravity, acting on a solid body during a given unit of time, is a constant quantity, and that the displacement of the fluid by the weight of the body, being a quantity that increases both with the velocity and the quantity of that displacement, must ultimately be equal in quantity, as it is opposite in direction, to the pressure of the solid downwards by gravity.

SECTION II.—*On the Motions that are communicated to the Particles of a Fluid by the Motion of a Floating Body.*

Many of the attempts which have been made to verify by experiment, or to discover empirically, the laws of the motion of floating bodies, have been defeated by the untoward circumstance of the *disturbance* which is caused by the protrusion of a solid into the space occupied by the fluid. The particles which are thus displaced are thrown aside by the anterior part of the body, and then collapse upon other parts of it; or they are thrown forward before the body, which is afterwards protruded on them a second time; or they are thrown up in heaps in certain forms of equilibrium, and the accumulations and irregularities of pressure which are thus occasioned give rise to currents of the fluid and mutual collisions amongst divided masses of it, and surges and other phenomena, all of which entering at once as elements of the resistance into the production of the resulting phenomena, do so modify them, as to give results that are totally inconsistent with theory, and are apparently at variance with each other, That theory, therefore, which will venture to assign the measure of the resistance of a fluid to a solid, upon the supposition that the surface of the fluid remains horizontal, and that the anterior part of the solid finds the surface of the liquid a level plane, will proceed upon imperfect data.

The only one of these disturbing causes which has hitherto been investigated in theories of hydrodynamics, is the lateral current proceeding from the stem towards the stern of the moving body.

The elements which I have added to those previously investigated are the Anterior, Posterior, and Central Waves.

I was first led to investigate the disturbances produced by the entrance of a floating solid into a quiescent fluid, by encountering a series of anomalous irregularities in my attempts to measure the immersion and resistance of the floating body at different velocities. There were certain velocities at which the body appeared to be almost buried in the water, and was so much impeded, that any force employed to accelerate the velocity of the body seemed only to accumulate resistance upon it, while at other velocities greater or less than these, the body would suddenly change its position, and instantly emerge out of the trough of the fluid to a considerable height above its statical elevation. But what happened in one portion of fluid did not occur in a different portion of the same fluid even at the same velocity. The resistance would sometimes exceed the third power, and again in another portion of fluid fall below the first power, for the very same velocity. These were disruptions of the law of continuity which the gradual law of an emersion as a function of velocity and gravity alone could not solve. I therefore entered upon a series of inquiries, directed solely to the subject of discovering and determining the unknown constituents of the disturbance of the equilibrium of the fluid occasioned by the presence of the floating body. The results of my inquiries I am now to state as briefly as clearness will allow, premising, at the same time, that the facts which presented themselves appeared at first to myself as extraordinary as they may now probably seem to those who may learn them for the first time; but

I may add, that, in the same degree in which they appeared wonderful to me at first, do they now appear to me the necessary and most satisfactory results of elementary and axiomatic principles.

In directing my attention to the phenomena of the motion communicated to a fluid by the floating body, I early observed one very singular and beautiful phenomenon, which is so important, that I shall describe minutely the aspect under which it first presented itself. I happened to be engaged in observing the motion of a vessel at a high velocity, when it was suddenly stopped, and a violent and tumultuous agitation among the little undulations which the vessel had formed around it, attracted my notice. The water in various masses was observed gathering in a heap of a well-defined form around the centre of the length of the vessel. This accumulated mass, raising at last a pointed crest, began to rush forward with considerable velocity towards the prow of the boat, and then passed away before it altogether, and retaining its form, appeared to roll forward alone along the surface of the quiescent fluid, a *large, solitary, progressive wave.* I immediately left the vessel, and attempted to follow this wave on foot, but finding its motion too rapid, I got instantly on horseback and overtook it in a few minutes, when I found it pursuing its solitary path with a uniform velocity along the surface of the fluid. After having followed it for more than a mile, I found it subside gradually, until at length it was lost among the windings of the channel. This phenomenon I observed again and again as often as the vessel, after having been put in rapid motion, was suddenly stopped; and the accompanying circumstances of the phenomenon were so uniform, and some consequences of its existence so obvious and important, that I was induced to make *The Wave* the subject of numerous experiments.

It very soon began to appear probable, that the existence of this phenomenon of *the solitary wave would exercise very great influence on the quantity and nature of the resistance of the fluid to a body moving with a given velocity, according as that velocity was equal to, or greater, or less than, the velocity of the wave.* And on making this the subject of a series of experimenta crucis, the correctness of the anticipation was established, and it appeared that *the velocity of the motion of the solitary wave had a peculiar relation to a certain well-defined point of transition in the resistance of the fluid.*

In prosecuting the inquiries to which this discovery gave rise, I found that, *in every instance of progressive motion of a solid in a fluid, the displaced fluid generated waves of the fluid that were sent in the direction of the motion of the body, and propagated with a constant velocity, which was quite independent of the velocity of the motion of the body, and that the magnitude, disposition, and velocity of these waves formed very important elements in the resistance of the fluid to the floating body.* I therefore directed my investigations to the discovery of the law of the genesis and motion of such waves, and the nature of their interference with the resistance of the fluid.

SECTION III.—*On the Laws which Regulate the Genesis and Propagation of* The Progressive Wave *which is created by the Motion of a Floating Body.*

It is very necessary that *The Wave* be carefully distinguished from certain elevations on the surface of a fluid which may likewise be included under the generic title of Wave, as observers who do not make this discrimination will be led into great confusion. I have observed at least four species of Wave,—the Ripple or Dentate Wave,—the Oscillatory Wave,—the Surge Wave,—and "The Wave" "par excellence," the solitary, progressive, great wave of equilibrium of the fluid. In regard also to the vessel, I have observed several waves. The Great Primary Wave of Displacement,—the Secondary Wave of Unequal Displacement,—the Great Posterior Wave of Replacement,—and the Secondary Waves of Replacement. It is the Great Primary Wave of Displacement which alone belongs to the species of the wave which I am now to examine.

The wave has been generated in two ways. By the addition of a solid to a limited portion of quiescent fluid, and by the addition of a given quantity of fluid. A loaded vessel being suddenly drawn with considerable force towards the mouth of a narrow channel, sends forward the displaced water into it in the form of the wave. A vessel being in the course of its motion made to vary suddenly, either made to move more rapidly or more slowly, or suddenly stopped, will send forward a sensible wave; and at all times, in smooth water, when moving with a velocity less than that of the wave, there will be perceived a series of waves preceding the vessel. If, also, there be made, by means of a sluice or otherwise, a sudden and considerable addition to the waters of a limited channel, the elevation will be transferred along the surface in the form of the wave.

It was found that the mode of the genesis of the wave, whether by a large or small vessel, by a long or short vessel, by a sharp or obtuse vessel, by a deep or a shallow vessel, whether by the addition of a quantity of water in one manner or in another manner, that the mode of genesis did not in any way, except in magnitude, as a great or small wave, produce any modification of form or velocity in the resulting wave. It was remarkable, also, that the velocity of the motion of the generating body did not in any way affect the velocity of the resulting wave, a wave, for example, of 8 miles an hour being produced alike from bodies moved at the rate of 2, 5, 6, and 12 miles an hour.

A very simple and early observation convinced me that the velocity of the propagation of the wave was owing chiefly to the depth of the fluid. After having propagated a given wave that had a velocity of 8 miles an hour, it was traced to a point at which the channel became deeper, and here its velocity was suddenly accelerated. The channel was also constructed as to become alternately narrower and wider, but no sensible effect was produced by the change; and when the wave once more reached that part of the channel which was of the original depth, its velocity returned to the original quantity.

Another observation equally simple served to shew that a large or high wave had a greater velocity than a small one. When a small wave preceded a large one, the latter invariably overtook the other, and when the large wave was before the less, their mutual distance invariably became greater.

In channels of rectangular section, the velocity was found by numerous experiments not to differ sensibly from that which is acquired by a heavy body in falling freely by gravity through a space equal to half the depth of the fluid.

In channels of variable depth in the transverse section, the velocity was found to be diminished below that which was due to the maximum depth, and to be equal to the mean of the velocities due to the differential depths.

The experiments on the magnitude of the wave shewed, that the velocity of larger, that is, of higher waves, appears to be greater than that of smaller ones, nearly in the ratio which is obtained by supposing the depth of the channel to be increased by a quantity equal to the height of the wave above the level of the surface of the quiescent fluid.

Experiments on the age and history of the wave, that is, upon the time which has elapsed, and the distance which has been travelled, and the route which has been described by it from the time and place of generation to the time and place of observation, shew that, after having traversed spaces from 100 to 2500 feet long in a sinuous channel, the wave remains unchanged in form and in velocity.

As the full investigation of the laws of the Genesis and Propagation of Waves forms a very extensive subject, in which I am at present engaged as a separate investigation, I have not loaded this paper with such observations as belong more properly to *that* subject. But I have given in this paper those examples which have peculiar reference to those experiments on resistance which I have now occasion to discuss in connection with the wave of the channels in which they were made.—(See Parts II. and III.)

The experiments on resistance were made in a channel 5.5 feet deep in the middle, but of irregularly diminishing depth towards the sides. The velocity of the wave in these experiments is about 8 miles an hour, being from 11 to 12 feet per second, varying with the height of each wave according to the law already given.

Very small waves, whose height does not exceed 0.1 of the depth of the quiescent fluid, are considerably retarded below the velocity due to the length, and move slower than the larger waves in a less depth.

The following extracts from the tables of a separate series of investigations directed exclusively to the examination of the laws of the wave, will serve to shew the degree of correspondence of the phenomena with the law already mentioned, and the connection between the velocity of the wave and the depth of the fluid, the fourth column being formed by adding to the first the mean of the second and third.

THE WAVE.

In a rectangular channel 13 inches wide, 75 feet long.

Depth of the Fluid at Rest in inches.	Heights of the Wave above the level of the Fluid in inches.		Total Depth reckoned from the top of the Wave.	Time for 70 feet. s.	Velocity in Feet per sec.
3.25	1.2	0.6	4.15	23.0	3.04
4.0	1.3	0.8	5.1	21.5	3.26
4.5	1.0	0.5	5.25	20.5	3.47
5.5	1.5	1.3	6.9	18.	3.9
6.25	2.5	1.5	8.25	16.5	4.49
6.25	3.5	2.5	9.25	15.5	4.52
9.0	2.3	1.0	10.65	14.5	4.82
9.0	3.0	2.5	11.75	14.0	5.00
9.0	3.5	2.3	11.90	13.5	5.19
9.5	1.0	0.6	10.3	14.5	4.82
9.5	2.5	1.2	11.3	14.0	5.00
13.0	1.0	0.5	13.75	14.0	5.00
13.0	2.0	1.1	14.55	13.0	5.38
13.0	3.0	1.4	15.2	12.0	5.83
37.0	9.0	5.0	44.0	*	10.598
66.0	4.0	4.0	70.0	†	14.087
66.0	6.0	6.0	71.0	†	14.284
66.0	9.0	9.0	75.0	†	14.727

SECTION IV.—*On the Form which is given to the Surface of a Fluid by the Motion of a Floating Body.*

It is only in a state of perfect rest that the surface of a limited reservoir of liquid can be considered as a horizontal plane. The displacement of any portion of that fluid deranges the equilibrium of all the particles in the vicinity of the disturbing cause, and it is only after the lapse of a considerable interval of time, and by means of an extensive series of interchanges of motion and position that the equilibrium is readjusted and the horizontal plane restored.

When a floating body is made to pass from one point in a fluid to another, it communicates motion to all the particles in the vicinity of its path. Such particles of the fluid as lie directly in that path are removed from it by immediate contact; these impart motion to those upon which they are protruded, and the

* This experiment was in a channel 12.3 feet wide.

† These three examples were in a channel 12.3 feet wide.

agitation is thus extended to particles remote from the body. In certain cases motion is thus communicated to particles before the body, so that when it reaches them it neither finds them in a state of rest nor terminated by a horizontal plane. This change of form must constitute an important element in the resistance experienced by the floating body.

The form which a fluid assumes when disturbed by a body moving with a velocity less than that of the wave, is very different from that which it takes when the velocity of the body is greater than that of the wave.

The phenomena attending velocities less than that of the wave, which are most general and important, are the *Great Anterior Wave of Displacement*, the *Posterior Wave of Replacement*, and the *Lateral Current.* The secondary wave of excessive displacement and the secondary wave of replacement, are phenomena of a peculiar and accidental nature, resulting from the form of the disturbing body.

The great anterior wave of displacement is produced by the translation of the fluid from the path of the solid—the mass of displaced fluid forms an elevation towards the anterior parts of the vessel, which is propagated continually forwards in the direction of the motion in the form of the wave, and with the velocity due to half the depth of the fluid. This anterior accumulation is constantly maintained by the continual displacement of the moving body, and forms a smooth well defined wave, extending many feet forward from the bow of the vessel, and across the whole width of the channel. The rounded summit of this wave is placed at low velocities considerably anterior to the stem of the vessel. At low velocities also the wave is small, but the wave increases with the increase of the velocity of the vessel, and at the same time the vessel is brought forward towards the highest part of the wave.

The lateral current of the fluid around the vessel from the stem towards the stern is a phenomenon that always accompanies the anterior wave. The elevation of the fluid anterior to the solid by its introduction into the space occupied by the anterior fluid, and the removal of the posterior part of the solid from the space previously occupied by it, form an elevation and depression, of which the inequality of the pressure determines a current with a given velocity in a direction opposite to that of the motion of the solid.

The great posterior wave of replacement is totally different in the nature of its generation and the law of its propagation from the anterior wave of displacement, and ought not in any way to be confounded with it. It is of the nature of an oscillatory wave, and frequently degenerates into a surge or breaking wave. It is formed in the following way: The motion of the solid having sent forward the particles of the fluid before it in the form of the anterior wave, there remains, when the posterior part of the body is withdrawn from a given part of the channel, a vacancy and corresponding depression of the surface of the fluid,—into this vacancy two currents are determined in opposite directions, the lateral current from the stem towards the stern sent backwards by the pressure of the anterior fluid, encounters near the stern a current in the opposite direction, sent forward by the pressure of that portion of the fluid behind the vessel which has regained its original altitude. The collision of these opposite currents generates around the point where they unite an accumulation of fluid, which performs a series of successive oscillations, until the equilibrium of the fluid under a horizontal plane is at last restored. At each velocity this posterior wave maintains a constant position in regard to the stern of the vessel. The velocity of this wave is equal to that of the vessel, but its position varies with the velocity, approaching nearer to the middle of the vessel at the slower velocities, and falling further behind as the velocity increases, so as to be frequently at a considerable distance behind the stern of the vessel.

While the velocity of the floating body continues to be small, the stem surge may be recognised in a gentle short undulation following in the wake of the vessel at the stern or near it, and followed at short intervals by a series of smaller waves of the same species. With an increase of velocity the crest of the surge rises in a sharper line, elevated to a greater height above the surrounding fluid, until it forms, at an increased distance, behind the stern, a high crested breaker, which foams and dashes along after the vessel with a loud roaring noise, tearing up the sides of the channel.

The form given to a fluid in which the velocity of the wave was found to be $8\frac{1}{4}$ miles nearly, is represented in the sections below, which represent the phenomena as observed at velocities of 4, 6, and $7\frac{3}{4}$ miles an hour, and compared with the fluid in a state of rest.

Fig. 3.
(at rest.)

Fig. 4.
(at 4 miles an hour.)

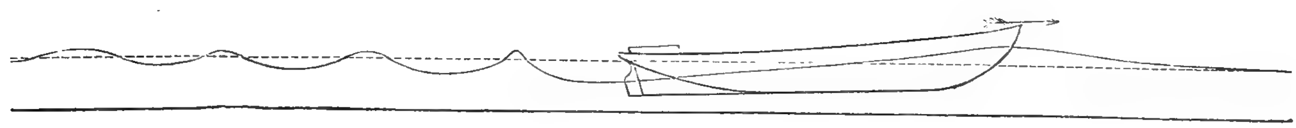


Fig. 5.
(at 6 miles an hour.)

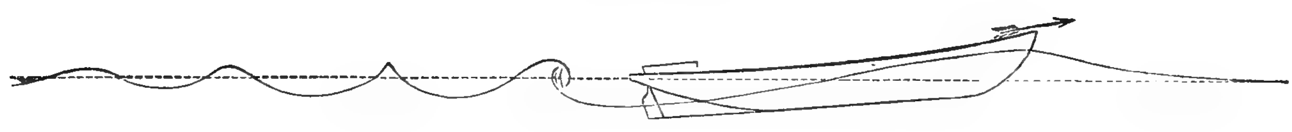
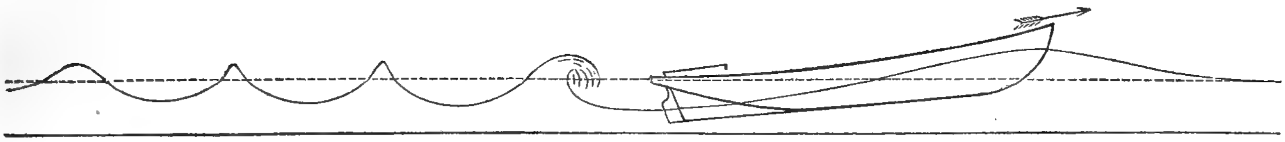


Fig. 6.
(at 7¾ miles an hour.)



The form given to the fluid in a channel about 8 feet deep, having a wave of the velocity of 10 to 11 miles an hour, is given in Plate II. fig. 1, the velocity of the vessel being about 7 miles an hour.

When the velocity of the solid is *greater than* the velocity of the wave of the fluid, the nature of the motion communicated to the fluid is totally different from that which is given to it by a lower velocity. The anterior wave no longer presses forward before the vessel, but the prow enters water that is smooth and undisturbed. The displaced fluid does not now accumulate at the prow, but is left on either side, forming a lateral elevation of fluid, which has the effect of increasing the depth of the fluid around the sides of the vessel, and forming a wave, on the summit of which the vessel may be poised in a position of equilibrium. This is in fact the wave formed by the displaced fluid, but moving with a less velocity than the vessel, and therefore posterior to the prow of the vessel instead of anterior to it, as in the former case, where the velocity of the vessel was less than that of the wave. It constitutes, therefore, a great central wave of displacement.

At velocities greater than that of the wave the stem surge has now disappeared. The wave of displaced fluid, instead of being sent forward, was to leave a vacancy in that part of the channel from which it was displaced, remains heaped up on the sides of the vessel until it has passed, and then collapses into the space which it had previously filled. The channel is therefore merely rendered fuller for the time being than it had formerly been.

Since, therefore, it appears that the form of the fluid is changed by the protrusion of a solid floating upon its surface, and is no longer bounded by horizontal plane; since, also, the form of the fluid is different when the velocity is less than that of the wave of the fluid, from its form when induced by a velocity in the solid greater than that of the wave; since, also, the mode of displacement and replacement are different, it may be expected that the law of resistance will exhibit a very important change at the point of transition. This will form the subject of the ensuing section.

Section V.—*On the Nature of the Increased Resistance experienced at Velocities less than that of the Wave.*

From the great change that is effected on the form of the fluid by the motion of a floating body with a velocity less than that of the wave, it is now very obvious that a vessel placed behind the wave, is in circumstances exceedingly different from the hypothetical condition of being drawn in a horizontal position along the surface of a level quiescent fluid. The prow of the vessel is pressed into the anterior wave, the stern is depressed into the hollow of the wave, the keel is inclined upwards in the direction of motion, at an angle amounting in some cases to 20°, an additional surface of horizontal displacement is presented, which increases as the sine of the angle of elevation of the keel. On attempting still farther to accelerate the velocity of the vessel in the vicinity of that of the wave, the variations which are thus produced in the condition of the vessel increase still further the causes of these variations, the increased immersion of the bow in the wave, augments the anterior wave formed by the displacement of the fluid, and the enlarged oblique surface now presented in the bottom of the vessel presses forward with increased velocity the wave on the slope of which it is elevated, and increases the elevation of that slope, becoming more depressed also at the stern, and giving rise to more rapid currents, and a higher stern surge. In short, it appears, that increased force applied gradually to the vessel for the purpose of rendering the velocity of the body equal to, or greater than that of the wave, has the effect at the same time of increasing at a more rapid rate the retarding forces, and a limit is soon reached, which it has in many cases been found impossible to pass. It is the circumstance of the very rapid increase of the resistance in approximating to the velocity of the wave, that has led to the false idea that there is a final and low limit to velocity on shallow water. There are circumstances in which this limit is final, the channel being very shallow, and the boat very bluff in its formation, I have seen in such an extreme case, when the depth of the channel was about five feet, the channel laid bare in the stern hollow behind the wave, so that the stern of the vessel no longer floated but rested on the bottom, while the bow was elevated and buried in a large anterior wave, rising more than two feet above the level, and overflowing the banks, and the posterior wave rushed on furiously behind, roaring and foaming, tearing up the banks of the channel, and threatening the destruction of the vessel, which, indeed, on stopping, it nearly accomplished. In such a case the persons in the vessel were not visible from the shore, being sunk in the hollow between the great anterior and posterior waves.

Any increase of velocity behind the wave is therefore accompanied by the following

Elements of Increased Resistance.

1. Increased Immersion of the bow in the anterior wave.
2. Inclination of the longitudinal axis of the floating body, so as to change the form of the displacing body.
3. Increased vertical section opposed to resistance $\doteqdot$ the sin of the inclination.
4. Increased velocity of the lateral current.

The following Table, extracted from the experiments of 1835, will serve to shew the rapid increase of resistance which is experienced in approaching the velocity of the wave, which in these cases was 8 miles an hour.

Example I.

Velocity in Miles.	Resistance in Pounds.
5.05	52.25
5.45	78.5
5.68	82.5
6.49	111.0
6.81	125.0
7.57	255.0
7.5 to 8	330.0
8 miles an hour = Wave's vel.	

Example II.

Velocity in Miles.	Resistance in Pounds.
5.05	95
5.45	100.5
6.19	152.0
6.49	312.0
6.81	386.0
6.81 to 7	392.0
8 miles an hour = vel. Wave.	

The following examples will shew the very slight increase of velocity in the vicinity of the wave, even when the increments of force are considerable. (See Experiments XLIII. and XXXIX.. 1835.)

Space. Feet.	Time. Secs.	Force. lb.	Space. Feet.	Time. Secs.	Force. lb.
100	10	124.7	100	9.5	172.2
100	10	127.5	100	9.25	200
100	10	150.5	100	9.25	212.2
100	10	157.5	100	9.0	227.7
100	10	197.7	100	9.0	239.7
100	10	207.0			

Velocity of the Wave being 100 feet in 8.5 seconds nearly:

Section VI.—*On the Nature of the Diminished Resistance which is experienced at Velocities greater than that of the Wave.*

Having now understood the manner in which a floating body moving behind the anterior wave deranges the equilibrium, and alters the form of the fluid, so as to cause a rapid accumulation of the elements of excessive resistance, it will be readily perceived that the annihilation of these elements, which takes place at velocities greater than that of the wave, will prevent the continued increase of the resistance derived from them, and it will also appear that the new arrangement of the particles of the displaced fluid renders the wave an element of diminished resistance. By how much, in fact, the wave was at a lower velocity, a + element of resistance will it now act as a — element of resistance.

Let it now be supposed that the vessel had created by its motion an anterior wave, and let it be supposed possible to lift the vessel entirely out of the water, and place its centre on the top of the wave, the stem being anterior to the wave, and the stern behind it, and suppose the vessel to be of such a form as to remain in a position of stable equilibrium on the surface of a fluid having the form of the wave, and suppose such a velocity to be given to the vessel as to keep it in the same relative position to the wave, then the following results would be obtained.

(1.) The vessel would be permitted to recover the horizontal position, and would present the minimum transverse section of resistance.

(2.) The immersion of the vessel being increased by the height of the crest of the wave around its centre of gravity, the anterior and stern displacements would be diminished, the total immersion being a constant quantity, by the amount of excessive central displacement.

(3.) The velocity of the vessel being now increased beyond that of the wave, the waves of displaced fluid falling continually behind the points where they were raised, would form a continued series of great central waves, bearing the vessel up upon their summit.

Such are precisely the circumstances of a vessel moving with a velocity greater than that of the wave, as shewn in section in the following illustration.

Fig. 7.—Behind the Wave.

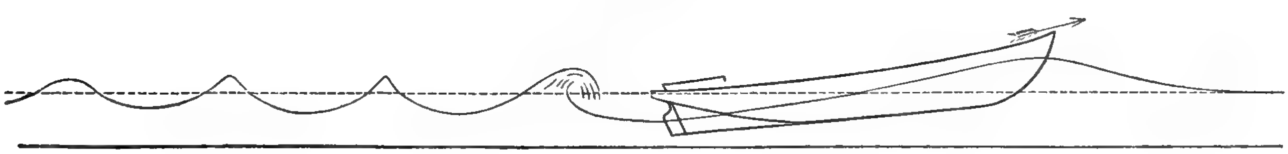
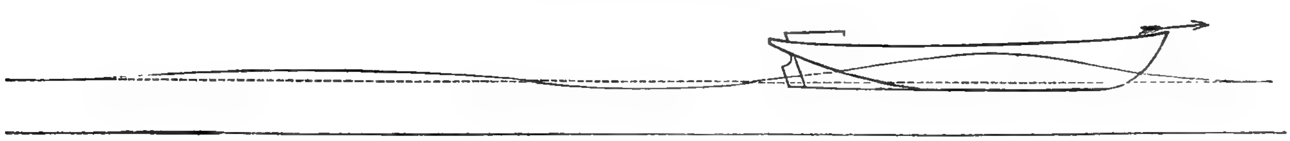


Fig. 8.—Upon the Wave.



But it will be inquired, how is a vessel to be placed in such circumstances? How is the extreme resistance of the anterior wave to be vanquished, and the vessel planted on its summit? This is admitted to be a practical problem, often of extreme difficulty; sometimes it is impracticable. There are some forms of vessel that do not admit of a position of stable equilibrium on the top of a wave. Still, however, it is a practical problem practically solved every day on all canals navigated on the Scotch system. Vessels of a greater length than the wave, having a fine entrance, built of light materials, and drawn by well trained highly bred horses, and guided by experienced postillions, are raised by a sudden and powerful jerk to the top of the wave (at from 6 to 8 miles an hour), and are drawn along on the summit of the wave with greater ease at 10 and 12 miles an hour, than at 6 or 7. (See Section IX.)

The progression of the resistance from 0, up to a velocity greater than that of the wave, follows therefore a very intelligible order. Suppose the velocity of the wave to be about 8 miles an hour, at the lower velocities of 2 and 3 miles an hour, the resistances bear to one another nearly the ratio of the squares of the velocities. But with the increase of velocity the excess of the resistance above that due to the velocity also increases, and nearly in the inverse ratio of the difference between the velocity of the vessel and the velocity of the wave, so that the ratio compounded of these two ratios accumulates very rapidly to a very high limit in the vicinity of the wave, which limit may in certain cases be infinity, but where it is not infinite, the resistance will suddenly diminish to a less quantity than the slower velocities under the wave, and will only increase in a ratio which will be less than that of the square of the velocity from two causes, from the diminished immersion due to the velocity (as in Sec. I.), and from the diminished anterior immersion explained in this section as the effect of the central wave; the resistance will then obtain a maximum and minimum as given in Sec. I.

The following experiment made with a simple dynamometer, giving only round numbers, will shew the manner in which horse-power may be exerted at velocities greater and less than the wave, and the exertion required to place the vessel on the wave. The velocity of the wave being 8 miles an hour, and the weight of the vessel and its load = 12,579 lbs. Two horses were used.

	Space. Feet.	Time. Secs.	Resistance. lb.	Velocity. In miles an hour.
Behind the Wave.	100	11.5	180	5.92
	200	11.0	200	6.19
	300	11.0	250	6.19
	400	10.0	300	6.81
	500	9.0	300	7.57
	600	9.0	350	7.57
	700	9.0	400	7.57
	800	9.0	500	7.57
Upon the Wave.	900	8.0	400	8.52
	1000	7.5	300	9.04
	1100	7.0	270	9.04
	1200	7.0	280	9.04
	1300	7.0	270	9.04
	1400	7.0	280	9.04
	1500	7.0	270	9.04

Although this experiment does not give accurate measures of force due to various velocities, it shews simply what was intended, the manner in which the force of horses is exerted to "overcome the wave" (as it is called). The following Table is made up of very correct experiments, continued through considerable spaces, upon the same basin of fluid, and with the vessel which is named the Raith in Part III., the weight of the vessel and load being $=10,239$ lbs. 17th October 1834.

		Space described.	Time.	Velocity in Feet per second.	Velocity in Miles per hour.	Moving Force.
		Feet.	Secs.			Lbs.
Behind the Wave,	Experiment I.	2640	387	6.8	4.72	112
	Experiment II.	2640	302.5	8.6	5.92	261
	Experiment III.	2640	295.5	8.9	6.19	275
Upon the Wave,	Experiment IV.	1000	74.0	13.5	9.04	250
	Experiment V.	1000	65.0	15.3	10.48	268.5

The resistance here is greater at 6 miles an hour behind the wave, than at 9 miles an hour upon it; and the resistance at $10\frac{2}{5}$ miles, is little more than at $5\frac{9}{10}$ miles an hour.

It is easy to see how the wave influences the resistance in the cases where the vessel has been raised upon it, and is drawn along at precisely the same velocity; but it is perhaps not quite so clear at first sight what are the phenomena which accompany velocities that are greater than that due to the wave, because, in that case, the vessel would leave the wave behind. But it should be observed, that a new wave is formed at every successive instant by the motion of the vessel through the water, whatever be the velocity of its motion; for the displaced fluid thrown aside at the bow, generates a series of waves, which move with a less velocity than the vessel, and fall back to a position behind the bow. The displaced fluid, which, in the case of motion with a less velocity than that of the wave, passed forward before the vessel, causing an extensive accumulation, cannot now pass forward with a velocity greater than that due to the depth and to the wave, and is therefore left behind, to fill up the vacuity which will remain when the stern of the vessel shall have passed on. The displaced fluid is therefore pushed aside by the bow of the vessel, and forms lateral accumulations on both sides of it, in the form of a continuous wave, upon the ridge of which the centre of the vessel is sustained in a position of station of stable equilibrium. The buoyant force of this ridge is the cause of the diminished anterior section of resistance.

It is always found that the commotion produced in the fluid is much greater at velocities less than the wave, than at velocities which are greater than it. The stem of the vessel, in the latter case, enters water which is perfectly smooth and undisturbed, because no wave has previously passed forward before the vessel to produce any anterior derangement; the water which is pushed aside by the bow of the vessel, forms a lateral accumulation proportioned to the increase of volume arising from the sudden entrance of the solid; and when the vessel has passed forward, the subsequent collapse of the lateral ridge restores the equilibrium. The disturbance of an anterior wave is thus rendered impossible, and the cause of the destructive stern surge is removed; for the displaced water remains to fill up that vacuity into which a stern surge would otherwise have been driven.

It is evident, therefore, that the nature of the motions communicated to a fluid at velocities greater than the velocity of the wave, are radically different in their nature from those of the less velocities. Lateral currents, breaking surges, can no longer exist. The fluid is simply divided by the entrance of the vessel, stands aside until it has passed, and gently subsides to the original level when the separating body has passed away.

The practical applications of these facts and phenomena are of great value in the navigation of canals and shallow rivers. (See Sec. VIII and IX.)

Tables of resistance at various velocities are given from seventeen different forms of the immersed portion of the floating body, at velocities from 3 to 15 miles an hour, in Parts II. and III.

Section VII.—*On a General Expression of the Law of Resistance of a given Solid in a given Limited Fluid.*

If the immersion of a floating body were like that of a solid wholly immersed in the fluid, a constant quantity, and if the surface of the fluid remained horizontal and plane, and if the particles of the fluid remained at rest until immediately acted on by the solid, and were the motion given by the solid to the displaced fluid horizontal without vertical excursions, then the usual simple expression,

$$R = \frac{v^2}{2g} \cdot m s \rho \quad . \quad . \quad . \quad . \quad (1.)$$

would represent the resistance R, being the weight of a column of fluid having the height due to the velocity v, g being the measure of gravity, s the anterior transverse section of the immersed part of the solid when at rest, m being a constant representing the modified resistance derived from the form of the anterior part of the solid, and ρ the density of the fluid, to which might also have been added a constant quantity for adhesion, but we have omitted it, for the sake of simplifying the expression.

If, however, we include the element of diminished immersion, we shall have by (Sec. I.)

$$R' = \frac{v^2}{2g} \cdot \rho \,.\, m\, s \left(1 - \frac{v}{2g}\right) \quad . \quad . \quad . \quad . \quad (2.)$$

If we now include the element of change in the position of the body, and of increased anterior immersion when behind the wave, we shall have, by using θ for the angle of elevation of the axis of the solid, δ for the difference of the anterior section or height of the wave forming on the solid, modified by the constant n, for the form of that part of the solid, and measured at a given unit of velocity in relation w the velocity due to the wave, we shall have

$$R'' = \frac{v^2}{2g} \cdot \rho \left\{ ms \left(1 - \frac{v}{2g}\right) \cdot (1 + \sin\theta) + \frac{n\delta v}{w - v} \right\} \quad . \quad . \quad . \quad (3.)$$

When the velocity is less than that of the wave, the quantity $\frac{n\delta v}{w-v}$ is positive, w being greater than v, and the effect of the wave is then to increase the resistance; as v increases, $w - v$ diminishes, still remaining positive. If the sides of the channel and of the vessel were infinitely high, and the increments of force uniform and very slow, the phenomena would give the case represented, when

$$\frac{n\delta v}{w - v} = \frac{n\delta v}{0} = \infty\,,$$

the resistance being infinitely great; and when the velocity v becomes greater than that of the wave w, $\sin\theta$ being $= 0$, the expression $\frac{n\delta v}{w - v}$ becomes negative, its denominator having become negative, and the expression is reduced to

$$R''' = \frac{v^2}{2g}\, \rho \cdot \left\{ ms \left(1 - \frac{v}{2g}\right) - \frac{n\delta v}{v - w} \right\} \quad . \quad . \quad . \quad (4.)$$

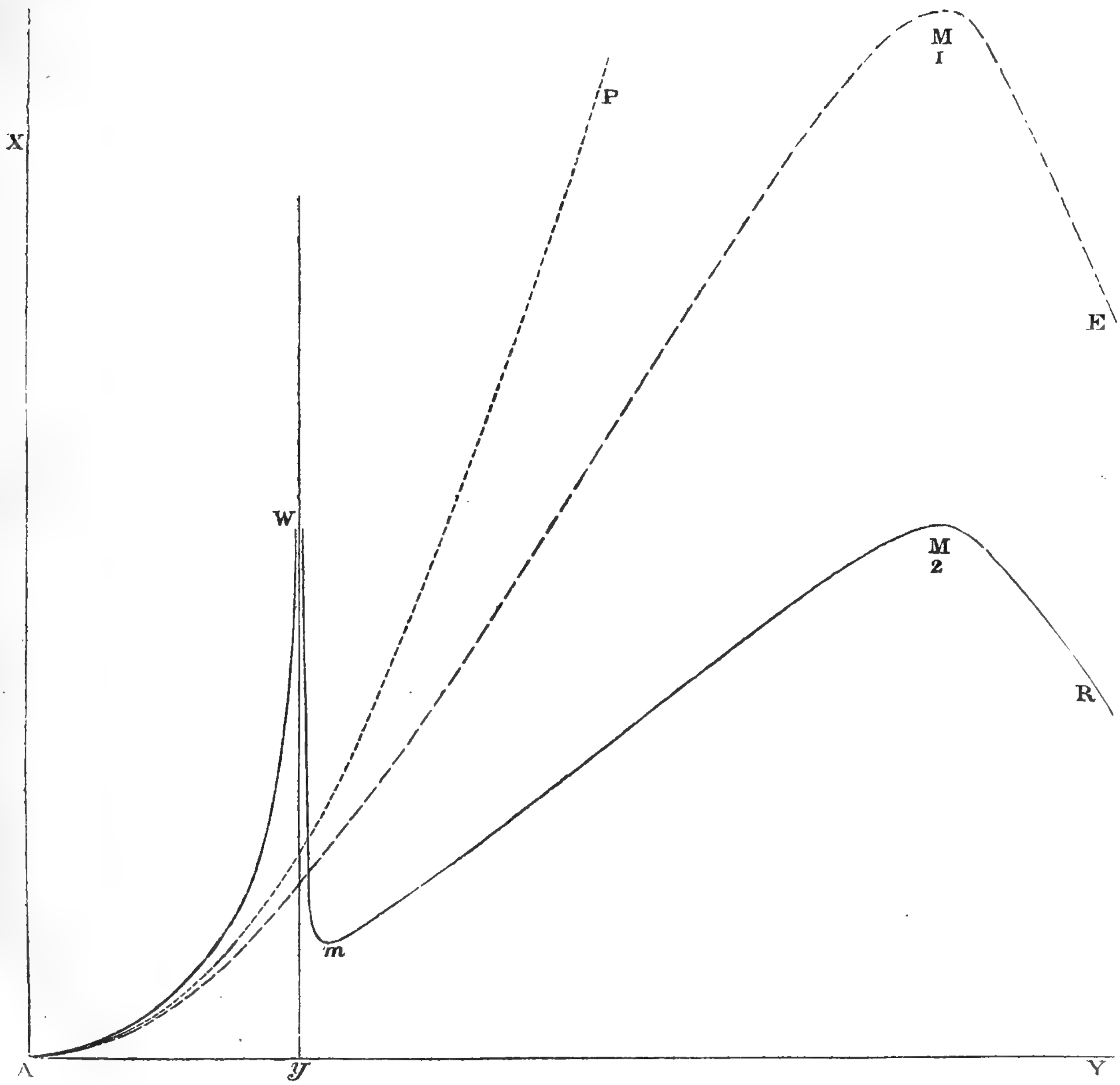
the expression of the case when the velocity is greater than that of the wave.

The line of resistance AP corresponding to Eq. (1.), is a parabola, AX being the axis of the parabola, and AY the tangent of the vertex, the velocities being represented by the ordinates parallel to AY, and the resistances being represented by the abscissæ parallel to AX; A being the origin.—(See Fig. 9.)

The line of resistance AM_2E corresponding to Eq. (2.), has all its abscissæ less than those of the former curve, and a point of maximum when $v = \frac{4}{3}g$, and of minimum when $v = 2g$.—(See Fig. 9.)

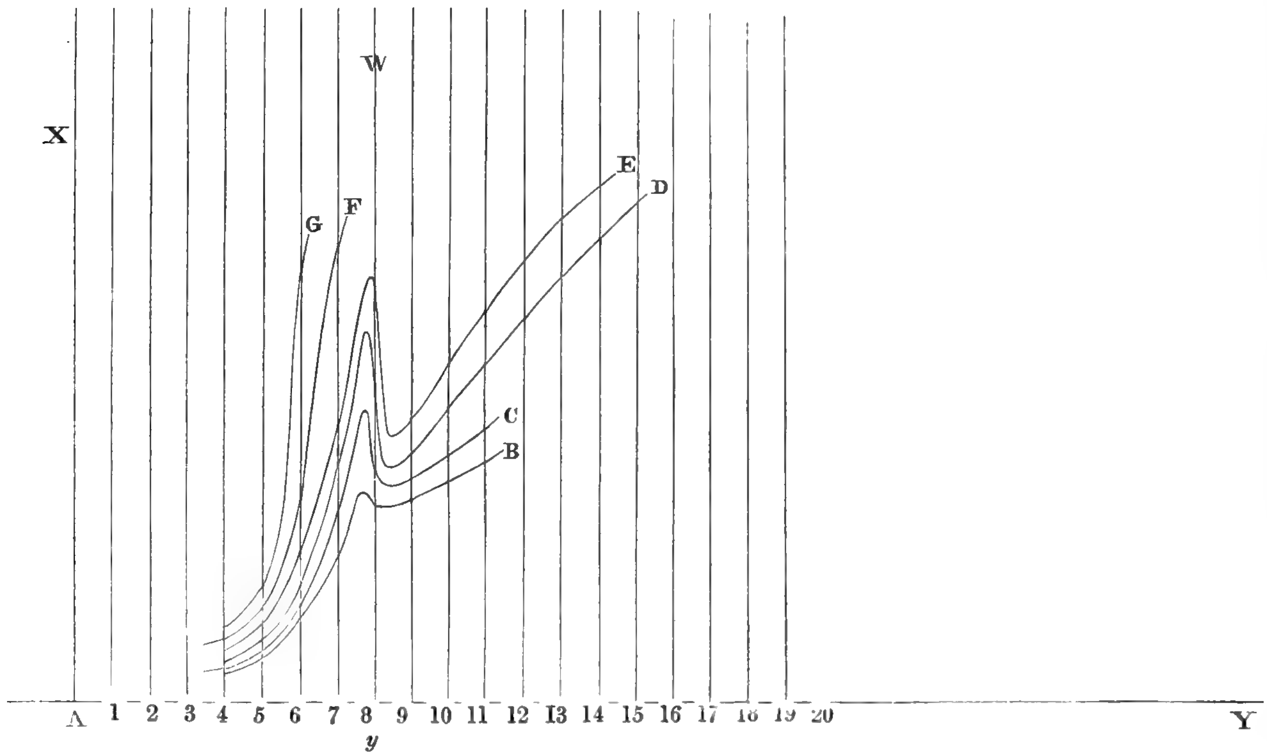
The line of resistance ($\mathrm{A\,W}m\mathrm{M_2R}$) corresponding to Eq. (3.), lies above the parabola, when $\frac{n\,\delta v}{w - v}$ is greater than $ms\left(\frac{v}{2g} - \sin\theta\right)$, becomes infinite when $v = w$ and falls below the parabola, when the velocity becomes greater than that of the wave, or when $v < w$.—(See Fig. 9.)

Fig. 9.



The lines of resistance derived from the experiments in Part III. are given in Fig. 10;—velocities being measured on AY, and resistances being taken parallel to AX, the lines of resistance AB, AC, AD, AE, AF, and AG, being formed from the Analysis of the Experiments of 1835, in pp. 100 and 101.

Fig. 10.



Section VIII.—*Practical Applications and Illustrations of the Law of the Wave in the Navigation of Rivers and Shallow Water.*

Experienced river boatmen are well aware of a fact which receives its explanation from Sect. III., that every large vessel moving with considerable velocity sends forward through the water an intimation of its approach while it is yet at a considerable distance, so much as several miles. This intimation consists of a wave or series of waves, propagated in the fluid, with a greater velocity than that of the vessel. A vessel that has ceased its motion, or suddenly varied its velocity, will send forward a very large and defined wave with the velocity due to the depth of the canal, and quite independent of its own velocity. These waves arrive at the place to which the vessel is moving long before she reaches it, and sensibly increase the depth of the water in the channel. I have in this way observed on the River Clyde the approach of a large steam-vessel, while yet at the distance of 2½ miles, the motion of the wave finding a beautiful index in the oscillations of the tall masts of vessels at anchor as it passed them in successive tiers. I have frequently been surprised by the appearance of such an indication, when I had no reason to suspect the approach of a vessel, and have invariably found it followed by the unexpected vessel. A distant storm in the ocean frequently gives a similar announcement, the waves moving with a velocity of 50 or 60 miles an hour, breaking in a heavy ground-swell upon a remote beach.

The effect of the formation of waves with a greater velocity than that of the vessel in forming an anterior accumulation, a posterior depression, and a stern surge, as shewn in Sects. IV.—VI., furnishes a satisfactory explanation of the phenomena of shallow navigation.

It is well known that it is extremely difficult to row or sail well in shallow water; this is the consequence of increased section of resistance from being behind the wave. But if by strong impulse the vessel were placed on the wave, it would then become easier than in deeper water at the same velocity. In water two feet deep, it is very difficult to row 4 and 5 miles an hour, and comparatively easy at six, the one less, the other greater, than the velocity of the wave. It is also found that in shallow water the stern of the vessel invariably takes the ground while the bow remains free, although drawing at least equal depth of water; this is the direct result of the depression between the anterior wave and the stern surge, as in Sects. IV. & V.

The difference of the immersion of a vessel below the surface of the fluid when on the summit of its wave, or in the depression behind it, Sect. IV., accounts satisfactorily for a long list of phenomena otherwise inexplicable. It has long been observed, that a vessel in motion will take the ground in water that is more than sufficient to float her when at rest, and it is equally well known that there are circumstances in which a vessel when in motion, will pass over a shallow without touching the ground, while it is not covered with the depth of water necessary for her statical floatation. Now, it is obvious, in the first instance, that if the vessel move with a velocity less than that of the wave, the prow being on the anterior wave, and the stern in the succeeding depression, the vessel will take the ground, and probably carry away her helm; while, if the vessel were poised on the summit of the wave in a horizontal position of equilibrium, and with the diminished immersion due to the velocity, much less depth of water would be sufficient, than with the slower motion, or even than is required to float the vessel in a state of rest. I have seen a vessel in five feet water, and drawing only two feet, take the ground in the hollow of a wave, having a velocity of about 8 miles an hour, whereas at 9 miles an hour, the keel was not within four feet of the bottom.

A highly scientific friend of mine, Mr Smith of Philadelphia, member of the Franklin Institute, has frequently observed in the Dutch canals, boats carrying passengers, kept in floatation by communicating to them rapid motion in shallow parts of canals, where they would otherwise have taken the ground, thus taking the advantage of moving on the summit of the wave.

I have also been informed, on the best authority, of the following fact, observed by a gentleman, who was surprised by the phenomenon, although unable to account for it, " The steam-boat Trenton, on the Delaware, in the United States, by passing over shallow portions with a high velocity, carries with it a body of water sufficient to float her over portions on which she would not have been floated if at rest." The body of water was of course the wave, the velocity of the vessel being above 13 miles an hour.

The navigation of the River Clyde presents excellent examples of the effect of the motion of the wave, and affords ample opportunity for the application of the principles developed in the preceding portions of the paper.

The wealth of an enterprising commercial community has enabled engineers to convert one of the worst rivers for navigation into a good, although as yet only shallow, channel. When the tide leaves the river, there are not more than six or seven feet of water in many parts of the channel, the wave having a velocity of about 9 miles an hour. Any observer looking attentively from the deck of a vessel on the sloping bank of the river, will see delineated there very distinctly the anterior wave preceding the bow of the vessel, the posterior depression, called by sailors " the suction of the vessel," and the stern surge, as delineated in Plate II. Fig.1, rushing along the bank with fury into the vacuity. It is invariably necessary for vessels of considerable size to lower their velocity very much in such places, to prevent their grounding in the depression of the wave. When two vessels pass each other this effect is much more sensible, as at the instant when the waves coincide, their elevation is equal to the sum of both, and when the depressions coincide, the hollow is equal to the sum of both; hence, although both may have floated previously, at the instant of passing either may take the ground, or both. It has on this account been found necessary to enact, that at low-water vessels before passing each other shall diminish their velocity. But if a velocity greater than the wave, such as 11 or 12 miles an hour, could be attained, there would no longer be any danger of grounding, or of lessening the speed, and thus the navigation be materially improved. Another very curious fact I have also observed on the Clyde, namely, that a vessel of greater power and velocity passing one of less power and velocity, will take the less powerful along with her in the depression, the more rapid sending forward the wave before the bow of the slower vessel, so as to obviate the immersion of the bow, and give her the impetus of the stern surge. It is further evident, that in a river so shallow as the Clyde, velocities of 7, 8, or 9 miles an hour behind the wave are very disadvantageous; whereas, if a vessel could be started over the wave, her progress would be so greatly facilitated, as to enable her with the same force to reach velocities of 12 or 13 miles an hour, when the stern surge would cease, and the danger of taking the ground cease along with it.

In shallow rivers, where the water is in motion, very singular phenomena result from the wave, from which it will be apparent, that a given velocity against the stream may, in certain circumstances, require less force to produce it, than the same velocity in the direction of the stream. Thus I have seen the current moved at the rate of about 1 mile an hour, and the wave about 4 miles an hour, on the surface of the water; when the velocity of the vessel, drawn against the stream, was 4 miles an hour in regard to the land, it was before the wave with diminished resistance, and when it was drawn with the stream also at the rate of 4 miles an hour, the vessel being then behind the wave, experienced the direct resistance arising from that cause, the velocity of the wave in regard to the land being in the one case 3, and in the other 5 miles an hour. Analogous phenomena to this, of a very curious nature, are to be recognised in the motion of a wave against a current. I have seen a wave move in the opposite direction to a stream, until it reached a rapid in which there existed a part of the stream where the current had a velocity equal to that of the wave, and in the opposite direction, and there, in consequence of the equality of velocities in opposite directions, I have seen the wave come to rest, and retaining its form unchanged, remain as a stationary heap of fluid, until, by the adhesion of the successive portions of water, it was at last rendered insensible. From these remarks it will be apparent, that the navigation of rivers may, in certain cases, be much facilitated by the action of the wave.

SECTION IX.—*Applications and Illustrations of the Law of the Wave in the Practical Navigation of Canals.*

Canal navigation furnishes at once the most interesting illustrations of the interference of the wave, and most important opportunities for the application of its principles to an improved system of practice.

It is to the diminished anterior section of displacement, produced by raising a vessel with a sudden impulse to the summit of the progressive wave, that a very great improvement recently introduced into Canal transports owes its existence. As far as I am able to learn, the isolated fact was discovered accidentally on the Glasgow and Ardrossan Canal of small dimensions. A spirited horse in the boat of WILLIAM HOUSTON, Esq., one of the proprietors of the works, took fright and ran off, dragging the boat with it, and it was then observed, to Mr HOUSTON's astonishment, that the foaming stern surge which used to devastate the banks had ceased, and the vessel was carried on through water comparatively smooth, with a resistance very greatly diminished. Mr HOUSTON had the tact to perceive the mercantile value of this fact to the Canal Company with which he was connected, and devoted himself to introducing on that canal vessels moving with this high velocity. The result of this improvement was so valuable in a mercantile point of view, as to bring, from the conveyance of passengers at a high velocity, a large increase of revenue to the Canal Proprietors. The passengers and luggage are conveyed in light boats, about sixty feet long, and 6 feet wide, made of thin sheet-iron, and drawn by a pair of horses. The boat starts at a slow velocity behind the wave, and at a given signal it is by a sudden jerk of the horses drawn up on the top of the wave, where it moves with diminished resistance, at the rate of 7, 8, or 9 miles an hour.

It was a natural consequence of this successful mode of transport on this one canal, that it should be immediately attempted on others, and numerous experiments were accordingly made with varying results. In some canals, and with certain vessels, similar phenomena were observed, and the like favourable results obtained. But in others the experiment totally failed, as it was not found that the tumult of the water subsided as in former cases, or that the resistance experienced any similar diminution. The cause of these variations was not then known. Many experiments were made, which failed in eliciting any solution of the difficulty. Many scientific and practical men, unable to account for such discrepancies, satisfied themselves with an unqualified denial of their existence, while those who were eye-witnesses of the fact could not assign any satisfactory cause.

It will not be difficult for us to account for these discrepancies, by what we have brought to light regarding the law of wave. In the canal where the fact was originally observed, having a depth of 3 or 4 feet, and a wave moving at about 6 miles an hour, it is obvious that the resistance of the anterior wave would only be encountered at velocities less than that of the wave, and the diminished resistance would be obtained by moving upon the wave, at a velocity of more than 6 miles an hour. Now, in making the same attempt in canals that were 5 or 6 feet deep, with a wave moving at the rate of eight miles an hour, the resistance would not be observed to suffer any diminution, till the velocity exceeded that of the wave; but would accumulate rapidly up to that point. While in canals that had a depth of 8 or 9 feet, and a wave moving at eleven miles an hour, no diminution could be observed till a velocity above that of the wave had been obtained, after which, the advantage of diminished anterior section could be acquired. Since the discovery of the law of the wave, I have had the experiments tried in such cases, the wave being passed, and the boat carried along on its summit at the rate of thirteen miles an hour.

When once the summit of the wave is attained, or its velocity exceeded, a comparatively small force may sustain the motion. But the resistances increase so rapidly in the vicinity of the wave, that this may become impracticable. If the increase of the velocity up to that of the wave be very slow and continuous, the waves will be closely crowded, and deeply accumulated around the bow of the vessel, so that an additional force will only increase the magnitude of the wave, and thus adding to its velocity, prevent the vessel from penetrating through or rising upon it. What I have stated accords perfectly with the experiments I have given, and with the experience of practical men. In these experiments it will be seen, that immediately behind the wave large increments of force are not accompanied with similar increase of velocity, while at the instant of passing the wave, the velocity makes with a given force a sudden transition to a higher velocity. And so in experience it is found very difficult, or quite impracticable, to pass the wave with a slowly accelerating motion. A sudden impulse from a low to a high velocity is found to be the easiest mode of effecting the change, and the method used is not to make the change immediately from a very high velocity behind the wave, to a very high velocity before it; but when it is intended to start a vessel over the wave, the speed must first be allowed to diminish, until it become nearly half of that of the wave, by which means the anterior wave is allowed to pass away with its proper velocity from before the bow of the boat, the stern surge is permitted to overtake it, and fill up the cavity behind the wave, and the surface of the water is reduced more nearly to a plane; and if now, in these circumstances, a sudden impulse be communicated to the vessel, it will easily attain a velocity greater than that of the wave.

A change in the depth of a canal produces a very marked change in the resistance in the vicinity of the wave. Certain portions of the Glasgow and Ardrossan Canal have their depth suddenly increased, and when a vessel that has been moving on the summit of the wave reaches these points, it finds its velocity diminished, in consequence of the wave having acquired a greater rapidity due to the increased depth.

The wind acting on the surface of a long canal has a force sufficient to send away so much of the fluid from one of its extremities, and accumulate it towards the other, that in a canal running about twenty-five miles in a direction east and west, a strong westerly wind will occasion a difference in depth of two feet, being at the east end one foot more, and at the west end one foot less than five feet, the average depth of the canal. It is observed in this case, that to maintain the vessel over the wave, a greater force is required at the deeper end, and a lessened force at the other.

In canals where the power of horses is applied to vessels navigating at high velocities, much inconvenience will be experienced, and much loss incurred, by giving to the water a depth which will produce a wave of so high a velocity, as to approach the limit of the available speed of horses. When the depth exceeds seven or eight feet, the struggle to conquer the wave will take place at or above nine miles an hour, being a velocity at which horses cannot advantageously exert much force above what is required for the transport of their own bodies;

and in such a case, in order to prevent any irregularity in the application of the force from permitting a wave to pass on before the vessel, the velocity will require to be maintained at twelve or thirteen miles an hour. Now, when the depth is so much less as to comprise the velocity of the wave within the limits of moderate exertion on the part of the horse, the higher velocities are gained without injury to the animal, and a rate of nine or ten miles an hour is maintained with certainty.

Two or three years ago, it happened that a large canal in England was closed against general trade by want of water, drought having reduced the depth from twelve to five feet. It was now found that the motion of the light boats was rendered more easy than before; the cause is obvious. The velocity of the wave was so much reduced by the diminished depth, that instead of remaining behind the wave, the vessels rode on its summit. I am also informed by Mr Smith of Philadelphia, that he remembers the circumstance of having travelled on the Pennsylvanian canal in 1833, when one of the levels was not fully supplied with water, the works having been recently executed, and not being yet perfectly finished. This canal was intended for five feet of water, but near Silversford the depth did not exceed two feet, and Mr Smith distinctly recollects having observed to his astonishment, that, on entering this portion, the vessel ceased to ground at the stern, and was drawn along with much greater apparent ease than on the deeper portions of the canal.

In a canal where the velocity due to the wave is nearly coincident with that velocity of transport which is found to be most desirable for the species of traffic, (for example, ten or eleven miles an hour, as has been the case recently on the Forth and Clyde Canal, whose maximum depth is about nine feet), in such a case this velocity is either impracticable or very disadvantageous, giving rise to a constant struggle with the wave. To solve the problem, however, the following mode has been found efficient: one mile is performed at the rate of eight miles an hour, being so far behind the wave as to suffer little from its accumulation on the prow, and at the end of that mile the boat is brought to the bank where the canal is shallow, and by starting the horses to a gallop of 13 or 14 miles an hour for another mile, being in advance of the wave, and this process being continued in alternate miles, a mean velocity of ten and a-half or eleven miles is attained in the transport, at a resistance whose mean is less than the resistance of the mean of the two velocities intermediate.

In every canal there must be two velocities, at which principally the transport is conducted, one sufficiently far behind the wave to render its interference inconsiderable, and another sufficiently in advance to give security against its passing in small changes of moving power; at a velocity one-half of that of the wave, and at another one-fourth part greater than the said velocity, both of these objects will be attained.

When a canal is to be constructed for a given kind of transport, such a depth ought to be selected as will admit of those velocities above and below the wave, which are required for the trade of the canal, the velocity of the wave being as far removed as possible from the velocities below it and above it.

When vessels only of a small draught of water are required for the trade, the canal should be as shallow as possible, and when larger vessels are desirable, the depth should be increased as much as possible, so as to remove the wave to a distance beyond the velocity of the motion of the vessels, and prevent anterior accumulation.

The breadth of the canal materially affects the resistance produced by the wave, although it does not directly affect its velocity. By preventing the diffusion of the wave, the narrowness of the canal increases the height of it, in consequence of which the resistance to the lower velocities is augmented, and facilitation in the higher velocities increased. But in general the depth is of much more consequence than the breadth of the canal, as the retardation or facilitation produced by the vicinity of the wave, is a quantity which may be made to bear an almost infinite ratio to the other elements of resistance.

For slow velocities alone, a broad and deep canal, but especially deep, should be made; and for high velocities, a narrow and shallow one, especially shallow, that the range of velocities may be extensive, and the velocity at which the wave is to overcome small.

There are also certain relations to be observed between the velocity of the wave, and the dimensions of the vessels of easiest transport, also between the form of the vessel and that of the wave; but this is an inquiry which I have not yet completed, but hope soon to terminate successfully. Relations have been distinctly indicated, but not accurately defined.

It is perhaps worthy of remark, that a vessel on the summit of the wave is more easily directed by the helm, than when behind it. In the latter case, the vessel by her anterior immersion is prevented from answering the helm, while in the former case, this obstruction being diminished, and the displaced fluid collected around the centre of gravity, horizontal rotation on the vertical axis passing through the centre of gravity is less resisted by the fluid than formerly, in proportion as the third power of their present distance from the particles of the wave is less than the third power of their former distance from the centre of rotation.

Another circumstance still more curious than the foregoing is, that at the instant of passing one another at high velocities, vessels are much more delicately poised than at any other time; the waves coinciding form a wave equal to their sum, on which the centres of gravity receive an additional elevation.

It appears from the experiments of 1835, that a vessel has conveyed on a canal given weights with the following forces:—

Moving Force.	Weight Moved.	Velocity.
71.5 lbs.	19,222 lbs.	4 miles an hour.
86	19,222	4.5
112.7	19,222	5.2
243	8,022	11.3
264	19,262	13.6
331	10,262	15.1

The examples are taken from the experiments made with the vessel named "The Wave," which was constructed according to the form which I have assigned as the solid of least resistance.

PART II.

The Experiments of 1834.

The experiments of 1834 were directed chiefly to the determination of the velocity of the wave, the emersion due to the velocity, and the amount of animal force required to overcome the resistance of the water at various velocities. The experiments of 1835 were the result of the experience of 1834, in consequence of which a vessel of a peculiar form had been constructed, and a mode of estimating the absolute and comparative resistance of the fluid at various velocities, with different vessels, and at several degrees of immersion, had been obtained, giving results more accurate, more uniform, and more worthy of confidence than those of the former year.

On the Velocity of the Wave.—The Progressive Wave, which forms the subject of these experiments, differs entirely in its nature and laws from the small undulations or oscillations of a fluid which are occasioned by the sudden elevation or depression of a small portion of the fluid, in which case we have a series of successive small undulations and depressions succeeding each other at nearly equal intervals. The progressive wave, sent forward by a floating body in rapid motion, is not necessarily preceded nor followed either by a depression, or an elevation, or any series of such depressions or elevations, but is a single elevation, of a well defined form, moving with an uniform velocity along the surface of the fluid; the forms of the fluid vary, but maintain an obvious relation to one another; they are of the same family of waves, or may be resolved into compounds of the members of the same family. A few of those that have been carefully and frequently observed, are given in Plate I. figs. 2, 3, 3 and 4.

The three first examples appear to be simple examples of the trochoid, a curve that appears to comprehend all the elementary forms of the wave. Other forms which make their appearance, seem to be compounds of these, into which they may be resolved by a very simple analysis, as is done in the succeeding figures. When one portion of such a compound wave is higher than another, I have invariably observed the higher portion move more rapidly than the rest, and finally separate itself, leaving the rest behind, and assuming a definite elementary form. Figs. 5, 6, and 7, shew the outline and analysis of some compound waves, which afterwards resolved themselves into simple ones of the forms given in Figs. 2, 3, or 4.

The first series of experiments on the wave, were directed to the determination of the relation between its velocity and the form and dimensions of the channel. A sheltered situation and calm day were selected, so that the form of the waves might be sufficiently perfect to enable the observers to mark with precision the place of the summit of the wave. At the termination and the commencement of distances that had been accurately measured, graduated rods were placed in a vertical position, and careful observers, furnished with assistants and accurate chronometers, were stationed opposite to each of them. A wave was generated by giving rapid motion to a vessel, and then depriving it of motion at a given distance from station A; and at the instant of the coincidence of the summit of the wave with the rod at A, a signal was communicated by sound to station B, the time of the transit being recorded at A, and the time of the sound at B. The wave now passed on towards B, and at the instant of its arrival time was observed at B, and the time of the signal of arrival communicated to A was also registered by chronometer A. Thus, without calculating the velocity of sound or δ, the time of describing the space, or s, was determined; for

$$s - \delta = \text{difference of times at B.}$$

$$s + \delta = \text{difference of times at A.}$$

$$\therefore \frac{s + s + \delta - \delta}{2.} = \text{true time corrected for the velocity of sound.}$$

The following observations were made at the experimental station at Hermiston, where also almost all the experiments on resistance were subsequently carried on.

Experimental Station.—Union Canal.

Breadth at top	= 40 feet,	See Fig. E, Pl. II.
Breadth at bottom	= 30 feet,	
Maximum depth	= 5.5 feet,	
	Clayey bottom.	

Exper.		Secs.	
1.	Space = 1000 feet,	85	Vel. = 11.730; 7.8473 miles.
2.		85	
3.		85	
4.		86	
5.	Space = 700 feet,	61.5	Vel. = 11.352; 78473 miles.
6.		62	
7.		61.5	
8.		61.5	
9.		61.5	
10.		62	
11.	Space = 800 feet,	63	Vel. = 11.7713; 7.8359 miles.
12.		69	
13.		68	

Paisley and Ardrossan Canal.—Dumbreck Bridge.

Breadth at top	= 28.27 feet,	See Fig. D, Pl. II.
Breadth at bottom	= irregular,	
Mean depth	= 3.3 feet,	
	Muddy bottom.	

Exper.		Secs.	
14.	Space = 556 feet,	61	Ve = 9.114; 6.0972 miles.
15.		61	
16.	Space = 820 feet,	90	Vel. = 9.111; 6.0952 miles.

Slateford Aqueduct.—Union Canal.

Breadth at top	= 12.33 feet,	See Fig. A, Pl. II.
Breadth at bottom	= 12.0 ...	
Maximum depth	= 5.6 ...	
	Smooth iron bottom.	

Exper.		Secs.	
17.	Space = 486 feet,	34	Vel. = 14.352; 7.5944 miles.
18.		34	
19.		33	
20.		33	
21.		35	
22.		34	
23.		36	

Same station.

Depth diminished until = 3.4 feet.

Exper.		Secs.	
24.	Space = 150 feet,	14	Vel. = 10.593; 7.0867 miles.
25.		14	
26.		15	
27.		14	
28.		14	
29.		14	
30.		14	
31.		14	
32.		14.5	

This experiment was made under the superintendence of Mr Ellis.

Glasgow and Ardrossan Canal.—Port Eglinton.

Breadth variable, with vertical sides.
Depth 5.5 feet.

Exper.		Secs.	
33.	Space = 501 feet,	40	Vel. = 17.431; 8.3163 miles.
34.		41	
35.		40	

Union Canal.—Tunnel.

Breadth at top	= 17.75,	See Figs. B & C, Pl. II.
Breadth at bottom	= 11.00,	
Depth	= 5.5 nearly	

Rocky bottom, irregular.

Exper.		m s	
36.	Space = 2038 feet,	2 35	Vel. = 13.208; 8.8361 miles.
37.		2 35	
38.		2 35	
39.		2 35	
40.		2 33	
41.		2 33	

On the velocity of waves in regard to their height above the surface of the water, the following experiment was made.

Experimental Station.—Hermiston.

Exper		Height.	Secs.
42.	Space = 700 feet,	6 in.	61.5
43.		5 ...	61.75
44.		3.5 ...	62.5
45.		2 ...	63.5

The following series of experiments were made with the view of determining whether the velocity of the wave remained unchanged during the whole of its progress, or varied with the distance over which it had travelled. I may observe as a matter of some interest, that when the wave had to traverse 1000 feet before arriving at the first station of observation it had to encounter a change in the direction of the canal, equal to a curved deflexion of 90°; and where it passed over

2500 feet, it had been deflected through double that quantity. The spaces marked as the distances of generation are exclusive of the distance between the stations A and B = 700 feet.

Experimental Station.—Hermiston,

Space traversed by the wave from A to B = 700 feet.

Exper.		Height A. Inches.	Height B. Inches.	Time. Secs.	
46.	Wave generated close to A.	7	5	61.5	Mean Velocity = 11.359 feet per second = 7.59315 miles.
47.		6	5	61.5	
48.		6	5	61.5	
49.		5	5	62	
50.	Wave generated 500 feet from A,	6	4.5	62	Mean Velocity = 11.290 feet per second = 7.553010 miles.
51.		3	2	62	
52.	Wave generated 1000 feet from A,	3	2	62.5	Mean Velocity = 11.200 feet per second = 7.49280 miles.
53.		4	2	62.5	
54.		4	2	62.5	
55.		0	2	62.5	
56.	Wave 1500 feet from A,	2	2	63.5	Mean Vel. = 11.023 ft. per sec. = 7.37438 miles.
57.	Wave 2500 feet from A,	2	2	64.5	Mean Vel. = 10.852 ft. per sec. = 7.259988 m.

In these examples no particular velocity was employed for generating the wave. A vessel was put in pretty rapid motion by a couple of horses, over a space of about 500 feet, and was then suddenly stopped, so as to allow the water it had set in motion to move forward before the vessel in the form of a wave, and the velocity of the wave was then measured from a mark at a station of observation to that of another station whose distance was known. These examples which have been given comprehend the waves of a considerable variety of velocities of motion. The following observations were made with this view alone, of determining whether the velocity of the vessel had any influence on that of the wave, from which the influence appears to be insensible.

Exper.		Velocity of Boat. Miles per hour.	Time. Secs.
58.	Space 700 feet,	5	62
59.		3	61
60.		10	61
61.		7	62
62.		7	62
63.		4	61.5

From these experiments it appears that the velocity of the wave, is that acquired by a heavy body falling through a space equal to half the depth of the fluid, and that the velocity appears to vary with the magnitude of the wave very nearly in the ratio which is obtained by supposing the depth of the fluid increased by a quantity equal to the height of the wave, so that the variations of velocity in a given depth may be traced to the varying height of the wave, the mean height in these experiments having been three or four inches. When the depth of the cross-section of the channel varies, the velocity is nearly the mean of the velocities due to the depths. For the more perfect determination of the laws of the motion of waves, I have begun a series of experiments extending through a much more extensive range of dimensions; those made in 1834 having been almost exclusively made in reference alone to their connection with the law of resistance.

On Resistance and Immersion.—For the purpose of conducting the inquiries regarding the immersion of bodies moving at high velocities, and the resistance of the fluid at these velocities, an experimental vessel was constructed, a very light skiff, capable of containing four or six observers, with the apparatus of experiment. The "skiff" was constructed of iron plates, extremely thin, and only weighed 430 lbs. The length of the skiff was 31.25 feet, breadth 4 feet, and her figure as given in Plate III. This vessel has been drawn by a highly-bred horse, at a rate of more than 20 miles an hour. The skiff carried the following apparatus.

(1.) Two forms of the tube of Pitot P′ P″ P‴, Plate III. Fig. 5. P′ being the aperture exposed to the water, P′ P″ a long tube, separating into two branches communicating by stop-cocks with P″ P‴, two vertical glass tubes carrying graduated scales, one connected with the open tubes being graduated in inches and decimals, zero being at the level of the fluid, to be used for low velocities, and the other for higher velocities, graduating so as to indicate, by the compression of air in a ball on the top of the tube, the height to which the water would have been sustained in an open tube of unlimited length. The observations made with Pitot's tube do not in any respect vary from those given by others, and generally received as correct. The tubes of Pitot were only useful as giving an index of velocity of considerable extent, and giving variable indices of velocity cotemporaneous with the variations of moving force. The observations with the tubes served to confirm those of the chronometers as indices of velocity.

(2.) *Gauges of Immersion.*—Many modes have been attempted of determining whether the immersion of a floating body in motion be variable or constant. Rods have been applied vertically between the gunwale of the vessel and the surface of the water, but the change of form caused by the currents of the fluid and its waves have interfered with this method. Lines stretched above the vessel, so as to measure the distance between the summit of the vessel and the fixed string in the two states of motion and rest have indicated an elevation, but have not given the means of distinguishing whether this elevation consisted of the fluid rising along with the vessel, or the vessel emerging from the fluid, or both of these causes united or modifying each other. The method I have used is this: into apertures pierced in the bottom of the vessel glass-tubes, open at both ends, and graduated in decimals of an inch, were inserted, as T_1, T_2, T_3, T_4, T_5, T_6, Plate III. Fig. 5, into which the external fluid was pressed up to the level of the surface of the quiescent fluid. The action of these gauges was found very delicate, a slight variation in the position of the tube, or a trivial error in the formation of its aperture at the bottom, giving irregular results. The value of the indications of the tubes was determined by drawing the vessel in opposite directions, which immediately shewed which of the tubes were affected by errors of position:—those which were free from such errors were selected for observation, and their indications are given in the following experiments.

(3.) *Dynamometers.*—Much has been written on the subject of dynamometers, and much in praise of a species of that instrument, in which the minor oscillations of the moving force, or the variations of the resistance, are suppressed, and only some unknown function of these variations supposed to approximate to their sum, or rather their mean is exhibited, this effect being produced by the application of the well-known principle of the retardation of a fluid passed through a very small aperture. I made trial of a very simple dynamometer formed on this principle, which was very accurately constructed for me by Mr John Adie. A helical spring contained in a cylinder, was compressed by a piston, which communicated through the piston-rod with the moving power. The cylinder being closed was filled with oil, and a communication between opposite ends of the cylinder effected through an external tube, governed by a stopcock. The stopcock gave the means of retarding or facilitating the passage of the fluid, and enabled the observer to render the position of the index more or less stable, by turning the stopcock in such a way as to facilitate or retard the motion of the fluid in the variations of force. I had at first considerable faith in this species of instrument. It certainly accomplished the purpose of giving a stable instead of an oscillatory indication, an indication easily observed. But it may be questioned, whether it be really a desideratum to obtain indications which have not the variations of the subjects themselves that are to be measured. The indications of the instrument are in truth false, or at least they only shew what effect the action of a desultory force produces on the motion of the fluid of the instrument itself. In applying this instrument to the measurement of the resistance of fluids, when the resistance is by no means very desultory, it is most desirable that the variations of the power should be apparent, instead of being rendered latent. It is obvious that the force communicated by jolts to a body in motion, produces effects that are so widely different from those of uniform pressure, that the sum of the impulses due to a given velocity is very different in its effect from a uniform pressure equal to that sum.

The disadvantages of using a desultory power like that of horses in producing motion, to which the resistance is like that of a fluid continuous, are very great. The following examples at velocities almost precisely equal, made with the same bodies, and in the same circumstances, as indicated by the compensating dynamometers, will shew the comparative effects of a variation in the action of the power.

Example I.

	Force indicated.	
Trotting,	95 lb.	Mean = 101 lb.
	100 ...	
	108 ...	
	Same velocity.	
Cantering,	138 ...	Mean = 101 lb.
	152 ...	
	155 ...	

Example II.

	Force indicated.		
Trotting,	107 lb.	67^s.	Space = 1000 feet.
	108 ...		
	108 ...		
	108 ...		
	108 ...		
	108 ...		
Cantering,	130 ...	66^s.	Space = 1000 feet.
	118 ...		
	130 ...		
	128 ...		
	128 ...		
	135 ...		

The specimen of the dynamical effect of trotting which I have given in Example II., is the most perfect specimen I have ever been able to obtain, and was obtained by a very powerful well-bred, well-trained horse, which was ridden in a very superior manner. Out of an immense number of experiments made with horse-power, I have been able to obtain comparatively few in which the differences of the successive impulses were sufficiently small to admit of an arithmetical mean being used to represent a constant force. All the others were of course comparatively valueless, except as illustrative of the manner in which the power of horses was applied in overcoming the peculiar mode of resistance of the fluid.

Although, therefore, during 1834, I made a very great number of experiments on the resistance of various vessels, in various conditions of immersion, and at many different velocities, in which the direct power of horses was applied, and measured by the action of the dynamometer I have described as the fluid dynamometer, and with the ordinary dynamometer, I am now disposed to place little faith in those where the application of the force deviated widely from uniformity, especially when absolute measures of the resistances are required, or delicate comparisons instituted. For observations on which we may rely implicitly as measures of resistance, I refer with perfect confidence to the experiments of 1835, which were made with continuous power, and under the improved arrangements which the experience of 1834 had dictated. I give here, however, a set of experiments of 1834, which were obtained after long experience had enabled us to render the variations of our desultory power as small as possible, assigning to them that degree of value only which the approximation to uniformity may appear to entitle them.

Experiments of 1834, with the Skiff;

The motion being produced by the direct application of the power of horses.

Weight of the Skiff,	810 lb.
Weight of observers, ballast, &c.	430 ...
Total Weight moved,	1240 ...
Depth of immersion by the gauges at the centre P″ when at rest,	2.7 inches.
Length of the Skiff on the gunwale,	31.25 feet.
Length of the Skiff on the keel,	30.75 ...
Maximum breadth,	4.21 ...

For the form of the Skiff, see Plate III. Fig. 5.

The first column contains the space described during the experiment; the second column consists of the resistances as registered at the commencement of the space, at the end of the space, and at equidistant points of division; the third column consists of the time in which the space was described; the fourth the velocity in miles per hour; and the fifth contains the indications of the gauges of immersion.

Space. Feet.	Moving Force. Lbs.	Time. Secs.	Velocity. Miles per hour.	Immersion. Inches.
500	10, 10, 10	113	3.0168	2.6
500	17, 18.5, 17.5	85	4.0096	2.5
500	16, 19, 18.5, 18.5	80	4.2613	2.4
1000	25, 24, 27, 26, 31, 27, 27	132	5.1652	2.2
500	35, 45, 35, 40	60	5.6816	2.2
1000	45, 48, 45, 53, 55, 55	116	5.9288	2.0

Space. Feet.	Moving Force. Lbs.	Time. Secs.	Velocity. Miles per hour.	Immersion. Inches.
500	55 55 65 65 65	53	6.4318	1.9
500	78 80 84 84	47	7.2534	1.8
1000	60 70 73 74 70 79	85	8.11	2.2
1000	85 87 87 87 87 87 85 87	75	9.164	2.3
1000	85 85 85 86 86 87 88	74	9.16	2.0
1000	107 108 108 108 108 108	67	10.237	2.0
1000	100 115 118 95 100 108	58	11.755	1.9
500	Not observed.	17	20	1.5

In the last experiment the vessel had been lightened, so as to draw only 2 inches, and retained only one individual, who executed the duties both of directing the vessel, and observing the gauges of immersion. It was not found practicable to observe the oscillations of a moving force so impetuous.

The experiments which have been given, afford the means of estimating the value of the following Table. To remove the appearance of affectation of superior accuracy, where the nature of the indications, and the subject of measurement, would not bear out the degree of apparent precision given by such figures, fractions of a pound have been omitted, as the experience of making these experiments has shewn me that errors of directing the vessel, and the resistance of the helm, must have affected the result to an amount greater than any such fraction. In general, I have endeavoured to render the apparent precision of number indicated a measure of the precision of the observation, having never allowed the indications to be read off with greater minuteness than the total of the probable error.

The following Table; formed from the Experiments of 1834, with the Skiff, regards only the resistance of the vessel. The vessel was steered within three feet of the line of motion of the horse, the line was sixty feet long, reducing the correction for the deviation of the line of traction from the line of the keel to a very small quantity; this correction has not been applied in the following table, which contains the experiments exactly as made. The first column gives the time of describing 500 feet; columns second and third giving the velocity; the fourth column contains the mean motive force applied at these velocities; the fifth column shews the number of feet described from which the observations have been drawn; and the sixth column is a table of squares of velocities, and affords the means of comparing the law of the squares with the real law of resistance. From the small immersion of the Skiff, her wave was small, and the velocity of a wave equal to that generated by her motion is given in the table as observed by experiment.

The Skiff.

Time to 500 Feet.	Velocity in Feet.	Miles per Hour.	Motive Power in lbs.	No. of Experiments made.	Ratio of Squares of Velocities.	Remarks.
s. 113	s 4.42	3.0168	10.1	Five	9.1011	Weight of Skiff 3 cwt. 94 lbs.; load in Skiff 7 cwt. 26 lbs. Total 1240 lbs. Length of Skiff 31 feet 3 inches, gunwale 30 feet 3 inches keel; maximum breadth 4 feet $2\frac{1}{2}$ inches.
85	5.88	4.0096	17.6	Three	16.095	
80	6.25	4.2613	18.6	Three	18.158	
66	7.58	5.1652	26.7	Seven	26.669	
64	7.81	5.3248	27.5	Four	28.353	
60	8.33	5.6816	39.0	Four	32.280	
59	8.45	5.7777	48.0	Two	33.382	
57.5	8.62	5.8789	50.2	Four	34.561	
58	8.69	5.9288	51.2	Six	35.151	
56.5	8.85	6.0335	55.	Nine	36.403	
53	9.43	6.4318	61.	Five	41.368	
48.5	10.30	7.0290	79.		49.407	Velocity of Wave 2 inches high, = 10.853 feet, = 7.3997 miles.
47	10.64	7.2534	82.5	Seven	52.612	
42	11.90	8.1168	82.5	Three	65.882	
40	12.50	8.5228	81.1	Nine	72.368	
37.2	13.44	9.1642	86.3	Twenty-one	83.982	
35	14.28	9.7403	89.2	Twelve	94.873	
33.3	15.01	10.2370	107.	Nine	107.86	
29	16.90	11.7555	111.	Six	138.19	

PART III.

The investigations of 1834 had established the principal points in the relation between the resistance of a fluid, the diminished immersion, and the velocity of the wave. The prominent features in the representation of the law had been traced, but the outline being in many parts faintly and ambiguously given, required to be retouched, corrected, and filled in. The power of horses, which had been used as the moving force, was desultory in its action, so that the measure of the force employed did not always afford the means of obtaining even a tolerable approximation to an accurate measure of resistance at a uniform velocity. Yet the power of horses had this advantage, that it could be continued for a much greater length of time, and over a much longer space, than that obtained by the action of a falling weight, or any other convenient mechanical means. For small models, indeed, it would have been sufficiently simple to provide, as has frequently been done, the means of applying a continuous moving force; but I was not, in 1834, in possession of any plan by which this object could be accomplished, so as to obtain a continuous moving power, acting through a great space, to generate high velocities in vessels of large size carrying considerable weights. In 1835, I had, however, attained this desideratum.

The means of obtaining the continuous action of a moving force with great power and through a great space, were very simply and conveniently obtained; and as the method may be useful to other inquirers, I shall on that account describe it more particularly than might perhaps be necessary for the mere purpose of appreciating the experiments conducted with it. The method which has been previously used for obtaining the power by means of a weight, has been by suspending that weight from an elevated structure by strands of rope passing over pulleys, by which a given weight, in falling through a given space, acts through one of the strands so as to move the end of the rope through a space greater than the space through which it falls by as much as the number of strands exceeds unity. In this case the weight to be raised, in order to obtain a given power, increases so rapidly with the increase of the space and the friction of the pulleys, and the effect of rigidity increases so rapidly along with it, that the limit of practicability, and, at all events of inconvenience, is very soon attained. Further, after one experiment has been obtained by an apparatus of this kind, considerable time must elapse before the weight is again elevated, and the rope drawn out to its former station for commencing another experiment. In the method which I have adopted, the weight never requires to exceed twice the moving force required, plus friction and rigidity, for five pulleys; the weight requires no increase for the space moved over, except for the friction of the additional horizontal rope on its supporters; and when one experiment has been completed, the arrangements are thereby made for instantly beginning the succeeding experiment. The mode was this: A pyramidal framing (see Plate III. fig. 6.), was raised to a height of 75 feet, formed by four logs of pine rising from the corners of a square of 45 feet, and firmly united at the apex, so as to give attachment to two fixed pulleys, and the structure was made rigid by an oblique framing of spars and ropes. This structure was placed on the bank of the sheet of water at the Experimental Station close to the Bridge of Hermiston, from which there extends a line of bank in a straight line of more than 1500 feet in length, which was accurately divided by painted rods into equal portions. Through the two pulleys (C) at the top of the framing were passed the two ends of the rope, and from the intermediate part of the rope, by means of a moveable pulley (D), was suspended the moving weight. The two ends of the rope that had been passed over the pulleys at the vertex of the pyramid were thus brought down to a point (B), raised 6 feet above the level of the water, where they were passed through two other pulleys fixed in the pile of masonry forming one of the piers of the bridge. This forms the whole of the apparatus for the application of the moving force.

The pyramid being placed at one end of the station, the vessel subjected to experiment was brought to the other end, and one end of the rope of the pyramid was brought along the whole length of the station and attached to (E) the bow of the vessel. Horses were attached (A) to the other end of the rope, which was cut short after leaving the pulleys fixed in the masonry. The horses now started, and having first tightened the rope, began to elevate the weight towards the top of the pyramid. But the other end of the rope fixed to the bow of the vessel had to sustain a tension in raising the weight equal to the part acted on by the horses, and, in consequence of this action, the vessel would have begun to move at the same time at which the horses began to raise the weight, but the vessel had been previously fixed by the stern-post to the bank, and thus a reaction was obtained to sustain the weight. When, however, the observers in the vessel observed the weight rise to a given height in the pyramid, they withdrew a small catch in the stern fastenings of the vessel, and she immediately proceeded towards the pyramid. In the mean time, however, the motion of the vessel allowed the weight to fall towards the ground, which it would have reached when the vessel had moved through a space equal to twice its original elevation, had the horses been allowed, after having raised the weight, to stand still; but as they were urged to a motion at their end of the rope with the same velocity which the vessel acquired at the other end of the rope, the weight was kept at rest in the air; or if the horses moved either a little slower or a little faster than the boat, the effect was merely to allow the weight to ascend or descend in the pyramid with a velocity equal to half the difference of the velocities of the horses and the vessel, and thus the difference of the action of the horses was not sensible in the force acting on the vessel. When the horses had arrived at that end of the station from which the vessel had commenced its motion, the vessel had reached the point from which they had started.

The apparatus was now ready for the succeeding experiment, one end of the rope being at the pyramid and the other at the starting-point. The vessel was immediately drawn back to the starting-point while the horses were returning to the pyramid, and being again attached to the extremities of the rope, the next experiment was begun.

It was found that considerable time elapsed before the vessel attained the uniform velocity due to the moving force, and therefore the vessel was put in motion through a considerable space previous to making the observations. Where this proved inconvenient, a very simple mode was used of attaining a higher velocity, which was by the attachment of an additional weight (F) by a rope about 50 feet long to the former weight in the pyramid, so that this weight should rest on the ground, unless the principal weight were raised to a height greater than 5 feet, in which case alone the additional weight would be called into action. By this means it was easy, on commencing the experiment, to keep the principal weight so high as to raise the additional weight to produce the required acceleration, and afterwards, before commencing the observations, to allow it, by resting on the ground, to cease from acting on the vessel. The velocity due to the moving force was thus attained in a shorter time than would otherwise have been necessary.

The observations were made in the vessel upon time and resistance, the rope through which the propelling force was applied being attached to the vessel by the hook of an accurate spring dynamometer, indicated the resistance in pounds, and accurate chronometers gave the time. One observer being placed so as to have a line of sight at right angles to the line of motion, communicated by sound the instant of passing the rods placed at equal distances along the bank, and at the same instant the time was read off by a second observer, and written down on paper by a third; a fourth observer read off the indications of the dynamometer at the same instant, and they were registered opposite to the instant of time to which their observation corresponded; and, for the sake of accuracy, two copies of this register were kept. The indications of this register form the body of experiments of 1835.

The experiments of 1835 were conducted on the following vessels:—

The Wave, . .	No. I.
The Dirleton, .	No. II.
The Raith, . .	No. III.
The Houston, .	No. IV.

The first of these having been made the subject of experiment at seven different degrees of immersion, and each of the others at three, are equivalent to experiments upon sixteen vessels of sixteen different forms.

The forms of the vessels are shewn in Plate III., being projected at an angle of the line of vision sin. $-^1 = \frac{1}{3}$. The "water lines of the entrance and of the run" are shewn below the projections of each vessel, as taken at successive heights of 6 inches. The comparisons of form may thus be easily made.

The principal dimensions of the vessels were nearly identical. The maximum breadth at the gunwale being about 6 feet, and the length, exclusive of the helm, 69 feet, there being added to this in the case of the "Wave" a cutwater or very sharp part of the bow 6 feet long, and of very small capacity.

The Wave is a vessel of very peculiar form. My observations on the nature of the resistance of fluids in 1834 suggested a form of least resistance. The Wave was built of that form, and answered fully, and even surpassed the expectations I had formed of the facility of her motion. The lines of entrance are parabolic tangent arcs, having a point of contrary flexure between the maximum transverse section and the stem. The run is formed of elliptical arches, and is by no means so fine as runs usually are. It has long been matter of observation with me, that the maximum resistance to a vessel of ordinary form is experienced in the immediate vicinity of the stem,—that the water there is thrown aside with a velocity much greater than is requisite to remove the particles from the portions of space to be passed over by the succeeding points of the bow. This "head of water" at the bow, instead of being merely thrown aside, is also thrown upward and forward, so as very much to increase the resistance beyond what appears necessary for the transit of the vessel. It occurred to me as probable that a form of vessel might perhaps be obtained, which would not at any given velocity raise a head of water above the level, but merely give to the particles displaced the minimum possible of lateral motion required to permit the transit of the vessel. The theoretical law of least displacement, which I imagined gave me the equation of a curve, which appeared to me to be a curve of minimum resistance. That this curve would be the curve of least resistance I could not a priori determine; but it appeared to me that an experimentum crucis might decide the question after the vessel was built. The experiment was simply to give the vessel a very high velocity, such as 17 miles an hour, and if it should then be found that no particle of water had any motion communicated to it except simply what was necessary for the passage of the vessel, if no spray were thrown up before the vessel or dashed aside by the prow; if, in fact, the vessel, on entering smooth water, should pass into it leaving the surface still unruffled, and producing no motion among the particles but what was the necessary result of mere repletion, by the presence of an additional body, then I should be warranted in denominating such a body the *solid of least resistance.* This experiment was actually tried. The vessel was built of this form (as given in Plate III. fig. 1), and was named "The Wave;" and it is a remarkable fact that, even when deeply laden, and when urged to a velocity of 17 miles an hour, there is no spray, no foam, no surge, no head of water at the prow, but the water is parted smoothly and evenly asunder, and quietly unites after the passage of the vessel, without having changed the natural relations of its particles to one another. Adhesion alone to the surface of the vessel drags forward a film of adjacent fluid, all else remains quiet and smooth.

The three other vessels, the Dirleton, the Raith, and the Houston, were built on the models of Mr John Wood of Greenock, a gentleman of much scientific knowledge and great practical skill; they are much more nearly analogous to the ordinary forms given to sea-going vessels. The Dirleton is the most recent and the best vessel; the Raith and the Houston are inferior and older.

It is worthy of remark, that the Wave is the sharpest vessel, the Dirleton next to her, the Raith third, and the Houston the most bluff in the entrance; that the Wave is fullest, the Dirleton next to her, the Raith next, and the Houston most fine in the run. From the experiments it would seem, that a fine entrance is of much more importance to velocity than has been hitherto supposed, and that a fine run is by no means entitled to the importance that has been attached to it. It should also be observed, that the increase of immersion causes a very great increase of resistance in the three last vessels, and comparatively little in the Wave; and that the water-lines become bluff as they descend, but retain the original curve in the Wave.

Table I. contains the Results of the Experiments of 1835, arranged in reference to the Velocity of the Wave of the Fluid, and is deduced from Tables II, III, IV, and V.

Table II. contains the Original Experiments of 1835 on the Wave, the form of vessel given in Plate III. fig. 1.

Table III. contains the Original Experiments of 1835 on the Houston. The form of the vessel is given in Plate III. fig. 3.

Table IV. contains the Original Experiments of 1835 on the Dirleton. The form of this vessel is given in Plate III. fig. 2.

Table V. contains the Original Experiments of 1835 on the Raith. The form of this vessel is given in Plate III. fig. 4.

ANALYTICAL TABLE of the RESULTS of the Experiments of 1835, upon the Resistance of four Vessels a

TABLE I. WAVE.—PL. III. FIG. 1.

		LIGHT.	I. TON.	II. TONS.	III. TONS.	IV. TONS.	V. TONS.	VI. TONS.	
Total mass moved in lbs.		5,782	8,022	10,262	12,502	14,742	16,982	19,222	Total mass.
Immersion in the fluid,		11.0 in.	13.5 in.	15.0 in.	16.5 in.	18.0 in.	19.0 in.	20.0 in.	Immersion.
	Velocity in miles an hour.	Resistance in pounds.	Resistance in pounds.	Resistance in pounds.	Resistance in pounds.	Resistance in pounds.	Resistance in pounds.	Resistance in pounds.	Velocity in feet per second.
Velocities less than the Velocity of the wave of the fluid,	3.7879	...	...	...	...	...	...	...	5.55
	3.8961	...	...	...	...	40.2	58.5	...	5.71
	4.0107	30.0	32.0	...	...	...	...	...	5.88
	4.1322	...	...	39.0	...	...	...	...	6.06
	4.2613	32.0	33.7	...	42.7	...	64.6	...	6.25
	4.3988	...	...	...	...	...	...	74.0	6.45
	4.5454	34.4	35.5	...	...	52.3	79.0	81.3	6.60
	4.7229	...	...	...	...	...	...	...	6.89
	4.8702	40.6	42.8	47.5	...	...	90.0	...	7.14
	5.0508	...	...	...	...	...	95.0	95.5	7.40
	5.2448	44.0	51.0	...	76.0	...	...	...	7.69
	5.4545	...	...	...	...	83.3	100.5	121.5 155.0	7.99
	5.6818	53.8	56.0	76.0	...	84.0	...	188.0	8.33
	5.9289	...	...	82.5 108.0	111.0 127.5	...	112.0	312.0 384.0	8.69
	6.1983	73.3	76.0	91.0 165.0	186.0 214.0	...	142.0 332.0	...	9.09
	6.4935	88.5	95.7 110.0	103.0 193.0	193.0 266.0	...	300.0 372.0	...	9.52
	6.8182	93.5 119.0	120.0 197.0	133.5	232.0 298.0	...	386.0 392.0	...	10.00
	7.1770	100.0 131.7	166.0 199.3	206.0 240.0	...	...	...	...	10.52
	7.5758	131.7 174.0	217.0 265.0	225.0 320.0	329.0 336.0	...	...	...	111.1
The Velocity of the Wave of the Fluid in these Experiments was about Eight miles an hour.									
Velocities greater than the velocity of the Wave of the fluid,	8.5227	168.0	214.0	...	...	...	...	...	12.50
	9.0491	...	210.0	...	...	...	...	...	13.33
	9.6955	189.5	227.3	235.0	...	...	...	...	14.28
	10.4895	...	...	...	...	...	...	...	15.38
	11.3634	212.3	245.5	333.0	...	344.0	...	...	16.16
	12.3967	222.0	...	352.0	...	408.0	...	...	18.18
	13.6364	...	...	...	...	...	...	...	20.00
	15.1515	...	...	444.0	...	...	...	...	22.22

NOTE.—The double observations shew a variation in the resistance at the same velocity of which the cause is to be found in the Histo given velocity its height is small, and the resistance is also small; but when the velocity has been acquired by small increments du and the resistance arising from it is also increased; thus in the vicinity of the wave, and immediately behind it, the history of the w tions.

Depths of Immersion, giving measures of resistance for sixteen forms of the floating body.

RLETON.—Pl. III. Fig. 2.			HOUSTON.—Pl. III. Fig. 3.			RAITH.—Pl. III. Fig. 4.			
IT.	II. Tons.	IV. Tons.	Light.	II. Tons.	IV. Tons.	Light.	II. Tons.	IV. Tons.	
9	10,339	14,819	6,076	10,556	15,036	5,859	10,339	14,819	Total mass.
in,	13.5 in.	16.0 in.	7.3 in.	11.0 in.	15.0 in.	7.5 in.	11.0 in.	15.0 in.	Immersion.
ce in ls.	Resistance in pounds.	Resistance in pounds.	Resistance in pounds.	Resistance in pounds.	Resistance in pounds.	Resistance in pounds.	Resistance in pounds.	Resistance in pounds.	Squares of Velocity.
	...	42.0	...	...	...	...	...	...	14.3482
	...	...	...	...	...	...	...	...	15.1796
	47.3	46,5	...	...	...	...	...	...	16.0857
	...	...	...	...	...	...	...	..	17.0753
)	...	...	...	...	60.0	...	...	...	18.1592
	52.5	...	...	...	...	...	...	...	19.3497
	...	...	37.5	...	66.0	40.2	...	...	20.6612
	54.0	...	...	...	...	...	...	...	22.1106
	...	...	44.0	64.5	...	...	81.0	...	23.7182
	...	109.5	53.0	...	...	...	...	...	25.5076
	...	114.0	...	72.7	...	60.0	83.0	116.0	27.5075
	...	117.0	...	...	...	...	...	...	29.7521
	99.7	126.0 144.0	83.0	...	166.5	...	103.5	148.0 180.0	32.2838
	114.7	...	...	...	172.5 255.0	...	...	...	35.1511
.5	133.5 169.0	169.0 235.0	98.0	197.0	252.0 279.0	87.0 98.0	126.7 218.0	110.0 294.0	38.4195
	177.0 210.0	...	...	195.0 258.0	288.0 324.0	100.0 156.0	...	306.0 357.0	42.1656
	216.0 285.0	...	195.0	...	...	...	...	..	48.4876
3	...	...	181.0 210.0	...	...	...	...	...	51.5098
.0	255.0 342.0	...	...	306.0 330.0	...	163.0 189.0	252.0	...	57.3921

ve feet per second; the form and dimensions of the channel are given in Plate II. Fig. E.

0	328.0	...	...	...	...	...	...	...	72.6369
0	...	...	234.0	...	...	...	...	...	81.8869
0	...	...	...	...	...	225.0	...	...	94.0628
.	...	...	241.5	...	...	...	...	...	110.0290
0	...	...	...	...	...	...	...	...	129.1325
.	...	...	...	...	...	...	...	...	153.6784
.	...	...	...	...	...	...	...	...	185.9504
.	...	...	...	...	...	...	...	...	229.5683

. The *magnitude* of the wave depends upon the *wave's age* in such a manner, that when the vessel has been rapidly brought to a
ble interval of time, the anterior waves have accumulated in the direction of the motion, and the height of the wave is increased
variations in the resistance of the fluid to a given body moving with a given velocity, and is the source of the double observa-

TABLE II. THE WAVE.

No. of Experiments.		Weight of Ballast.	Total Weight Moved.	Depth of Immersion.	Weight on Pyramid.	Time of commencing Observation.	Resistance at 0.	100 Feet.		200 Feet.		300 Feet.	
								Time.	Force.	Time.	Force.	Time.	Force.
		Lbs.	Lbs.	Inches.		h m s	Lbs.	s		s		s	
7th July 1835,	III.	0000	5782	11	2 cwt.	8 6 53	30.0	21	30	20	30.0	19	29.3
	IV.	0000	5782	11	3 cwt.	8 26 46	32	16	32	16	35.5	13	39.0
	V.	0000	5782	11	3 cwt.	9 48 22	35.5	16	37.3	14	44.0	14	45.7
	VI.	0000	5782	11	3 cwt.	10 2 4	28.0	15	37.3	15	39.0	15	39.0
	VII.	0000	5782	11	4 cwt.	10 23 9	39.0	13	40.3 / 41.5	12	41.5 / 42.7	12	47.5 / 51.0
	VIII.	0000	5782	11	4 cwt.	10 40 16	52.3	13	53.5 / 54.7	12	54.7 / 70.0	10	70.0 / 73.0
	IX.	0000	5782	11	4 cwt.	11 0 30	47.5	12	53.5 / 56.0	13	56.0 / 56.0	11	56.0 / 63.0
	X.	0000	5782	11	5 cwt.	11 17 55	85.7	10	87.5	11	87.5	11	93.5
	XI.	0000	5782	11	5 cwt.	11 45 15	66.5	11	87.5	10	93.5	10	98.0
	XII.	0000	5782	11	5 cwt.	12 7 30	82.5	11	84.0	10	81.7	10	100.0
	XIII.	0000	5782	11	6 cwt.	12 21 1	93.5	10	108.5	10	116.3	10	131.7
	XIV.	0000	5782	11	6 cwt.	1 25 20	...	9	100.0	10	108.5	10	131.7
	XV.	0000	5782	11	6 cwt.	1 46 33	119.0	10	119.0	10	131.7	9	137.0
	XVI.	0000	5782	11	7 cwt.	2 13 1	131.7	9	133.3	9	133.3	10	147.0
	XVII.	0000	5782	11	7 cwt.	2 27 32	124.7	9	140.0	9	140.0	9	163.5
8th July 1835,	XVIII.	0000	5782	11	7 cwt.	10 1 20	*157.5	10	161.0	10	163.5	8	†174.5
	XIX.	0000	5782	11	7 cwt.	11 7 15	*122.5	9	168.0	9	168.0	9	168.0
	XX.	0000	5782	11	8 cwt.	11 23 20	*154.0	9	200.0	9	186.0	9	195.0
	XXI.	0000	5782	11	8 cwt.	12 7 10	*147.0	9	150.5	9	163.5	9	193.0
	XXII.	0000	5782	11	8 cwt.	12 18 36	*147.0	9	147.0	9	159.3	†8	187.7
	XXIII.	0000	5782	11	8 cwt.	12 37 26	*140.0	9	...	9	154.0	9	186.0
	XXIV.	0000	5782	11	8 cwt.	12 50 20	122.5	9	124.7	8	143.5	9	154.0
	XXV.	0000	5782	11	8 cwt.	1 19 34	131.7	9	137.0	9	159.3	9	163.5
	XXVI.	0000	5782	11	8 cwt.	1 36 23	...	11	161.0	8	...	6	200.0
9th July 1835,	XXVII.	0000	5782	11	10 cwt.	8 30 36	210.5	7	210.5	6	210.5	7	210.5
	XXVIII.	2240	8022	13.5	2 cwt.	10 26 13	41.5	...	...	...	40.3	15	41.5
	XXIX.	2240	8022	13.5	2 cwt.	11 5 49	32.0	17	33.7	16	35.5	15	39.0
	XXX.	2240	8022	13.5	3 cwt.	11 24 49	106.0	11	...	11	...	...	...
	XXXI.	2240	8022	13.5	3 cwt.	11 39 0	...	...	52.3	15	52.3	14	63.0
	XXXII.	2240	8022	13.5	3 cwt.	11 51 10	* 41.5	16	45.7	15	49.3	13	52.6
	XXXIII.	2240	8022	13.5	4 cwt.	12 2 15	* 53.5	49	...	...	...	...	...
	XXXIV.	2240	8022	13.5	3 cwt.	12 59 26	* 32.0	...	...	...	39.0	14	51.0
	XXXV.	2240	8022	13.5	4 cwt.	1 21 36	76	12	76	11	76	11	81
	XXXVI.	2240	8022	13.5	4 cwt.	1 31 8	82.5	...	82.5	...	82.5	10.5	82.5
	XXXVII.	2240	8022	13.5	5 cwt.	1 42 54	124.7	...	124.7	...	124.7	10.5	124.7
	XXXVIII.	2240	8022	13.5	3 cwt.	1 52 50	24.5	18	28.0	18	28.0	16	30.0
	XXXIX.	2240	8022	13.5	5 cwt.	2 10 28.5	*131.7	11.5	119.0	10	122.5	10	124.7
	XL.	2240	8022	13.5	6 cwt.	2 20 2	*131.7	10	131.7	9	141.7	9	145.3
	XLI.	2240	8022	13.3	6 cwt.	3 38 0.5	*105.5	...	152.3	9.3	162.3	9.7	168.0
	XLII.	2240	8022	13.5	6 cwt.	3 46 49	111.0	10	135.5	9.5	140.0	10	167.0
	XLIII.	2240	8022	13.5	8 cwt.	3 57 40.5	*170.0	9	170.0	9.5	172.3	9.3	200.0
	XLIV.	2240	8022	13.5	8 cwt.	4 0 2	174.5	9.3	176.7	9.2	191.3	9	210.3
	XLV.	2240	8022	13.5	8 cwt.	4 19 39	173.0	10	173.0	8.5	191.3	8.7	197.0
	XLVI.	2240	8022	13.5	9 cwt.	4 31 26	195.0	† 8.5	200.0	8	214.0	7.5	234.7
	XLVII.	2240	8022	13.5	9 cwt.	4 40 1.5	197.0	† 5.5	197.0	8.5	238.0	8.5	238.0
	XLVIII.	2240	8022	13.5	9 cwt.	4 59 41	173.0	9	207.0	9.5	223.7	8.5	231.3
	XLIX.	2240	8022	13.5	9 cwt.	5 11 38.5	214.0	7.5	214.0	8.3	216.3	7.2	217.0
	L.	2240	8022	13.5	10 cwt.	5 21 5	*225.9	8	227.3	8	227.3	7	227.3
10th July,	LI.	4480	10262	15.0	2 cwt.	10 11 45	39.9	...	39.0	...	39.0	16	39.0
	LII.	4480	10262	15.0	2 cwt.	10 24 12.5	76.0	15.5	12.0	16	18.0	17	28.0
	LIII.	4480	10262	15.0	2 cwt.	10 35 44	39.0	18	32.0	17	39.0	16	39.0
	LIV.	4480	10262	15.0	3 cwt.	10 40 5	41.5	16	44.0	15	45.7	15	47.5
	LV.	4480	10262	15.0	3 cwt.	11 2 2.5	39.0	15.3	39.0	14	47.5	13.5	51.0

* In this experiment an accelerating weight was used to acquire velocity previous to the first observation.
† Point of transition from a velocity less than the wave to a velocity greater than it.
‡ These examples shew the variation of resistance at the same velocity, due to the history of the Wave.

THE WAVE.

….	500 Feet.		600 Feet.		700 Feet.		800 Feet.		900 Feet.		1000 Feet.		REMARKS.
…rce.	Time.	Force.	Time.	Force.	Time.	Force.	Time.	Force.	Time.	Force.	Time.	Force.	
0.0	17	30	16	30	16	31	15	35.5	13	42.3	14	56	The two first experiments were lost.
4.0	13	44	12	51	12	65.5	12	70.0	12	...	...	...	16^s = 32 lb.; 13^s = 44 lb.
1.0	12	52.6	12	59.5	11	64.7	11	76.0	11	91.0	11	100	14^s = 40.6 lb.; 12^s = 51.8 lb.
2.7	13	44.0	13	51.0	12	66.5	10	70.0	11	82.5	9	91	15^s = 34.4 lb.
4.3 5.0	11	59.5 66.5	10	70.0 78.5	12	84.0 91.0	9	100.0 103.0	11	119.0 122.5	9	131.3	The rope slightly entangled.
4.5 5.0	11	77.3 81.0	11	84.0 91.0	10	98.0 103.0	10	106.0 119.0	10	135.0 135.0	9	137.0	11^s = 73.3 lb.; 10^s = 98 lb.
3.0).0	11	74.3 78.5	11	79.7 91.0	10	98.0 98.0	10	98.0 103.0	10	119.0 131.3	10	140	12^s = 53.8 lb.; 11^s = 67.4 lb.
).0 1.3	10	106.0 119.0	10	124.3 134.3	9	138.5 154.0	9	157.5 186.0	8	193 200	...	...	10^s.5 = 88.5 lb.; 10^s = 93.5...119 lb.‡
3.0	11	124.3	9	127.7	10	127.7	10	163.5	9	179.0	...	...	10^s = 93.5...127 lb.‡
4.5	10	131.7	10	137.0	10	137.5	9	174.5	8	193.0	8	114.0	10^s = 93.7...137 lb.‡
).3	10	157.5	8	163.5	†8	166.0	8	193.0	7	232.7	...	...	10^s = 93.5...116.3 lb.‡
7.0	9	157.5	9	166.0	9	186.0	†8	203.5	8	232.7	7	256.6	9^s.5 = 100...131.7 lb.; 9^s = 131.7...
1.0	10	168.0	9	174.5	9	174.5	9	186.0	8	197.0	...	...	9^s = 168...174 lb.‡ [166 lb.‡
3.5	8	184.3	†7	186.0	6	186.0	7	114.0	6	114.0	...	...	9^s = 133.3...163.5 lb.;‡ 6^s.5 = 186 lb.
).0	9	182.5	†7	182.5	7	182.5	6	217.0	...	...	...	...	7^s = 182.5 lb.
7.0	8	197.0	7	198.5	7	198.5	7	222.0	...	...	...	...	8^s = 169 lb; 7^s = 197.5
3.0	8	182.5	7	197.0	7	200	6	...	6	...	6	...	9^s = 122.5...168 lb.;‡ 8^s = 168 lb.;
21.7	...	...	...	...	...	...	..	...	...	...	...	...	An accident. [7^s = 189.7 lb.
3.0	†8	194.0	7	195.0	6	207.0	5	238.0	6	...	6	...	7^s = 194 lb.
).5	7	189.5	7	189.5	6	198.5	6	245.0	6	341.6	...	...	8^s = 173.5 lb.; 7^s = 189.5 lb.
5.0	†8	199.3	8	203.5	8	217.0	5	245.0	7	266	6	...	9^s = 140...186 lb.‡
).0	†8	182.0	8	187.0	9	200.0	6	227.3	5	266	6	...	9^s = 122.5...179 lb.‡
5.0	8	194.0	†8	195	8	207.	6	224.5	...	...	...	...	The times in these two experiments
4.5	8	179.0	9	197	8	207	7	232.6	6	256	7	...	inaccurately observed.
.	6	236.0	7	....	6	161.0	7	193.0	7	193	7	...	See continuation in Exper. CIV.
1.5	14	42.7	14	51.0	12	56.0	12	56.0	12	56.0	...	...	14^s = 41.5 lb.; 12^s = 56 lb.
4.0	14	45.7	14	51.0	11	57.7	14	73.0	12	82.0	...	...	16^s = 33.7 lb. 14^s = 42.8 lb.
.	...	...	...	...	...	...	...	...	...	...	...	...	An accident.
5.5	12	66.5	11	70.0	11	91.0	11	91.0	12	106	12	...	
3.0	...	131.7	...	137.0	12	137	10	96.0	11	107.3	...	...	The accelerating weight accidental-
.	...	....	...	...	...	...	...	...	...	...	...	...	An accident. [ly raised.
1.0	13	53.5	11	63.0	11	76.0	12	...	...	...	...	...	
4	12	96.0	10	114.5	10	116.3	10.5	127.5	10	154.0	11	...	11^s = 76 lb.
4	12	84.0	10.5	93.5	11	98.0	10.5	119.0	10	150.5	...	...	10^s.5 = 95.7 lb.
7.0	10.5	147.0	9	163.5	11	168.0	9	198.5	10	207.0	...	...	
).0	15	30.0	15	40.3	14	47.5	12	51.0	12	53.0	...	...	
7.5	9.5	150.5	10	157.5	10	197.7	10	207.0	...	...	...	...	10^s = 122.5...197 lb.‡
5.7	9	179.0	10.5	193.0	8.5	216.3	10	233.9	8	...	...	...	
).0	10	188.3	9	217.0	9	217.0	†8	248.5	8	245.0	6	...	
).3	9.5	199.3	9.5	243.3	9	245.0	†7.5	280.0	7.5	327.3	7	...	9^s.5 = 166...199.3 lb.‡
2.3	8.5	217.7	9	239.7	†8	265.0	7	269.0	6.5	352.0	5.5	...	
7.0	9	238.0	†8	253.5	8	268.6	7	308	6	308.0	...	...	
3.5	9	216.3	†8.7	240.3	8	266.6	7	268.6	6	295.6	6	...	
4.7	7	235.7	6	245.0	6	267.5	6	150.5	...	...	...	...	8^s = 214 lb.; 6^s = 245 lb.
3.4	8.5	239.7	7.5	256.0	7	266.0	5.5	268	5.5	...	...	...	
5.0	8.7	280.0	7.8	282.0	7.3	332.5	5.2	336.0	6	...	...	...	
7.0	6.5	...	...	...	...	203.5	8	...	...	...	...	...	8^s = 226.5 lb.; 7^s = 227.3 lb.
7.3	5.5	223.6	6.5	137.0	7	222.0	6	186	7	140.0	...	...	See in continuation Exper. CXXII.
).0	14.5	44.0	14.5	51.0	13.5	140.0	11	140.0	11.2	96	11.8	...	
).0	16.5	39.0	14.5	39.0	13	124.7	12	106.0	12	76	12	...	
).	14.5	40.0	15	47.5	14	53.5	14	63.0	12.5	66.5	12.5	...	16^s.5 = 39 lb.; 14^s = 47.5 lb.
3.5	11.5	56.0	13	63.0	12.5	124.7	11	96.0	10.5	...	...	...	14^s = 46.6 lb.
3.5	12.5	70.0	11.5	108.5	10	99.0	11.5	106.0	10.5	...	...	...	13^s.5 = 51.0 lb.

* In this experiment an accelerating weight was used to acquire velocity, previous to the first observation.
† Point of transition from a velocity less than the wave to a velocity greater than it.
‡ These examples shew the variation of resistance at the same velocity, due to the history of the Wave.

TABLE II.—continued. THE WAVE.

Date	No. of Experiments.	Weight of Ballast.	Total Weight Moved.	Depth of Immersion.	Weight on Pyramid.	Time of commencing Observation.	Resistance at 0.	100 Feet.		200 Feet.		300 Feet.	
								Time.	Force.	Time.	Force.	Time.	Force.
		Lbs.	Lbs.	Inches.		h m s	Lbs.						
10th July 1835,	LVI.	4480	10262	15.0	3 cwt.	11 14 36.5	39.0	14.5	39	15.5	45.7	14.5	53.5
	LVII.	4480	10262	15.0	4 cwt.	11 28 24	78.5	13	78.5	12.5	78.5	11.5	85.7
	LVIII.	4480	10262	15.0	4 cwt.	11 38 21	76.0	13.5	76.0	12	76.0	12	82.5
	LIX.	4480	10262	15.0	4 cwt.	11 54 57	59.5	13	70.0	13	71.5	12	82.5
	LX.	4480	10262	15.0	5 cwt.	12 5 48	96.0	12	103.0	11	109.7	10	124.7
	LXI.	4480	10262	15.0	5 cwt.	12 15 4	91	11	102.0	11.5	107.3	10.5	111.0
	LXII.	4480	10262	15.0	5 cwt.	12 29 13	78.5	12	85.7	11	...	12	106
	LXIII.	4480	10262	15.0	6 cwt.	12 40 24	154.0	11	157.5	11	157.5	9	182.5
	LXIV.	4480	10262	15.0	6 cwt.	1 2 20	124.7	10	131.7	11	133.5	10	166.0
	LXV.	4480	10262	15.0	6 cwt.	1 13 14	137.0	11	138.5	10.5	140	10.9	161.0
	LXVI.	4480	10262	15.0	6 cwt.	1 23 25	137.0	11	163.5	10	174.5	10	182.5
	†LXVII.	4480	10262	15.0	6 cwt.	1 36 13	238.0	7.5	245	9	200.0	10.5	193
	LXVIII.	4480	10262	15.0	8 cwt.	1 49 0	...	...	...	...	...	...	...
14th July 1835,	LXIX.	6720	12502	16.5	2 cwt.	10 13 0	...	...	53.5	17	51.0	16	44.0
	LXX.	6720	12502	16.5	2 cwt.	10 30 14.5	44	18.5	39	17	41.5	16.5	42.7
	LXXI.	6720	12502	16.5	2 cwt.	10 44 55	32	20	33.4	17	33.7	17	39.0
	LXXII.	6720	12502	16.5	3 cwt.	10 56 0	51	16	51.0	14.5	57.7	15	60.6
	LXXIII.	6720	12502	16.5	4 cwt.	11 9 37	56.0	14	70.0	13.5	82.5	12.5	...
	LXXIV.	6720	12502	16.5	4 cwt.	11 20 40.5	81	13.5	81.0	13	81.0	13	81.0
	LXXV.	6720	12502	16.5	5 cwt.	11 34 45	124.7	11	127.5	10.5	137	11.5	154
	LXXVI.	6720	12502	16.5	6 cwt.	11 53 56	* 258.0	9	154	10	189.5	11.5	268
	LXXVII.	6720	12502	16.5	8 cwt.	12 7 42.5	* 261.3	8.5	266.0	9	209.0	11	287
	LXXVIII.	6720	12502	16.5	8 cwt.	12 20 9	* 207.0	9.5	193.0	10	214	10.5	232.6
	LXXIX.	6720	12502	16.5	9 cwt.	12 28 28	* 200	9.5	214	...	...	...	...
	LXXX.	8960	14742	18.0	2 cwt.	2 10 17	37.2	19	39	17	40.2	17	47.5
	LXXXI.	8960	14742	18.0	3 cwt.	2 23 2	63.0	13.5	51	14.5	51	13.7	66.5
	LXXXII.	8960	14742	18.0	4 cwt.	2 34 14	* 170.0	12	81	12	84	12	94.7
	LXXXIII.	8960	14742	18.0	5 cwt.	2 23 39	* 140.0	11	137.0	17	138.5	11	104.7
15th July 1835,	LXXXIV.	11200	16982	19.0	2 cwt.	7 24 13	...	15	...	20	...	19	50
	LXXXV.	11200	16982	19.0	2 cwt.	7 41 14	58.3	16	68	16	64.6	16	65
	LXXXVI.	11200	16982	19.0	3 cwt.	7 54 55	80.0	16	79	15	79	16	85
	LXXXVII.	11200	16982	19.0	4 cwt.	8 10 6.5	108.0	14.5	112	13	113	12	{102.5 102.5}
	LXXXVIII.	11200	16982	19.0	4 cwt.	8 23 43	...	13	108	12	113.5	12.5	130
	LXXXIX.	11200	16982	19.0	5 cwt.	9 47 41.5	164.0	11.5	165	10.5	174	11.5	192
	XC.	11200	16982	19.0	5 cwt.	10 6 26	...	12	182	10	184	12	206
	XCI.	11200	16982	19.0	6 cwt.	11 23 27	204	10	228	10.5	234	11.5	243
	XCII.	11200	16982	19.0	8 cwt.	10 36 1	264	12	276	9.5	276	10.5	321
	XCIII.	11200	16982	19.0	8 cwt.	10 49 14	270	10	288	10	300	10.5	312
	XCIV.	11200	16982	19.0	8 cwt.	11 2 41.5	280	...	...	...	299.3	11	329
	XCV.	11200	16982	19.0	10 cwt.	11 15 1	124.7	10.3	347.3	9.2	347.3	10	369.6
	XCVI.	13440	19222	20	2 cwt.	12 56 0	...	...	...	...	80	16	60
	XCVII.	13440	19222	20	2 cwt.	1 10 28.5	76.3	18	76.3	16.5	73.5	17	71
	XCVIII.	13440	19222	20	3 cwt.	1 23 30	79.5	18	79.0	14	80.5	15	82
	XCIX.	13440	19222	20	3 cwt.	1 33 40	73	11	73	16	74	15.5	75
	C.	13440	19222	20	4 cwt.	1 43 0	119	14.5	120.5	13.5	121.5	12.5	125
	CI.	13440	19222	20	5 cwt.	1 53 7	188.5	12.5	188	12	188	11	198
	CII.	13400	19222	20	6 cwt.	2 4 54	...	11.5	312	11.5	328.5	11	339
	CIII.	13440	19222	20	8 cwt.	2 37 54	268	10	245	11.5	245	11.5	245

* In this experiment an accelerating weight was used to acquire velocity previous to the first observation.
† Point of transition from a velocity less than the wave to a velocity greater than it.
‡ These examples shew the variation of resistance at the same velocity due to the history of the wave.

THE* WAVE.

eet.	500 Feet.		600 Feet.		700 Feet.		800 Feet.		900 Feet.		1000 Feet.		REMARKS.
Force.	Time.	Force.	Time.	Force.	Time.	Force.	Time.	Force.	Time.	Force.	Time.	Force.	
56.5	12.5	56.0	12.5	101.0	10.5	101.0	11	124.7	10.0	140.0	...	...	$11^s = 111$ lb.
96.0	12	107.2	11	111.0	10	131.7	10	147.0	10.5	...	...	...	$11^s = 107.2$
91.0	12.5	99.0	10	101.5	10.5	106.0	10	137.0	10	150.5	...	...	$12^s = 76$ lb.; $11^s.5 = 82.5$
82.5	11	87.5	10	100.0	10	103.0	11	124.7	12	143.5	10.5	...	$11^s.5 = 82.5$ lb.
124.7	10	140.0	11	145.3	10	155.7	9.5	186.0	11.5	193.0	...	...	$10^s.5 = 103 \ldots 193$ lb. ‡
124.7	10.5	149.3	10.5	150.5	10	162.6	10.5	189.5	10.5	207.0	...	...	$11^s = 91 \ldots 111$ lb.; ‡ $10^s.5 = 124.7 \ldots$
111.0	10	115	11	131.7	10	161	10	179	10.7	195	...	...	$11^s = 85.7$ lb. [189.5 lb. ‡
187.7	10	193.0	10.5	210.5	10.5	224.6	10	238.0	10	269.0	...	...	
168.0	10	195.0	10	207	11	217	10	248.5	10	...	...	...	$10^s = 133.5 \ldots 248.5$ ‡
193.0	10	210.5	10.5	224.6	10.5	238	10	256.6	10	261.3	...	...	
194.5	10	203.5	10	235.3	10	245.0	9	268	9	280.0	...	...	$10^s = 163.5 \ldots 235.3$ lb. ‡
200	10	203.5	11	217.0	10	229.5	10	250.2	9	266.0	10	...	See in continuation Exper. CXXX.
...	...	...	...	...	...	...	...	...	...	...	...	...	An accident.
51.0	13	53.5	14	59.5	12	63.0	13	70.0	12	54	13.5	96	
46.3	16	60.6	13	66.6	14	66.5	13.5	73	12.5	73	13	73	$16^s = 42.7$ lb.
46.3	15	47.5	14	56	14	70.0	13	76	12	87.5	13	...	$17^s = 33.5$ lb.
63.0	13	70	12	82.0	13	91.0	12	93.3	10.5	93.5	13.5	...	
96.0	12	...	...	...	...	...	...	...	...	...	...	...	
96.0	12.5	111.0	11	111.0	11	192.0	11	131.7	11.5	131.7	...	...	$11^s = 111$ lb.; $13^s = 81$ lb.
86	11	193	10	200	10	222	10.5	...	...	...	...	...	$11^s = 127.5$ lb.; $10^s.5 = 186.5$ lb.
14	11	238	10.5	269.3	10.5	266	10.5	287	10	...	15.5	...	
32.6	10.5	238	11.5	261.3	10.5	266	9.5	298.6	10	336	...	...	
38	10	266	10	287	9.5	329	9	352	9	336	...	...	$10^s = 193 \ldots 266$ lb.; $9^s = 329 \ldots 336$ lb.
...	...	...	...	...	...	...	...	...	...	...	...	...	An accident.
51.0	15	53.5	15	63	14	70	13	78.5	13.5	96	...	...	$17^s = 402$ lb.; $15^s = 52.3$ lb.
76.0	13.3	81.0	12.5	83.3	12	923	12.5	100.0	11.5	111.0	...	...	$12^s.5 = 83.3$ lb.
15.0	11	154.0	11	207.0	11	140.0	11.5	140.0	11.5	161	...	...	$12^s = 84$ lb.
79.0	11.5	154.0	10.5	154	11	215.5	10	227.3	11.0	...	...	...	See in continuation Exper. CXLVI.
50	19	48.0	16.5	58.5	17.5	58.3	16.5	59.7	15.5	69.0	13	...	$17^s.5 = 58.5$ lb.
70.9	15	76	15	83	13	90	14	90.0 / 90.0	13	96 / 100	15	...	$16^s = 64.6$ lb.
89	14	95	13.5	95	12.5	100.5 / 103.5	11.5	112 / 118.5	11.5	128 / 140	14	...	$16^s.5 = 79$ lb.; $13.5 = 95$ lb.
24 / 40	12.5	153	11	152	12	155 / 176	12	180 / 186	20.5	...	...	...	$11^s.5 = 112$ lb.
41	11.5	142 / 148	11	156 / 158	11	162 / 168	11	182 / 183	11	196 / 196	...	...	$11^s = 142 \ldots 195$ lb. ‡
02	11	220	11	232	11	140	11	60	15	...	...	...	$11^s = 202 \ldots 232$ lb. ‡
20	11.5	236	11	240	11.5	240	11	240	10.5	...	...	...	
55	10	270	10.5	288	12	300	11	312	10.5	...	...	...	
24	10.5	336	11	357	10.5	360	10.5	360	11.5	...	...	...	$11^s = 321 \ldots 336$ lb.
42	10.5	360	11	372	10.5	180	14	120	...	...	...	...	$10^s.5 = 300 \ldots 372$ lb.
38.8	10.5	354.5	10.5	359	10.5	371.9	10.5	386.3	13.5	...	...	...	
90	9	...	10	...	10	...	10	238	...	...	...	...	
..	...	...	...	...	...	...	...	78	15	78	...	...	
71	17	72	...	...	...	716	17	86.5	16	90	14.5	102	
89	14	102	13	104	13	113	12	118	13	122 / 136	15	...	$15^s = 81.3$ lb.; $13^s = 103$ lb.
79	15	91	18	95	13	96	14	112	13	122	13	...	$15^s.5 = 74$; $13^s.5 = 95.5$ lb.
33	11.5	156 / 152	13.5	164	13	168	11	180 / 188	12.5	188	...	...	$12^s.5 = 121.5 \ldots 156$ lb.
08	11.5	214	10.5	2[illegible]0 / 228	11	226 / 228	13	236 / 240	...	...	...	...	$12^s = 188$ lb.
42	12	354	11	369	11.5	372	11	384	11.5	384	...	...	$11^s.5 = 312 \ldots 384$ lb.
45	11	263.6	12	269	11	287	12	294	11	...	...	...	

* In this experiment an accelerating weight was used to acquire velocity previous to the first observation.

† Point of transition from a velocity less than the wave to a velocity greater than it.

‡ These examples shew the variation of resistance at the same velocity, due to the history of the Wave.

TABLE II.—continued. THE WAVE.

No. of Experiments.	Date	Weight of Ballast.	Total Weight Moved.	Depth of Immersion.	Weight on Pyramid.	Time of commencing Observation.	Resistance at 0.	100 Feet. Time.	100 Feet. Force.	200 Feet. Time.	200 Feet. Force.	300 Feet. Time.	300 Feet. Force.	400 Feet. Time.	400 Feet. Force.	500 Feet. Time.	500 Feet. Force.
		Lbs.	Lbs.	Inches.		h m s	Lbs.										
CIV.	26th July 1835,	0000	5782	11.0	2 cwt.	10 4 11.5	* ...	14.5	...	13	51.5	11	74	11	85	10	106
CV.		0000	5782	11.0	3 cwt.	10 15 55.5	* ...	...	...	...	...	...	...	...	...	...	...
CVI.		0000	5782	11.0	3 cwt.	10 40 30.5	* 100	11	625	10.5	77	10.5	77	11.5	112	8	124
CVII.		0000	5782	11.0	3 cwt.	10 53 24	54	11.5	100	11.5	100	10.5	92	10	116	9.5	92
CVIII.		0000	5782	11.0	4 cwt.	11 1 40	96	9	108	10	134	11	156	9	166	10	200
CIX.		0000	5782	11.0	4 cwt.	11 7 12	106	10	118	10	130	10.3	148	9.7	139	9	168
CX.		0000	5782	11.0	4 cwt.	11 19 30	224	7.5	96	10	112	7.5	134	10	162	9	170
CXI.		0000	5782	11.0	4 cwt.	11 27 59	62	11	72	11	78	10	94	11	112	10	132
CXII.		0000	5782	11.0	4 cwt.	11 35 13	164	8	96	9	112	10	128	9	170	9	170
CXIII.		0000	5782	11.0	5 cwt.	11 44 30	128	9.5	148	8.5	160	†8	180	7.5	192	7	...
CXIV.		0000	5782	11.0	5 cwt.	11 53 11	132	9	148	10	152	8.5	176	†8	212	7.5	218
CXV.		0000	5782	11.0	5 cwt.	12 1 53	132	9.5	150	8.5	160	9	180	†8	174	7.5	180
CXVI.		0000	5782	11.0	5 cwt.	1 36 57	140	9	152	9	172	8.5	190	†7.5	216	8	222
CXVII.		0000	5782	11.0	6 cwt.	1 45 27	204	†7	152	7	164	7	220	6	224	6	224
CXVIII.		0000	5782	11.0	6 cwt.	1 53 1	208	†8	...	7	...	6	...	6	...	...	...
CXIX.		0000	5782	11.0	6 cwt.	2 1 43.5	255	†6.5	255	7	234	7	252	6	300	6	348
CXX.		0000	5782	11.0	8 cwt.	2 10 42	219	†7	219	6	223.5	6	270	6.3	282	5.7	...
CXXI.		0000	5782	11.0	9 cwt.	2 38 0	...	...	...	...	...	...	...	...	...	...	...
CXXII.	27th July 1135,	2240	8022	13.5	2 cwt.	11 25 53	82	13	82	12.5	84	12	110	11	106	12	120
CXXIII.		2240	8022	13.5	3 cwt.	11 38 51	100	12	103	11	110	10.5	116	11	120	10	...
CXXIV.		2240	8022	13.5	3 cwt.	11 48 20	224	9.5	144	11.5	160	10	163	9.5	164	10	204
CXXV.		2240	8022	13.5	5 cwt.	11 56 17	216	10	224	†8	240	6	...	9	...	...	...
CXXVI.		2240	8022	13.5	5 cwt.	12 5 54	228	9	204	8.5	228	8.5	237	†8	243	7	288
CXXVII.		2240	8022	13.5	6 cwt.	12 14 3	243	†7.5	210	6.5	240	7	246	6	318	6	300
CXXVIII.		2240	8022	13.5	6 cwt.	12 21 8	210	†8	210	7	228	6.5	270	6	300	..	...
CXXIX.		2240	8022	13.5	8 cwt.	1 6 20.5	240	7	270	...	...	...	...	...	...	...	...
CXXX.		4480	10262	15.0	2 cwt.	2 0 29	60	15	66	14	68	13	80	13.5	83	12	90
CXXXI.		4480	10262	15.0	3 cwt.	2 9 57	106	12	106	11.5	110	11.5	118	11	120	11	120
CXXXII.		4480	10262	15.0	4 cwt.	2 15 36	166	11	164	11	164	10.5	164	11	170	10.5	172
CXXXIII.		4480	10262	15.0	5 cwt.	2 22 47.5	172	11	168	10.5	168	11	168	10.5	192	10.5	224
CXXXIV.		4480	10262	15.0	6 cwt.	2 30 15	206	9.5	240	9.5	240	9.5	...	9.8	...	7	180
CXXXV.		4480	10262	15.0	6 cwt.	2 40 34	225	9	225	9	270	9	300	9	306	9	330
CXXXVI.		4480	10262	15.0	7 cwt.	2 0 0	...	...	...	...	...	...	...	...	...	...	...
CXXXVII.		4480	10262	15.0	7 cwt.	3 22 43.5	246	9.5	303	10	350	...	...	...	...	...	...
CXXXVIII.	31st July 1835.	4480	10262	15.0	6 cwt.	10 43 6.3	177	11.7	192	8	203	11	220	10	240	...	...
CXXXIX.		4480	10262	15.0	8 cwt.	10 55 18	216	9.7	223.5	10	253.5	9.8	283.5	7.7	297	9.5	330
CXL.		4480	10262	15.0	8 cwt.	11 10 47	223.5	9	240	8.7	240.5	9.8	300	9.5	312	8	336
CXLI.		4480	10262	15.0	10 cwt.	11 24 13.5	284	8.5	300	†7.5	312	7.5	312	6.7	332	5.3	...
CXLII.		4480	10262	15.0	12 cwt.	11 39 21	352	6.5	404	6	440	...	480	...	160	...	..
CXLIII.		4480	10262	15.0	14 cwt.	11 58 19	392	6	392	6	400	...	200	6	200	6	280
CXLIV.		4480	10262	15.0	14 cwt.	...	...	...	...	...	...	...	...	...	...	...	...
CXLV.		4480	10262	15.0	14 cwt.	12 30 48	428	...	...	...	428	5.5	456	5	480	...	...
CXLVI.		8960	14742	19.0	10 cwt.	2 2 4.7	343.5	...	...	...	...	7.3	...	10	372	7.3	...
CXLVII.		8960	14742	19.0	12 cwt.	2 41 24	472.0	7.5	484	6.7	392	6	392	5.3	408	5.5	192
CXLVIII.		4480	10262	15.0	12 cwt.	3 32 19	400.0	7	360	6	344	6	344	5.5	400	6	400
CXLIX.		4480	10262	15.0	14 cwt.	3 55 13	348	5.3	352	7.2	...	...	...	...	...	...	...
CL.		4480	10262	15.0	14 cwt.	4 23 16.5	...	5.5	192	6	264	5	400	4.5	488	4.5	...

REMARKS.

Exp. CVI. $11^s = 77$ lb. Exp. CIX. $10^s = 106...148$ lb Exp. CXIII. $8^s = 162.6$ lb. Exp. CXIV. $8^s = 176$ lb. Exp. CVII. $7^s = 185$ lb.; $6^s = 224$ lb.

Exp. CXXIII. $10^s.5 = 110$ lb.; $10^s = 120$ lb. Exp. CXXIV. $10^s = 163$ lb. Exp. CXXVI. 8^s 237 lb. Exp. CXXVII. $7^s = 231$ lb.; $6^s = 24$[illegible] Exp. CXXVIII. $7^s.5 = 210$ lb.; $6^s.5 = 228$ lb.

Exp. CXXX. $13^s.5 = 75$ lb.; $13^s = 81.5$ lb.; $12^s = 86.5$ lb. Exp. CXXXI. $11^s.5 = 108$ lb.; $11^s = 119$ lb. Exp. CXXXII. $11^s = 165$ lb. Exp. CXXXIII. $10^s.5 =$ ‡ 168...192. ‡ Exp. CXXXIV. $9^s.5 = 206...240$. Exp. CXXXV. $9^s =$ ‡ 225...320. Exp. CXLI. $7^s.5 = 306$ lb.; $6^s = 322$ lb. Exp. CXLV. $5^s.5 = 428$ lb.

Exp. CXLVII. $5^s.5 = 408$ lb.

Exp. CXLVIII. $6^s = 344$ lb. Exp. CL. $4^s.5 = 444$ lb.

* In all the experiments from CIV. to CC. an accelerating weight was used to give velocity previous to the first observation.

† Point of transition from a velocity less than the wave to one greater than it.

‡ These examples shew a variation in the resistance at a given velocity which is due to the history of the Wave.

ABLE III. THE HOUSTON.

PERIMENTS.	Weight of Ballast.	Total Weight Moved.	Depth of Immersion.	Weight on Pyramid.	Time of commencing Observation.	Resistance at 0.	100 Feet.		200 Feet.		300 Feet.		400 Feet.		500 Feet.	
							Time.	Force.	Time.	Force.	Time.	Force.	Time.	Force.	Time.	Force.
	Lbs.	Lbs.	Inches.		h m s	Lbs.										
CLI.	0000	6076	7.3	2 cwt.	11 40 37	46	14	122	10.5	54	11.5	68	12.5	72	11.5	100
CLII.	0000	6076	7.3	2 cwt.	11 50 2	35	15	40	15	44	14	53	13.5	55	12.5	65
CLIII.	0000	6076	7.3	3 cwt.	12 0 9	83	12	95	11	101	10.7	117	10	148	10.3	164
CLIV.	0000	6076	7.3	4 cwt.	12 7 47	114	10	116	11	122	9.5	148	10.5	171	9.5	200
CLV.	0000	6076	7.3	5 cwt.	12 14 0	174	9	177	10	195	10	201	8.5	211.5	8.5	219
CLVI.	0000	6076	7.3	6 cwt.	12 20 7	181	9.5	210	9.5	222	7.5	234	8.5	246	5	276
CLVII.	0000	6076	7.3	7 cwt.	12 26 31.5	222	8	234	6.5	192	6	241.5	6	241.5	7	270
CLVIII.	0000	6076	7.3	8 cwt.	12 35 59	228	7	18	7	60	8	300	7	180	6	198
CLIX.	4480	10556	11.0	2 cwt.	1 46 23	60	16	63	14	64.5	14	66	14	69	13	76.5
CLX.	4480	10556	11.0	4 cwt.	1 56 36	189	11	195	11	225	10	234	11	252	10.5	258
CXI.	4480	10556	11.0	7 cwt.	2 4 27	267	10	288	9.3	306	9.2	330	9	336	8.5	348
CLXII.	8960	15036	15.0	2 cwt.	2 20 0	57	...	60	15.3	60	16	63	15	69	15	78
CLXIII.	8960	15036	15.0	4 cwt.	2 28 51	165	11.7	168	12.3	172.5	11	192	11	246	12	255
CLXIV.	8960	15036	15.0	6 cwt.	2 37 17.5	249	12	252	11	279	11	288	10.5	309	10.5	324

REMARKS.

.II. $15^s = 37.5$ lb.; $14^s = 44$ lb.; $13^s.5 = 53$ lb. Exp. CLIII. $12^s = 83$ lb.; $11^s = 98$ lb. Exp. CLV. $10^s = 195$ lb.; $8^s.5 = 211.5$ lb.
CLVI. $9^s.5 =$ ‡ 181...210; $7^s.5 = 234$ lb. Exp. CLVII. $6^s.5 = 241.5$ lb.
IX. $14^s = 64^s.5$ lb.; $13^s = 72.7$ lb. Exp. CLX. $11^s = 197$ lb.; $10^s.5 =$ ‡ 195...258 lb. Exp. CLXI. $9^s =$ ‡ 306...330 lb.
XII. $16^s = 60$ lb.; $15^s = 66$ lb. Exp. CLIII. $12^s = 166.5$ lb.; $11^s.5 =$ ‡ 172.5...255 lb. Exp. CLXIV. $11^s =$ ‡ 252...279 lb.; $10^s.5 =$
.324 lb.

BLE IV. THE DIRLETON.

ERIMENT.	Weight of Ballast.	Total Weight Moved.	Depth of Immersion.	Weight on Pyramid.	Time of commencing Observation.	Resistance at 0.	100 Feet.		200 Feet.		300 Feet.		400 Feet.		500 Feet.	
							Time.	Force.	Time.	Force.	Time.	Force.	Time.	Force.	Time.	Force.
	Lbs.	Lbs.	Inches.		h m s	Lbs.										
CLXV.	0000	5859	8.7	2 cwt.	9 47 54.7	33	17.3	38	16.5	41	16.3	44	13.2	50	14.5	50
CLXVI.	0000	5859	8.7	4 cwt.	10 1 53	91	10.7	98	11.3	101	10	103	10	96	11	138
CLXVII.	0000	5859	8.7	6 cwt.	10 10 52	132	9.5	150	9.5	162	9	178	9.3	178	10.2	162
LXVIII.	0000	5859	8.7	8 cwt.	10 20 9.5	204	8	216	7.5	216	7	234	7	150	7	138
CLXIX.	0000	5859	8.7	9 cwt.	10 42 30	210	8	210	8	213	6	228	6	150	7	150
CLXX.	4480	10339	13.5	2 cwt.	11 11 44	40.5	19	46.5	17	48	17	52.5	15.3	54	14.7	58.5
CLXXI.	4480	10339	13.5	4 cwt.	11 22 39	99	12	100	12	108	11.5	121.5	11.5	133.5	11	160.5
CLXXII.	4480	10339	13.5	6 cwt.	11 36 22	153	11	162	11	177	10	183	11	210	10	222
LXXIII.	4480	10339	13.5	8 cwt.	11 47 53	211.5	9	228	10.5	247.5	9.7	276.5	10.3	301.5	9.3	324
LXXIV.	4480	10339	13.5	6 cwt.	11 55 46	222	...	...	...	...	...	...	...	...	...	...
CLXXV.	4480	10339	13.5	8 cwt.	12 12 40	223.5	10	235.5	10	243	10	285.5	9	300	9.5	318
LXXVI.	4480	10339	15.5	10 cwt.	12 21 42	255	9	270	9	342	9.7	345	9.3	348	8.3	366
XXVII.	4480	10339	13.5	10 cwt.	12 39 12	300	8	306	8.7	330	8.7	360	7.5	240	7.5	180
XXVIII.	8960	14819	16.0	2 cwt.	1 40 51.7	31.5	18.8	33	18.7	34.5	18.8	42	18	46.5	17	48.5
LXXIX.	8960	14819	16.0	4 cwt.	1 51 24.5	109.5	13.5	114	13	114	12.5	120	12.3	126	12	144
LXXX.	8960	14819	16.0	6 cwt.	2 1 5	172.5	12	172.5	12	189	...	...	...	...	...	...
LXXXI.	8960	14819	16.0	6 cwt.	2 26 34.5	169.5	11	175.5	10.7	193.5	11	216	12.3	235.5	10.7	247.5

REMARKS.

XV. $16^s.5 = 38$ lb, Exp. CLXVI. $11^s = 94.5$ lb.; $10^s = 102$ lb. Exp. CLXVII. $9^s.5 =$ ‡ 132...178 lb. Exp. CLXVIII. $8^s = 204$;
$= 216$; $7^s = 225$ lb. Exp. CLXIX. $8^s = 210$ lb; $6^s = 228$ lb.
XX. $17^s = 47.3$; $15^s.5 = 52.5$; $14^s.5 = 54$ lb. Exp. CLXXI. $12^s = 99.7$. lb; $11^s.5 = 114.7$ lb.; $11^s = 133.5$ lb. Exp. CLXXII. $11^s =$
..162 lb.; $10^s.5 =$ ‡ 177^s...210; $10^s = 216$ lb. Exp. CLXXIII. $10^s =$ ‡ 228...276.5. Exp. CLXXV. $10^s =$ ‡ 223.5...285.5. Exp.
XVI. $9^s =$ ‡ 255...342.
XXVIII. $18^s = 42$ lb.; $17^s = 46.5$ lb. Exp. CLXXIX. $13^s5 = 109.5$; $13^s = 114$; $12^s.5 = 117$ lb; $12^s =$ ‡ 126...144 lb. Exp. CLXXX.
‡ 172.5...189 lb. Exp. CLXXXI. $11^s =$ ‡ 169.5...235 lb.

* In all the experiments from CIV. to CC. an accelerating weight was used to give velocity previous to the first observation.
† Point of transition from a velocity less than the wave to one greater than it.
‡ These examples shew a variation in the resistance at a given velocity which is due to the history of the Wave.

TABLE V. THE RAITH.

No. of Experiments.	Weight of Ballast.	Total Weight Moved.	Depth of Immersion.	Weight on Pyramid.	Time of commencing Observation.	Resistance at 0.	100 Feet. Time.	100 Feet. Force.	200 Feet. Time.	200 Feet. Force.	300 Feet. Time.	300 Feet. Force.	400 Feet. Time.	400 Feet. Force.	500 Feet. Time.	500 Feet. Force.
	Lbs.	Lbs.	Inches.		h m s											
CLXXXII.	0000	5859	7.5	2 cwt.	9 41 0	...	...	37	15	43.5	15	49.0	16	60	12	60
CLXXXIII.	0000	5859	7.5	4 cwt.	9 50 44	100	11	108	10	120	10.7	...	10.3	...	10	...
CLXXXIV.	0000	5859	7.5	4 cwt.	10 5 11	87	11	92	11	98	10.7	116	10.3	138	10.5	156
CLXXXV.	0000	5859	7.5	6 cwt.	10 13 9	163.5	9	169.5	9.5	189	8.5	202.5	8	276	8	288
CLXXXVI.	0000	5859	7.5	8 cwt.	10 34 17	270	6	261	7	228	...	...	...	222	6.3	222
CLXXXVII.	0000	5859	7.5	11 cwt.	10 0 0	...	...	...	...	...	...	...	...	...	...	...
CLXXXVIII.	0000	5859	7.5	10 cwt.	10 49 56	225.5	6	225	7	232	7	267	6.3	336	6.7	120
CLXXXIX.	4480	10339	11	2 cwt.	11 18 10	81	14	81	14	82.5	14	84	12	108	12	72
CXC.	4480	10339	11	4 cwt.	11 29 11	97.	12	103.5	12.5	109.5	12	121.5	10.5	132	11	163.5
CXCI.	4480	10339	11	6 cwt.	11 43 16.5	187	11	218	11	240	...	...	...	255	10.3	270
CXCII.	4480	10339	11	8 cwt.	11 55 0	...	...	...	...	252	9	255	9	294	10	324
CXCIII.	4480	10339	11	10 cwt.	12 6 8	319.5	8	334.5	8.7	339.5	7.3	360	7	...	6	...
CXCIV.	4480	10339	11	10 cwt.	12 21 4.3	207	8.7	212	7	224	7.5	226	6.5	162	7	255
CXCV.	8960	14819	15	2 cwt.	1 40 12.5	68	17	42	18	72	13	72	14.5	70	14.5	88
CXCVI.	8960	14819	15	4 cwt.	1 51 9.5	100	10.5	124	13.5	80	13.5	108	13	52	15	88
CXCVII.	8960	14819	15	4 cwt.	1 59 17	116	13	116	13	148	11.5	139	11.5	180	13	180
CXCVIII.	8960	14819	15	6 cwt.	2 0 2	98	13.5	110	11	126	11.5	228	11	236	11	...
CXCIX.	8960	14819	15	8 cwt.	2 16 13.5	222	11	240	10.5	294	11.5	282	11.5	294	11	342
CC.	8960	14819	15	10 cwt.	2 0 41.5	306	10	354	11	357	11	...	10.5	...	11	...

REMARKS.

Exp. CLXXXII. $15^s = 40.2$ lb.; $13^s = 60$ lb. Exp. CLXXXIII. $10.^s5 =$ ‡ 100...120 lb. Exp. CLXXXIV. $11^s =$ ‡ 87...98 lb.; $10^s5 =$ ‡ 68 ...156 lb. Exp. CLXXXV. $9^s =$ ‡ 163.5...189. Exp. CLXXXVIII. $6.^s5 =$ ‡ 225 ... 267.

Exp. CLXXXIX. $14^s = 81$ lb.; $13^s = 83$ lb. Exp. CXC. $12^s = 103.5$; $11^s. = 126.7$ lb. Exp. CXCI. $11^s. =$ ‡ 187.5 ... 218 lb. Exp. CXCII. $9^s =$ ‡ 252 ... 294. Exp. CXCIII. $8^s. = 334.$; $7^s. = 224$.

Exp. CXCV. $14^s. = 72$ lb. Exp. CXCVII. $13^s. = 116$ lb.; $12^s. =$ ‡ 148 ... 180 lb. Exp. CXCVIII. $11^s. =$ ‡ 110 ... 236. Exp. CXCIX. $11^s.$ = ‡ 222... 294. Ex. CC. $10^s5 =$ ‡ 306. ... 357 lb.

* In this experiment an accelerating weight was used to acquire velocity previous to the first observation.
† Point of transition from a velocity less than the wave to a velocity greater than it.
‡ These examples shew the variation of resistance at the same velocity due to the history of the Wave.

PLATE II.

Fig. 1.

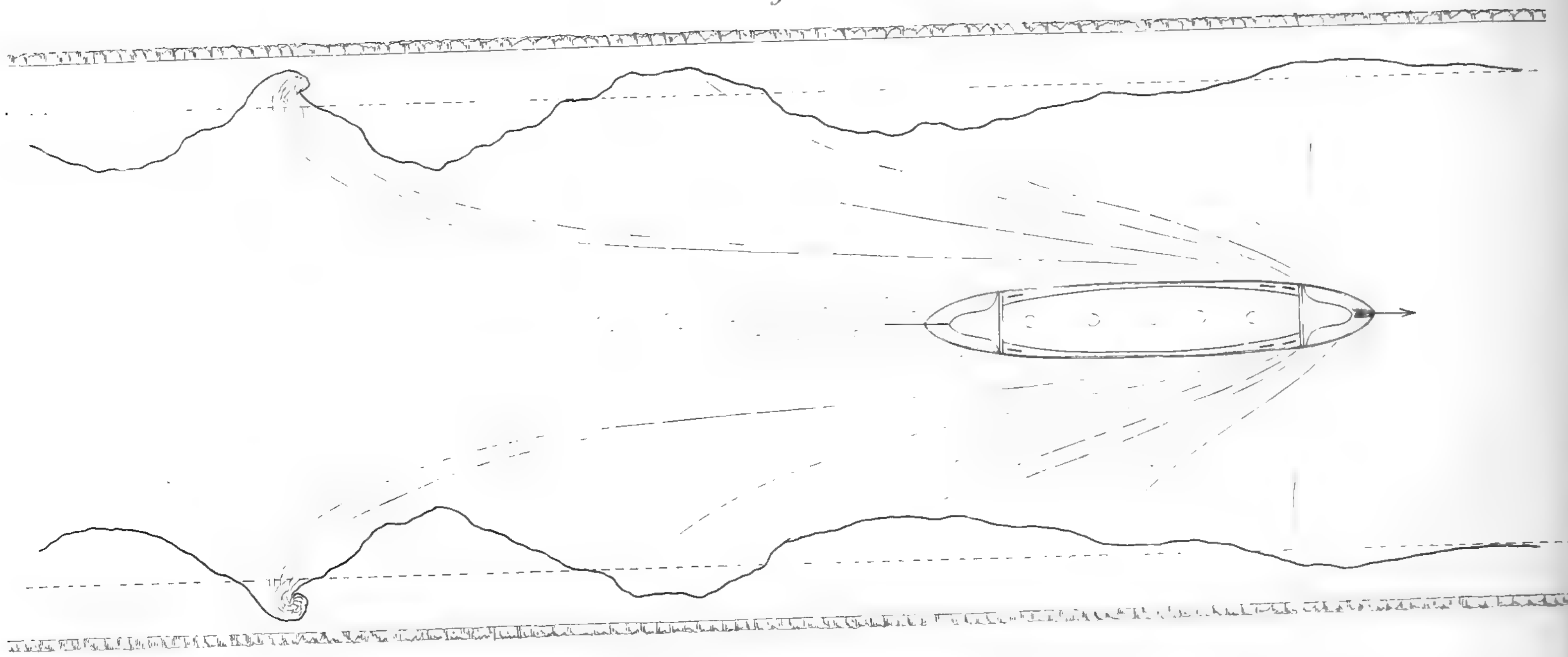


Fig. 2.

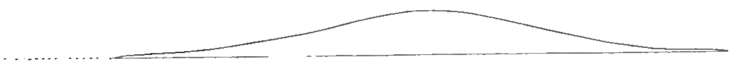


Fig. 3.

Fig. 4.

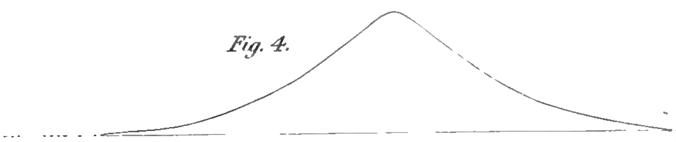
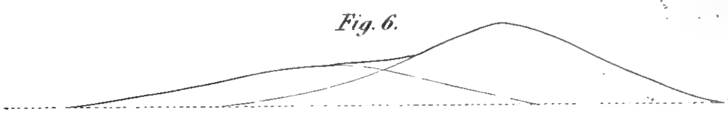


Fig. 5.

Fig. 6.



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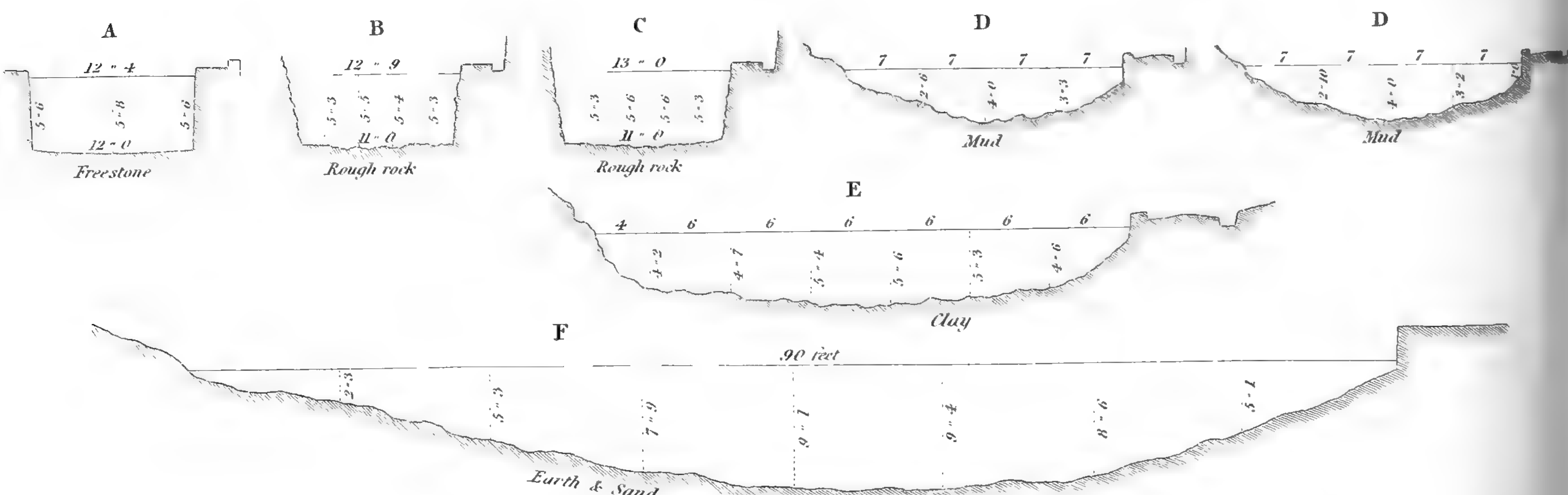


MR. CONNELL'S PAPER.

Fig. 1.

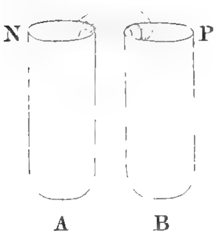


Fig. 2.

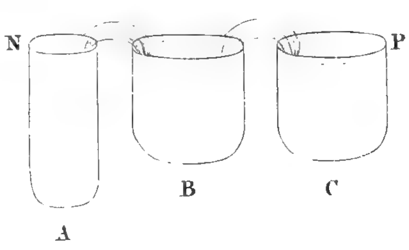


Fig. 3.

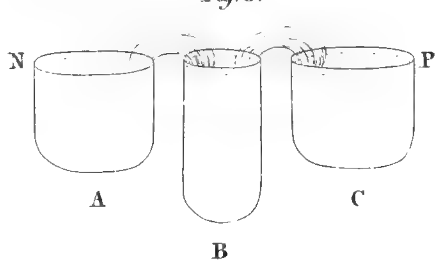


Fig. 4.

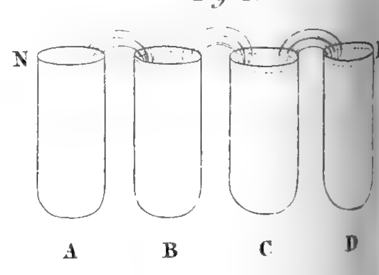
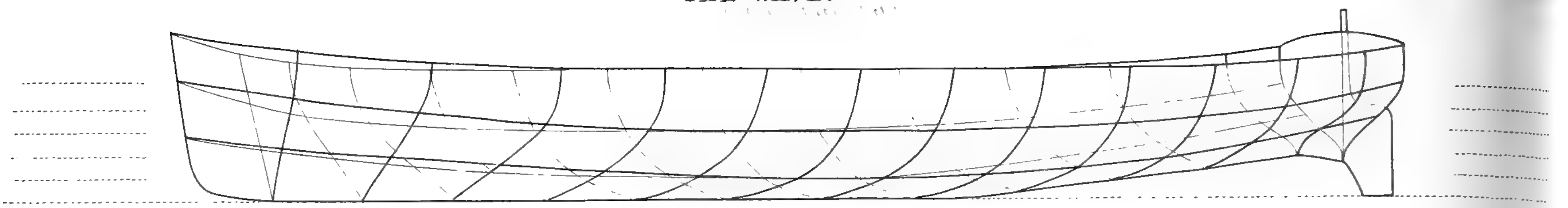


Fig. 1.

THE WAVE.



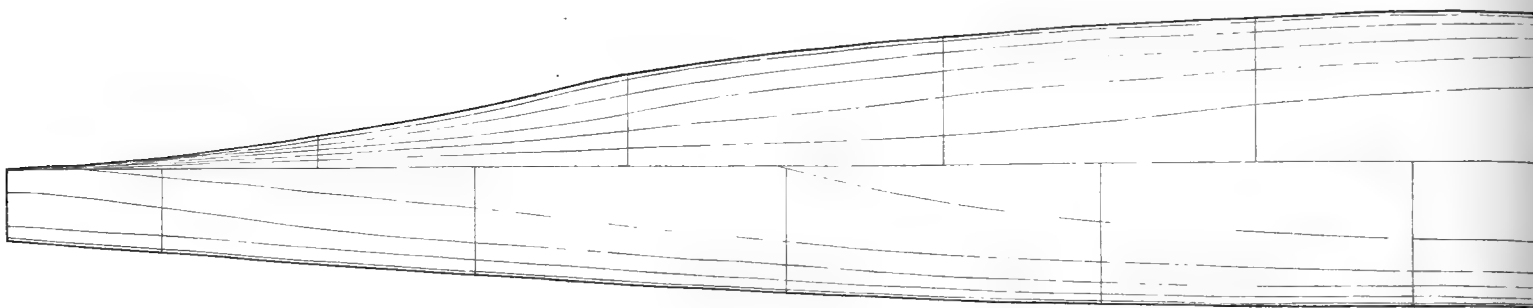
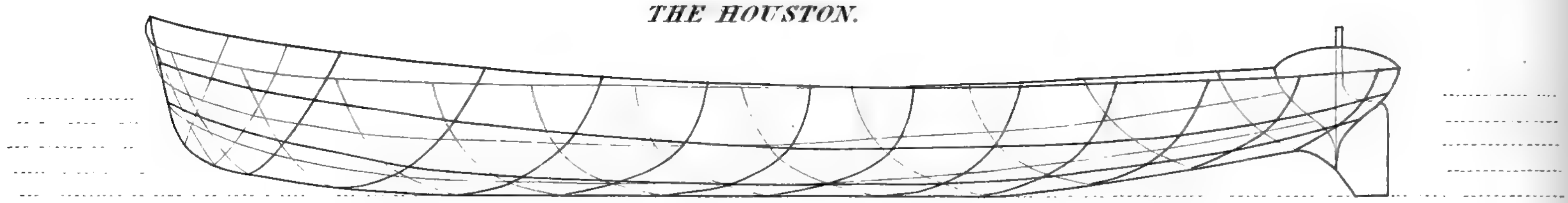
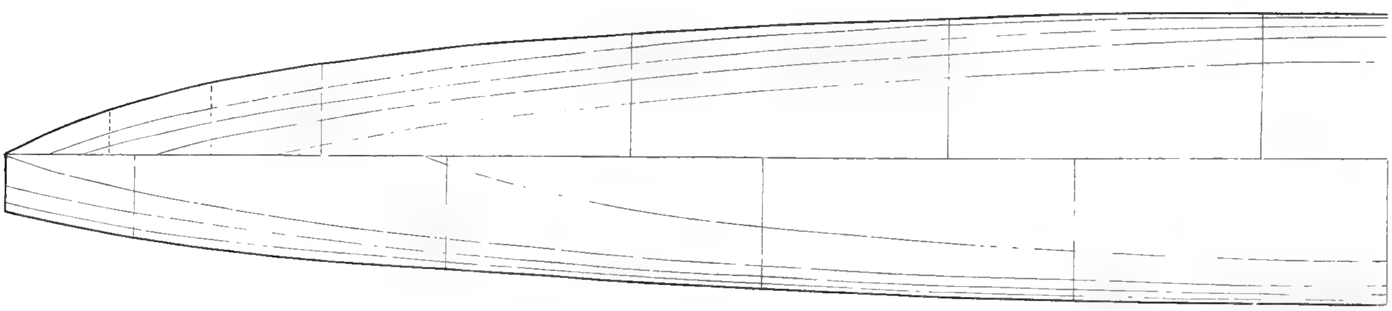


Fig. 3.

THE HOUSTON.





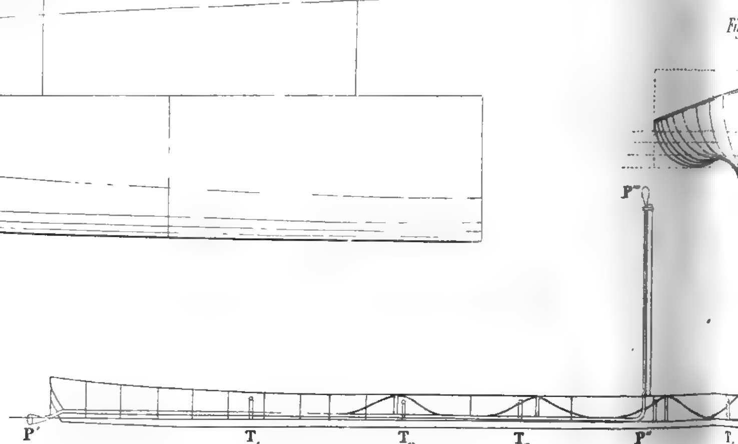


Fig. 5.

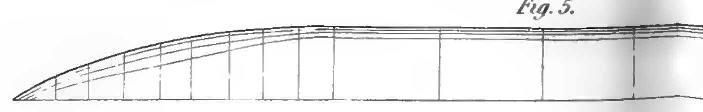
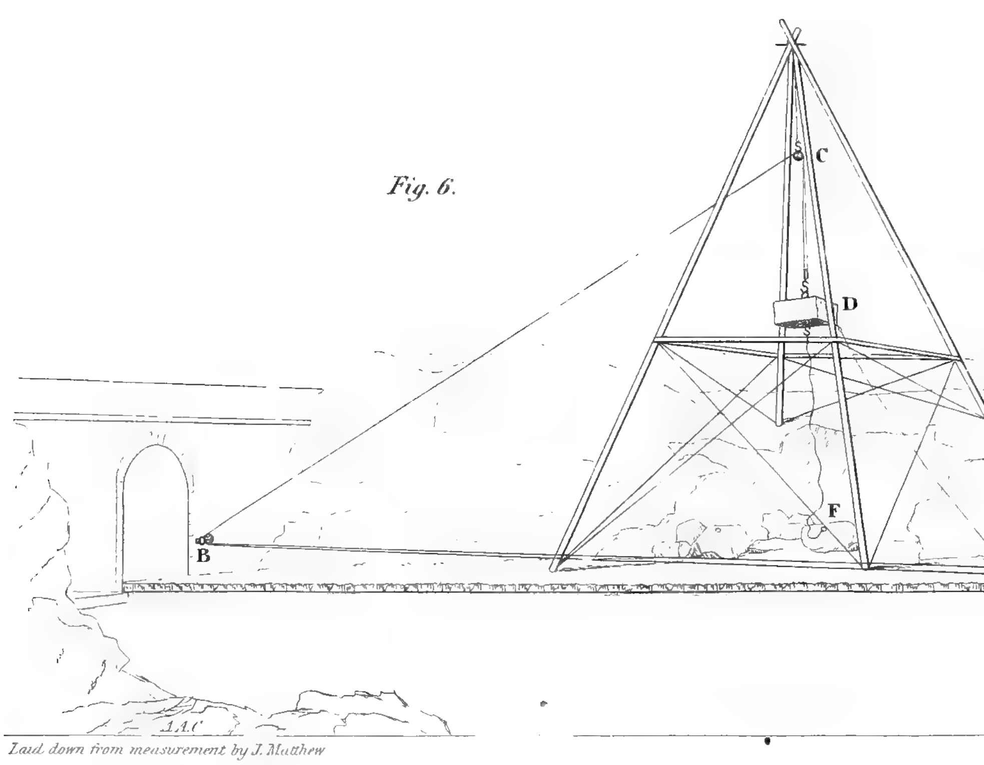


Fig. 6.



Laid down from measurement by J. Matthew

Fig. 2.

THE DIRLETON.

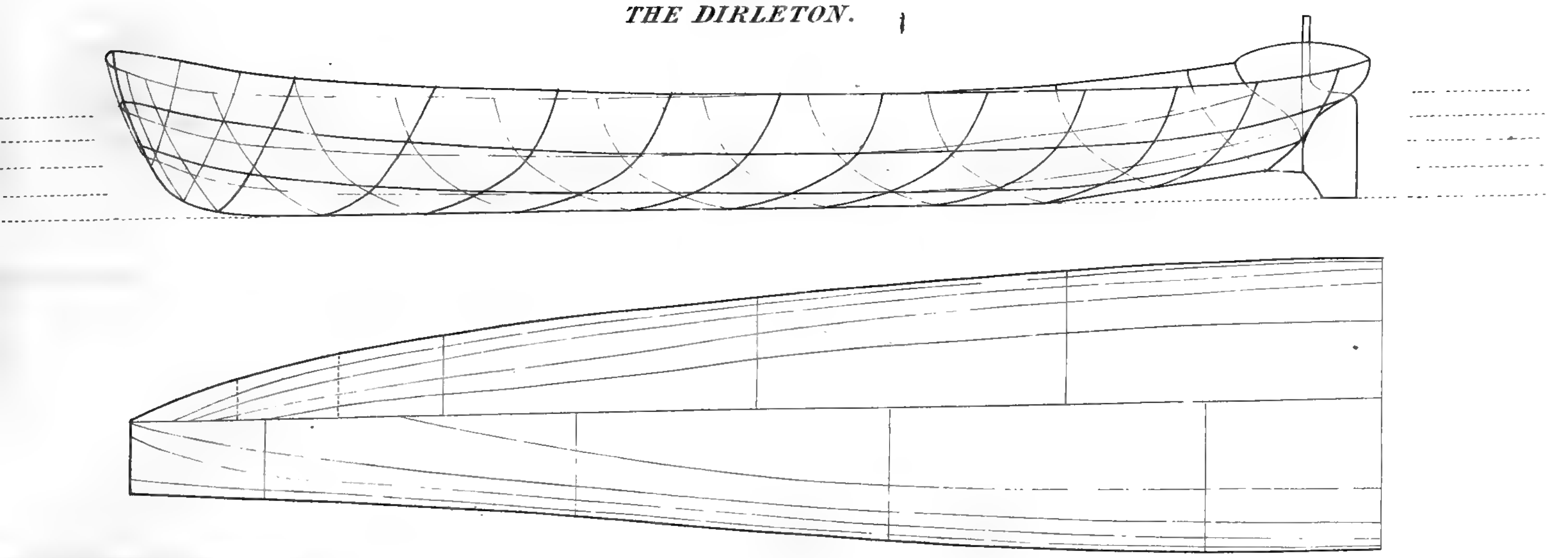


Fig. 4.

THE RAITH.

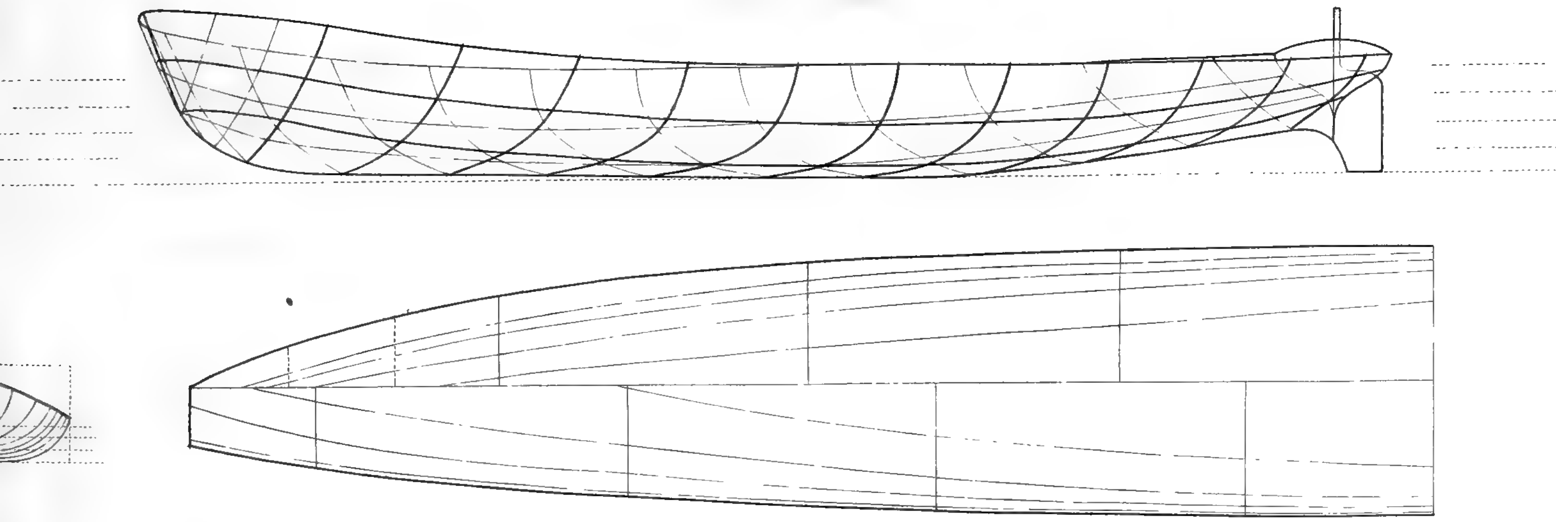


Fig. 5.

THE SKIFF.

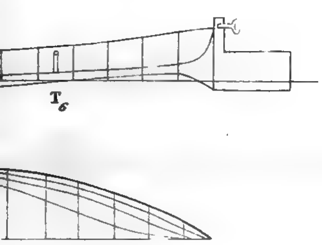
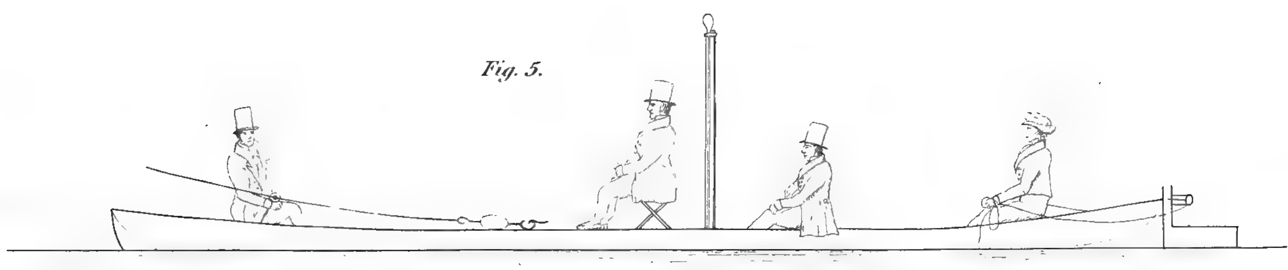
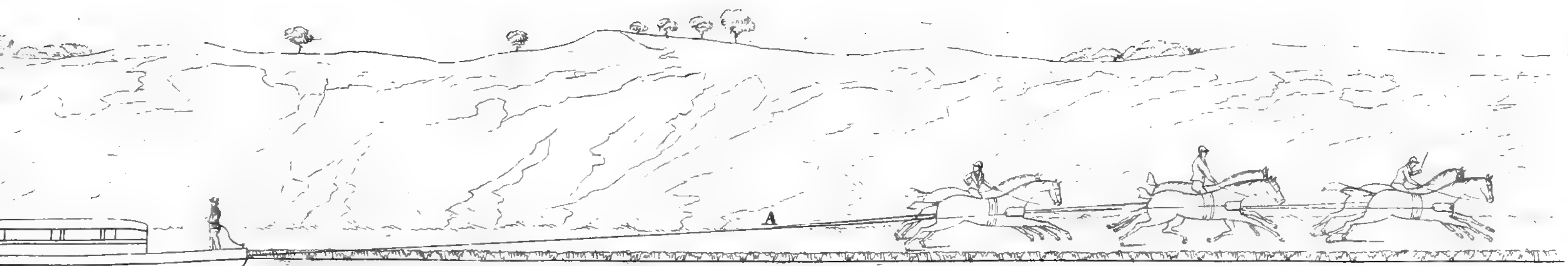


Fig. 5.





Engd. by W.H. Lizars

DESCRIPTION OF PLATES II. AND III.

PLATE II.

Fig. (1.) Represents the form given to the surface of a fluid by the motion of a floating body. The bed of the channel was nearly of the form given in Fig. F, which is a transverse section taken at right angles to the direction of motion of the floating body. The arrow at the stem of the vessel indicates the direction of the moving body, and on each side a dotted line shews the place of the fluid when at rest. The anterior wave at the bow of the vessel swells above and beyond the line of rest, the stern depression falls below and within it, the summits of the stern waves of replacement also protrude beyond and above it. The summits of the waves of unequal displacement, due to the improper form of the vessel, extend from it towards the banks, and give rise to undulations of the second order on the terminal line of the fluid.

Figs. (2.) (3.) and (4.) are the observed forms of the great primary wave of the fluid, in the channel of which the form is given in Fig. E, and in which the mean velocity of the wave is 8 miles an hour.

Figs. (5.) (6.) and (7.) are observed forms of compound waves which were afterwards analyzed, and gave the elementary and simple forms of Figs. (2.) (3.) and (4.) The outline represents the compound wave, the inner lines indicate the analysis.

Figs. A, B, C, D, E, and F. are sections of channels in which waves were propagated and other experiments made, and to which reference is made in the paper.

PLATE III.

Figs. (1.) (2.) (3.) and (4.) are projections of the forms of vessels made the subject of experiment They are simply fore-shortened, so as to diminish their length in the ratio of 3. : 1, or they are projected on an angle $\sin^{-1} = \frac{1}{3}$, so that the transverse sections are diminished in the ratio of the cosine of that angle, the dimension of depth remaining unchanged. The dotted lines at the sides are drawn for each six inches of immersion, so that a line may be drawn across the whole of each vessel at the depth of immersion given in the tables, for the purpose of shewing the parts of the vessel below and above the surface of the fluid. Below the projection are given the unprojected water lines for each six inches of immersion; the lines of the bow are placed above those of the stern.

Fig. (5.) consists of the transverse sections, longitudinal section, water lines, and elevation of the Experimental Skiff of 1834. P', P'' and P''', the position of the tube of Pitot. T_1, T_2, T_3, P_2, T_5, T_6, glass gauges of immersion.

Fig. (6.) Shews the improved mode of obtaining a continuous moving force as used in 1835. The power of horses is not applied directly to the object to be moved, but acts on the end of a rope at A, which rope extends directly from A to a fixed pulley at B, whence it passes to the summit of a pyramidal structure 75 feet high round another fixed pulley C, descends to a pulley at the weight D, and passing round it returns to a second pulley at C, descends once more to B, and is finally attached to a dynamometer at E, the bow of the vessel. The power of the horses is therefore used to sustain the weight, while its gravity overcomes the resistance of the fluid. An assistant at the foot of the pyramid prevents the weight from turning round, and a second weight F may be used for acceleration previous to the commencement of the observations.

IV.—*On the Action of Voltaic Electricity on Pyroxylic Spirit, and Solutions in Water, Alcohol, and Ether.* By Arthur Connell, Esq. F. R. S. Ed.

Read 20th March 1837.

The following paper contains a continuation of the experiments on the action of the voltaic pile on alcohol, and some other liquids, of which experiments a considerable number was described to the Royal Society in a former memoir.* At present it is intended, in the first place, to shew the perfect analogy between the electric action on pyroxylic spirit, and on alcohol, thereby confirming the interesting analogy already known to exist between these fluids in other respects: in the second place, to adduce a few farther illustrations of secondary voltaic actions in aqueous solutions; in the third place, to examine the nature of the changes produced in alcoholic solutions, under galvanic agency; in the fourth place, to inquire whether electric action does not throw light on the state in which the haloid salts are dissolved by water; and, lastly, to endeavour to suggest as a general law, regulating the electric decomposition of solutions of binary combinations of elementary substances in the principal solvents, that the dissolved body is not directly decomposed, but only the solvent, if itself an electrolyte.

I. *Voltaic Action on Pyroxylic Spirit.*

Previous to the examination of this liquid by MM. Dumas and Peligot, experiments had been made on it by several chemists, as by MM. Macaire and Marcet, Dr Thomson, and others. The gaseous hydrate of methylene of Dumas and Peligot, appears certainly to have been obtained, although in small quantity, by Macaire and Marcet, by distilling pyroxylic spirit, with three parts of sulphuric acid; but they mistook its nature, supposing it to be protocarburetted hydrogen, with a little hydrogen.† Still more important were the researches of Dr Thomson, which led him distinctly to infer the existence of the carbohydrogen CH^2, which may be viewed as substituted in the pyroxylic series for $C^2\ H^4$ in the alcohol series. By distilling a mixture of pyroxylic spirit and aqua regia, he

* On the action of voltaic electricity on alcohol, ether, and aqueous solutions. Edinburgh Transactions, vol. xiii. Part II.

† Bib. Univer. xxiv. 128.

obtained a new inflammable gas, which, after freeing it from nitrous and azotic gases with which it was mixed, he inferred from his experiments to consist of 1 vol. carbon vapour + 1 vol. hydrogen + $1\frac{1}{2}$ vol. chlorine; and to be a sesqui-chloride of the new carbo-hydrogen.* Although Dr Thomson, thus distinctly pointed out the existence of this carbo-hydrogen, he did not positively state that it existed in pyroxylic spirit, which indeed could hardly have been done at the time, as the existing analysis of the latter substance by Macaire and Marcet was inaccurate; but still he appears to have had some such idea in view.†

It is however undoubtedly to MM. Dumas and Peligot, that we are indebted for an accurate view of the nature and constitution of this liquid, and for the full development of the highly interesting series of relative substances. Amidst the multitude of detached and isolated facts, which organic chemistry at present offers, presenting little interest from analogies, or often supported by merely imaginary ones, the development of the pyroxylic series, by unfolding a beautiful analogy with alcohol and the ethers, possesses with some other examples, a high degree of importance.

It was to be expected that a similar connection should be observed between pyroxylic spirit and alcohol, in their galvanic relations as in their composition and general properties; and experiment fully established this farther analogy.

The pyroxylic spirit employed, was obtained from Glasgow, and for a commercial article possessed a high degree of purity. Its colour had a slight yellow tinge. Its specific gravity and boiling point, (taken after it had been a year in my possession) were, the former .851 at 62° F., the latter, in contact with mercury, 160° F., under 30 inches of pressure. Treated with slaked lime there was no evolution of ammonia either in the cold or by heat. By a single distillation from powdered quicklime, it became quite colourless and transparent. After a second distillation from quicklime, its specific gravity was .808 at 62° F. A third distillation from recently ignited and powdered lime brought it down to .801 at 62° F. Its boiling point in contact with mercury was then found to be 148°, under a pressure of 29.5 inches.‡ It was with the product of this last distillation, that the galvanic experiments were made.

* Transactions of the Royal Society of Edinburgh, vol. xi. p. 15.

† See Inorganic Chemistry, vol. ii. p. 295.

‡ The specific gravity of absolute Pyroxylic spirit is given by MM. Dumas and Peligot as .798 at 68° F., and its boiling point 151.7° F. at 30 inches of the barometer. In the case of alcohol, the difference of temperature at which the specific gravity was taken by the French chemists and myself, would very nearly account for the difference of specific gravity, and I did not conceive it of any moment to attempt to rectify the pyroxylic spirit more highly, because in regard to alcohol I had found that far greater differences in the density had no material influence on the voltaic results; and the analogy of the action on the two fluids was complete, as will soon appear.

In examining the nature of the voltaic action on pyroxylic spirit, it was unnecessary to go into the same minuteness of investigation as in regard to alcohol for the analogy of the two cases became immediately apparent, and all that then was necessary was to seize the leading points of resemblance. It will be proper briefly to recapitulate the principal facts which I had observed in regard to alcohol, and the conclusions which I had drawn from them.

1*st*, It was found, that, under powerful voltaic agency, absolute alcohol yielded hydrogen from the negative pole, and no elastic fluid from the positive.

2*d*, By dissolving minute quantities of certain acid, alcaline and saline bodies in absolute alcohol, this voltaic agency was greatly favoured, through an increase of the conducting power of the liquid; $\frac{1}{10,000}$th part of potash having a marked effect.

3*d*, By particular arrangements, as by operating in metallic vessels, elastic fluid also appeared at the positive pole.

4*th*, The quantity of hydrogen obtained at the negative pole was found to be the same as that given off from water, under the influence of the same electric current.

5*th*, Besides elastic fluid, there were formed in the liquid acted on, certain products, the same as, or analogous to, those often resulting from the oxidation of alcohol; such as resinous matter, carbonic acid which combined with the dissolved potash, &c.

6*th*, From these various facts it was concluded, that water contained in the alcohol was the immediate subject of the voltaic agency, its hydrogen being evolved at the proper pole, and its oxygen being engaged in giving rise, by a secondary action, to the products of oxidation, dissolved in, or precipitated from, the liquid.

7*th*, As a general inference from the whole, it was farther concluded that, as the phenomena were obtained with absolute alcohol, that fluid must necessarily contain water as an essential constituent; a view which, although previously very generally adopted, had not, it was conceived, hitherto received any direct experimental proof.

Such were the leading facts and conclusions which it will be easy to shew are all equally applicable to pyroxylic spirit. In these investigations, it was found, that less powerful currents are capable of producing the same effects on pyroxylic spirit as on alcohol; a circumstance probably due to the greater absolute quantity, although not greater atomic proportion, of water, in a given weight of the former of these liquids.

A little more than a dram of the rectified pyroxylic spirit was exposed in a tube,* with parallel platinum-foil poles, and adapted for collecting evolved elas-

* See fig. 2 of plate in former memoir. Ed. Trans. xiii. pl. xiii.

tic fluid, to the action of seventy-two pairs of 4-inch plates.* In a few minutes elastic fluid began to be evolved, and was collected over mercury; and the liquid after a time became warm, but did not boil. In order to favour the action, the foils were only about $\frac{1}{20}$th of an inch apart; which circumstance made it difficult to say with certainty, from the appearance alone, from which pole the gas came; but from the nature of the elastic fluid, as afterwards determined, as well as from the subsequent experiments, and the analogy of alcohol, little doubt could exist that it proceeded from the negative foil. After one and a-half hour's action, about one-third of a cubic inch was obtained, and, from the diminished action of the battery, the flow was a good deal slackened. This gas was analyzed in the voltaic eudiometer, and was found, as in the case of alcohol, to be hydrogen mixed with a little impurity, which was partly common air or its constituents, and partly a trace of vapour of the liquid acted on.

The pyroxylic spirit which had been acted on, when mixed with water and evaporated, shewed a little whitish matter mixed with it, and afforded a peculiar smell; and when the evaporation was carried to dryness, some yellowish-white resinous matter was left.

A minute quantity of pure caustic potash, when dissolved in the liquid, had, as in the case of alcohol, a wonderful effect in promoting the voltaic action. A similar quantity of the spirit, as in the last experiment, containing in solution $\frac{1}{350}$ of pure caustic potash, was acted on in the same apparatus by thirty-six pairs of 4-inch plates, the platinum foil poles being parallel to one another, and from one-eighth to one-tenth of an inch apart. Elastic fluid was immediately evolved, and from the greater distance of the foils it was easy to see that it proceeded entirely from the negative pole. The action was so intense that the liquid soon boiled. A cubic inch of permanently elastic fluid was collected over mercury in a quarter of an hour; and when two cubic inches had been obtained, the process was stopped, although gas was still coming over. A portion of this gas was analyzed as before, and found to be hydrogen in a state of nearly perfect purity.

The liquid during the action did not perceptibly change in colour. A little flocky matter had precipitated, but it did not appear to be carbonate of potash. Some of the liquid was mixed with water, and after being a good deal concentrated by heat, it became slightly muddy, and a pungent aromatic smell arose, and some brownish matter was left on evaporating to dryness.

When a little of the spirit containing about $\frac{1}{600}$ of potash was acted on in a watch-glass by fifty pairs of 2-inch plates, the platinum-foil poles being simply approached to one another horizontally, elastic fluid was evolved, as in the preceding experiment, from the negative pole, and none from the positive foil; but

* All the batteries employed in the experiments in this paper were, as formerly, on Cruickshanks' construction.

when a platinum capsule was substituted for the watch-glass, gas arose from both poles, as had also been observed in the case of alcohol, holding a similar minute quantity of potash in solution.

A small quantity of chloride of calcium, when dissolved in the spirit, had also the effect of increasing the action, gas appearing at the negative pole, and none at the positive in glass vessels.

The most minute quantity of potash which could be employed was found to have the effect of increasing the action. When only $\frac{1}{10000}$th part was dissolved, the evolution of elastic fluid at the negative pole could be distinctly observed in a watch-glass with fifty pairs of 2-inch plates; the pure spirit itself, under such circumstances, scarcely shewing the slightest action.*

Although no distinct formation of carbonate of potash was observed when small quantities only of potash were held in solution, the case was different when a strong solution of the alkali in pyroxylic spirit was acted on. A small quantity of such a solution was exposed to the agency of thirty-six pairs of 4-inch plates in a tube, with parallel platinum-foil poles placed at the distance from one another of about $\frac{1}{10}$th of an inch. In this case a copious evolution of elastic fluid took place from the negative pole as usual; but gas also arose, although in less quantity, from the positive, owing to the greatly increased action from the concentration of the liquid, and also to the *now* notable quantity of water in the hydrate of potash dissolved. The liquid boiled in a few minutes, and soon acquired a red colour; and a good deal of white matter was deposited, which proved to be carbonate of potash. The red liquid acquired a strong peculiar odour, and when mixed with water became muddy, and got a yellow tint, evidently from the separation of oily or etherial matter which had been formed during the action.

The true nature of the voltaic action in all the experiments which have been detailed appears to be sufficiently obvious. Water is decomposed, as was the case when alcohol was employed instead of pyroxylic spirit. Its hydrogen is evolved at the negative pole, whilst its oxygen is employed in giving rise by a secondary action to the formation of small quantities of resinous, oily, or etherial matter, and also carbonic acid when the action is energetic. In the last described experiment, the quantity of as generated at the positive pole being larger than in the

* If any one should imagine that the water of the hydrate of potash employed has any effect on these experiments, he is at liberty to calculate the quantity of water in $\frac{1}{10000}$th part of potash, held in solution by a few drops of spirit contained in a watch-glass. gain, the quantity of spirit acted on in the experiment in the preceding page, contained .16 of a grain of hydrate of potash, which contains .03 of water, equivalent to .154 of a cubic inch of hydrogen. But above two cubic inches of hydrogen were collected, and the process was stopped while the evolution was going on. Similar observations apply to the experiments with alcohol. The true action of the potash in these cases is just the same as when it is dissolved in water itself. It increases the conducting power of the liquid, aided, in the case of alcohol and pyroxylic spirit, by a circumstance to be noticed immediately.

others, the excess beyond what was required for the secondary action was liberated. This fact was more particularly determined in regard to a strong alcoholic solution of potash, as will be afterwards noticed, and the analogy was sufficiently obvious. The resinous, oily, or etherial matters were never found in sufficient quantity to admit a more particular examination of them; an observation which applies equally to the formerly detailed experiments with alcohol.

The extraordinary extent to which dissolved potash promotes the voltaic action on alcohol and pyroxylic spirit, appears to be in part due to its *disposing* affinity for the resinous or acid secondary products.

As a farther proof that water was the true subject of the direct voltaic action, an experiment was made with the volta-electrometer, as had been done in the case of alcohol. The current from thirty-six pairs of 4 inch plates was passed through pyroxylic spirit containing $\frac{1}{500}$ part of potash dissolved, and also through water containing the same quantity of potash, in the apparatus fig. 6. of former memoir, the pyroxylic solution being placed in the bent tube sealed at the negative end. Gas was evolved from all the poles except the positive of the spirit solution, and at the end of 1 hour 5^m there was found in

N of the pyroxylic solution	.10	cub. in.
N of the aqueous solution	.12	— —
P of the aqueous solution	.05	— —

Thus the quantities of hydrogen evolved from the two negative poles were sufficiently similar in amount to confirm the view, that water in both cases was the subject of decomposition.

It being thus in the whole circumstances sufficiently clear, that when pyroxylic spirit is submitted to voltaic agency, water contained in the liquid is resolved into its elements by the direct operation of the current, it is conceived that experimental proof is thus afforded that pyroxylic spirit, like alcohol, contains water as an essential constituent. When allowance is made for the difference of temperature at which the specific gravity of the spirit was taken by MM. Dumas and Peligot and myself, the observed densities probably hardly differ; and no material variation on the nature of the action occurred in the case of alcohol, under much more considerable diversities of specific gravity.

Since the substance in the pyroxylic series, corresponding to ether in the alcohol series, bears the gaseous form, I did not attempt to submit it to voltaic action; but I can hardly doubt that, had it been a liquid, the same analogy would have been shewn in its electric relations, with respect to ether, as pyroxylic spirit exhibited in regard to alcohol. Following out the general analogy between the two series, I am inclined to adopt the same view in regard to the constitution of the two liquids of the one series as with respect to those of the other; and as it appears to be sufficiently proved that pyroxylic spirit is a hydrate, it may be regarded as a hydrate of pyroxylic ether, in the same way that alcohol was viewed as a hydrate of sulphuric ether. In my former experiments, I had found that sulphuric ether resisted the action of the most powerful voltaic battery which I had at my command, nor could I discover any substance which, when dissolved, led under voltaic agency to any appearance which countenanced the idea that it contained water, as such, as a constituent; and I therefore concluded that water, as such, did not enter into its constitution. On those views, the formula of pyroxylic ether will be $H^6 C^2 O$, and that of pyroxylic spirit $H^6 C^2 O + H^2 O$. In this way no hypothetical radicle is assumed as the basis of which pyroxylic ether is considered as the oxide, the latter being viewed, like sulphuric ether, merely as a ternary combination of its constituent elements.* Neither am I acquainted with any experiment which proves the existence of the hydro-carbon $H^2 C$, as such, in the pyroxylic series, any more than that of the hydro-carbon $H^4 C^2$ in the alcohol series. Indeed, the former hydro-carbon has not yet been obtained in a separate form in a state of purity, so that even its own existence is still in some measure hypothetical. It may be questioned, therefore, whether we have really made much greater progress towards a knowledge of the existence of the hydro-carbon $H^2 C$ in combination as such, in consequence of the discovery of the pyroxylic combinations, than when we only knew of the sesqui-chloride of Dr Thomson. If we content ourselves with saying, that throughout the pyroxylic series certain elements are substituted for certain other elements throughout the alcohol series, we do little more than express a matter of fact, with scarcely any theory. The elements so substituted appear to me to be 6 atoms of hydrogen and 2 atoms of carbon in the pyroxylic series, for 10 atoms of hydrogen and 4 atoms of carbon in the alcohol compounds, and the analogy between the two series appears to be nearly as well preserved on this view as on any other. I have in contemplation some voltaic experiments on the compound ethers of both series, which may possibly throw some light on the nature of the combinations.

II.—*Voltaic Action on Aqueous Solutions.*

In my former paper I endeavoured to show that an electric current of sufficient intensity to decompose distilled water, did not cause the appearance of chlorine or iodine in aqueous solutions of the corresponding hydracids and haloid salts, where the evolution of oxygen at the positive pole did not actually take place in the solution, but in distilled water connected with that solution by as-

* Even should we assume that pyroxylic ether and sulphuric ether unite with acids after the manner of *bases*, this circumstance will not, I conceive, prove them to be oxides, consisting of radicles *as such* and oxygen, any more than the same circumstance proves the vegetable alkalies, although undoubtedly *bases*, to be *oxides*, or than the circumstance that the vegetable acids unite with alkalies, shews that they consist of radicles and of oxygen.

bestus; and that chlorine or iodine only appeared after a considerable time in the positive water, when acid had passed over into it, so as to afford room for a secondary action.* These experiments were made with such moderate voltaic powers, as 50 pairs of 2 inch plates, but still sufficiently energetic to decompose distilled water, and, therefore, far more capable of producing the ordinary appearances in solutions, of hydracids and haloid salts, when both poles were placed in the solution: and I have since had occasion fully to confirm the results, with stronger powers. The consequence of employing more powerful batteries is just what might have been anticipated. The chlorine, and particularly the iodine, make their appearance sooner, and why? because acid is sooner carried over into the positively electrified distilled water, as shown by test-paper, and because the reducing energy of the battery from evolved oxygen is increased.

Thus, when muriatic acid diluted with between twice and thrice its bulk of water was placed in a tube A, Fig. 1, Plate II, of the capacity of $1\frac{1}{2}$ dram, connected with the negative side of a battery of 72 pairs of 4 inch plates, and distilled water in a similar tube B, connected with the positive side, the tubes being connected with one another by a bundle of asbestus about $\frac{1}{4}$th inch thick, acid was detected at the positive pole within three or four minutes, with effervescence from both poles, and in six minutes a very doubtful trace of the smell of chlorine was discernible, but when test-paper was dipped into the liquids, no trace of bleaching was observed. After half an hour's action, the smell of chlorine in B was still slight, and the liquid in it showed an acid reaction, but no bleaching; whilst the liquid of the other tube neither had any smell of chlorine, nor did it bleach. The battery was now reversed without replenishing it; the platinum foil, which was in the water, and had formerly been positive, being now connected with the negative side of the battery, and the foil in the muriatic acid being now made the positive pole. An instant pungent smell of chlorine arose from the now positive tube, with brisk effervescence from the negative pole, and rather less from the positive; and test-paper was bleached at the positive pole as soon as the reaction was tried, which was in less than one minute.

When a moderately strong solution of hydriodic acid was next substituted for the muriatic acid, all other circumstances being exactly the same as in the commencement of the preceding experiment, and the voltaic power being the same and in fresh action, a commencement of browning, as from the formation of iodine, was observed, in about five minutes, in the liquid B, with effervescence from both poles, and at the same time a slight acid reaction was observed on the asbestus close to the same place. This browning went on increasing, and the acid reaction became quite obvious at the positive pole, the effervescence still continuing there, although considerably less than at the negative pole. In about

* Edinr. Trans. xiii., 339, et seq.

twenty minutes the battery was reversed as before described; instantly the positive foil was covered with red matter, without any evolution of gas from that pole, and dense red liquid continued to fall from it, a brisk effervescence going on at the same time at the negative pole.

In these experiments it is evident that the appearance of chlorine or iodine at the positive pole before reversal, is dependent on acid passing over into the positive water; and the appearance is sooner observed and much more marked in the case of iodine than in that of chlorine, because hydriodic acid is a much weaker and more easily reduceable combination than muriatic acid, although the reaction of the latter acid on the positive side is much more marked than that of the former. On reversal, chlorine or iodine alone appears at the positive pole, and that instantly, the oxygen being entirely employed in reducing the corresponding acid. In the experiments formerly detailed, when weaker powers were employed, the chlorine or iodine was much longer of appearing previous to reversal, just because acid was longer of being carried over in sufficient quantity to make its secondary decomposition visible under the less energetic oxidating agency. In this point of view moderate powers are perhaps best calculated for such experiments, because the apparent contrast between the results at the positive pole before and after reversal is more striking, although the appearances with more powerful batteries are, on a very slight reflection, equally indicative of a secondary action.

The experiment with hydriodic acid was varied by connecting two glass-cups, of the capacity of a quarter of an ounce containing water, with the tube A of $1\frac{1}{2}$ dram measure containing the acid, the acid being made negative by a battery of 72 pairs of 4 inch plates, and one of the water-glasses C positive; the other B, being intermediate, and all the three vessels being connected by asbestus, as in Fig. 2. Slight effervescence was observed at both poles, in one or two minutes. During the first fifty minutes not the least discoloration of any of the liquids was observed. A few minutes afterwards the positive liquid in C began to acquire a very slight brown tint, with slight acid reaction at the positive pole; and in ten minutes more the brown tint throughout the liquid in C was quite decided, without the slightest discoloration of that of B or A; and acid was also observed on the asbestus between B and C. The battery was then reversed, when the usual instant discoloration ensued at the positive pole without effervescence, while gas arose from the negative. Here, again, the iodine which appeared in the positive liquid before reversal, evidently owed its origin to a secondary action on the acid, which had travelled to the positive pole through the liquid in B.

A moderately strong solution of chloride of potassium was now placed in A, fig. 1, connected with the negative side of seventy-two pairs of 4-inch plates, and distilled water in B, connected with the positive side, asbestus being interposed as usual. In two or three minutes acid appeared at the positive pole, and near the positive extremity of the asbestus, with effervescence from both poles; and in about eight minutes a slight odour of chlorine was observed, and also acid reaction at the positive side of the liquid in A. In a quarter of an hour test-paper was not bleached when dipped into either tube. After upwards of twenty minutes, no smell of chlorine could be distinguished in A, and when the action was then suspended the odour in B was still only slight. The battery was now reversed as before described. An instant smell of chlorine arose, with effervescence from the positive pole, and test-paper was bleached there within one minute. Effervescence also from the negative pole.

A moderately strong solution of iodide of potassium was now substituted for chloride of potassium, all other circumstances, including the voltaic power, being the same as at the commencement of the preceding experiment. In five minutes there was acid reaction at the positive side of the liquid in A; and about the same time browning was observed to be just beginning in the positive liquid near the termination of the asbestus, and extending to the positive foil. This browning went on increasing in the positive liquid, with acid reaction on the neighbouring asbestus and positive side of the negative liquid. The colour of the negative liquid was not changed. In this particular experiment the battery was not reversed; but in numerous others with the smaller powers, it was always found that the reversal caused immediate production of iodine at the positive pole without effervescence, whilst gas arose from the negative.

These results are quite conformable to those with the hydracids. Chlorine or iodine appears in virtue of acid passing to the positive side and suffering a secondary action. The degree of observed secondary action is proportional to the facility with which the corresponding acid is decomposed by nascent oxygen, and not to the absolute quantity of acid which appears at the positive side, the secondary action being strongest in the case of iodide of potassium, whilst the quantity of acid on the positive side is much smaller than in the case of chloride of potassium. I shall afterwards describe an experiment similar to that with hydriodic acid and two other vessels of water, in confirmation of these views. I shall then also state the grounds which have led me to infer from the appearances under galvanic agency, that haloid salts do not exist in solution as such, but as *hydracid salts;* in other words, that when dissolved they decompose water.

It was shewn a few years ago by M. de la Rive, that when a mixed solution of bromide of iodine and starch in water was acted on voltaically, iodine appeared at the positive pole, and formed the usual blue combination with the dissolved starch.* By an experiment conducted on similar principles with those just described, I have been led to conclude, that the action is here also a secondary one. The mixed solution placed in a tube was connected with the positive side of fifty

* Annales de Chim. et de Phys. xxviii. p. 160.

pairs of 2-inch plates, and a solution of starch in another similar tube was connected with the negative side, asbestus intervening as usual. Effervescence speedily ensued from both poles, but after forty minutes' action not a trace of any blue colour was observed in either tube. The battery was then reversed. Within two minutes, the blue combination appeared round the negative foil now in the mixed solution, the effervescence ceasing at that pole, but continuing at the positive pole. Had the bromide been directly decomposed, iodine ought to have been liberated, and the blue colour produced in one or other of the tubes before reversal; but as this change did not occur till after reversal, the effect was due to nascent hydrogen at the negative pole. This hydrogen must have combined with bromine if the bromide is dissolved as such, as is usually held, or with oxygen if the whole or a part of it decomposes water, forming hydrobromic and iodic acids.* It is plain, however, that the experiment is equally effectual on this view as on any other for the purpose to which M. DE LA RIVE applied it, that of detecting iodine in bromine. It does not, however, prove that bromine when in combination with iodine is carried to the positive pole; but on passing the current from thirty-six pairs of 4-inch plates through liquid bromide of iodine, neither water nor starch being present, I found the galvanometer to be decidedly, although not powerfully, affected; and although in the course of a few minutes' action I could not notice the appearance of either bromine or iodine at the respective poles, yet the quantities carried to the poles may have been too minute for observation, or they may have been redissolved by the bromide as soon as carried to the extremities.

III.—*Voltaic Action on Alcoholic Solutions.*

In my former voltaic experiments on alcohol, the object was merely to promote the action by dissolving such minute quantities of different substances as served to increase the conducting power of the liquid. At present, it is intended to examine the nature of the changes produced by electric agency on alcoholic solutions of greater strength.

The appearances presented by solutions of acid, alkaline, and saline substances in alcohol under voltaic action have, generally speaking, a great resemblance to those offered by the corresponding aqueous solutions; and when we consider that water as such enters into the constitution of alcohol, and suffers its ordinary electric decomposition, it is not surprising that this resemblance in the phenomena should take place; the principal difference being, that oxygen is hardly ever

* I am quite aware that when both the poles are introduced directly into the mixed solution, the voltaic power being in fresh action, there is effervescence at both poles, along with the appearance of iodine at the negative; but in this case I apprehend that a *part*, although not the whole, of the hydrogen enters into the new combination.

evolved at its proper pole, being employed in there producing secondary effects either on the solvent or the dissolved body.

When an ordinary oxy-acid salt, of a powerful base, such as nitrate of lime, is dissolved in absolute alcohol, the acid and base go to their proper poles under voltaic agency, as in a similar aqueous solution, but much more slowly, effervescence taking place at the negative pole, and little or none at the positive. Where the base is not of difficult reduction, as in nitrate of zinc, the evolution of gas at the negative pole is diminished, and metal reduced by hydrogen separates at that pole, mixed with oxide.

When an acid whose elements are strongly united, as boracic acid, or an alcali, as potash, is held in solution, effervescence appears at the negative pole, but none at the positive, unless in the case of a strong solution of a hydrated alcali, when a slight evolution of gas also occurs at the positive pole; and in these cases, appearances do not indicate any decomposition of the dissolved body. When an alcoholic solution of a haloid salt is acted on, no gas is evolved from the positive pole; but in the case of an iodide there is immediate separation of iodine at that pole, which is dissolved by the solution, giving it a deep red colour. When the metal of the haloid salt is one of powerful affinities, as potassium, calcium, or magnesium, there is brisk evolution of hydrogen, and more or less appearance of the oxide of the metal at the negative pole; metal appearing in that case to be first reduced by a portion of the nascent hydrogen, which combines with the electro-negative element of the haloid, and then to react on the water of the alcohol; and a portion of the oxide, when it is soluble in alcohol, being also drawn from the positive pole, where it has been formed by another secondary action. Where the metal is of more easy reduction, as in the case of zinc, the effervescence at the negative is diminished, although it does not cease, and metal separates there, apparently in consequence of a part of the hydrogen combining with the electro-negative constituent of the haloid salt.

When absolute alcohol, holding in solution chloride of magnesium, prepared by Liebig's process, is acted on in a close tube, magnesia separates, after a few hours' action, as a transparent and colourless crystalline layer, covering the negative platinum-foil, and much resembling the native hydrate of magnesia.*

I formerly stated the changes produced by voltaic agency on alcohol, containing very small quantities of potash in solution; I shall now shew the action on a strong solution.

Rather more than an ounce measure of a saturated solution of hydrate of potash in alcohol was submitted to the action of thirty-six pairs of 4-inch plates, by parallel platinum-foil poles, placed at the distance of one-sixth or one-seventh

* When this transparent substance was heated, it gave off moisture, so that it appeared either to be a hydrate or an alcoate of magnesia, but the experiment had afforded too little to determine this point.

of an inch from one another. A brisk effervescence ensued from the negative pole, and considerably less from the positive, and the mixed gases were collected over mercury. After twenty to thirty minutes' action, the liquid began to deepen in colour, and when the foils were examined after nearly an hour's action, the positive foil was found to be fringed with white matter, which was afterwards ascertained to be carbonate of potash. In seven and a-half hours, after which time a moderate effervescence was still going on from the negative pole, and none from the positive, the liquid had acquired the colour of Port wine, and after twenty-two hours' action, when a feeble effervescence was still observed, the colour had become a very dark red; and carbonate of potash was collected at the bottom of the liquid, having doubtless gradually fallen from the positive foil, which was found to be still fringed with that salt. When the red solution was evaporated nearly to dryness, re-dissolved in water, and saturated with muriatic acid, an abundant precipitation of resinous matter ensued.*

The hydrogen which was collected during the second quarter of an hour of the preceding experiment, was found to contain about $\frac{1}{20}$ of oxygen, which had come from the positive pole, the difference between this proportion and that in water, having been employed in producing the secondary action, from which the resinous matter resulted.

I formerly shewed that when the same electric current was passed through absolute alcohol containing a small quantity of potash, iodide of potassium, or chloride of calcium, and through water either containing the same proportion of the same substance, or simply acidulated with sulphuric acid, the quantity of hydrogen evolved at the negative pole from both solutions was the same.† I

* It would appear that Döbereiner had observed the formation of resinous matter in small quantities, in a galvanized solution of potash in alcohol (Pog. Annal. xxiv. 609), but he says nothing of any evolution of elastic fluid at either pole; and although he regarded the formation of resinous matter as an effect of oxidation, he gives no more explicit opinion as to the source of the oxygen or nature of the action. On the other hand, M. Lüdersdorf (*Ib.* xix. 77), like Dr Ritchie, had observed that absolute alcohol, holding nothing in solution, gave off, under strong voltaic agency, elastic fluid from the negative pole; but he did not state that it was hydrogen, and, on the contrary, seems to have thought that it was not hydrogen, from the colour of its flame. I have found that the hydrogen evolved from pyroxylic spirit under electric action, when it contained a little of the vapour of the spirit mixed with it, burned with a blue flame, but when freed from that vapour, by being washed with solution of potash, it burned with a pale whitish flame. In analyzing, by the voltaic eudiometer, the gases obtained in such experiments, deceptive appearances, if we are not on our guard, may arise from the production of small quantities of carbonic acid, proceeding from the presence of vapour of the spirit which has passed over. I had read both Döbereiner's and Lüdersdorf's observations, when first published, but in the two or three intervening years they had escaped my memory, until again recalled to it by allusions to them which I met with in the course of my reading, subsequent to the publication of my former paper; and even if I had remembered them at an earlier period, they could not have superseded any part of my researches.

† Edinb. Trans. xiii. 327, et seq.

have since compared, in the same way, some other alcoholic and aqueous solutions. Thus, absolute alcohol, containing $\frac{1}{20}$ of dry nitrate of lime, was placed in the bent tube A, Fig. 6, of former memoir, and water acidulated with $\frac{1}{12}$ of sulphuric acid in the tubes and evaporating basin B, the positive pole of the one solution, and the negative of the other, being in metallic connection. The current from thirty-six pairs of 4-inch plates was passed through both solutions, and in an hour and forty minutes there was collected from the negative pole of the alcoholic solution .0375 of a cubic inch of hydrogen, and .039 from the negative pole of the aqueous solution, lime appearing at the same time at the negative pole of the alcoholic solution, and acid at the positive. The quantities of gas were thus very small from the feeble conducting power of the solution, but sufficiently similar in amount to shew that in both solutions water had been decomposed. The same experiment was now made, substituting an alcoholic solution of $\frac{1}{40}$ of boracic acid for the nitrate of lime solution, all other circumstances, including the voltaic power, being the same. The conducting power of this alcoholic solution was still more feeble than that of the other, insomuch so, that for some time I thought there would not have been any sufficient action to afford room for a comparison; although when an alcoholic solution of boracic acid is acted on in a tube with parallel platinum foil poles, the action is immediately seen. In three hours the charge of the battery was renewed, and the whole left for eighteen hours farther. No deposit was, during the whole time, formed on either foil in the boracic solution, nor was any gas evolved from the positive pole in it. At the end of the above mentioned time, there was collected from the negative pole in the alcoholic solution .025 of a cubic inch, and from that in the aqueous solution .035. Thus this result, from the very feeble conducting power of the solution, was much less regular than in any former trial; but still I think it will be admitted that it at least does not interfere with the conclusion, that in this case, as was evident in all the other cases, water was the subject of the voltaic agency; and there were no other appearances which indicated that boracic acid had been decomposed.

It will I hope be granted from the various phenomena, which have been described now and formerly, that when alcoholic solutions of acids, alkalies, and oxyacid salts are submitted to voltaic agency, the water of the alcohol is the subject of direct electric action, and that the dissolved body, with the exception of oxyacid salts, is not decomposed. In regard to alcoholic solutions of haloid salts, however, it might perhaps be held from the electro-negative constituent actually appearing, at least in the case of iodides, at the positive pole, that it is really the haloid salt which is directly decomposed, and that the definite quantity of hydrogen at the negative pole, arises from the reaction of the metal of the decomposed haloid, on the constituent water of the alcohol; a view which, of course, would afford equally satisfactory evidence as the other, of the existence of water as such in absolute alcohol. I have now, however, to describe a variety of experiments analogous to those made with aqueous solutions of haloid salts, from which I conceive it follows that water is directly decomposed in alcoholic solutions of such bodies, as well as in those of other substances, and that the appearance of iodine at the positive pole in such solutions, is a secondary effect; and I hope I shall be pardoned for some minuteness of detail, with a view to the general conclusion alluded to in the commencement of the paper.

Absolute alcohol containing in solution as much dry and pounded iodide of potassium as it took up in three quarters of an hour's digestion, at a temperature of about 130°, which was about $\frac{1}{40}$th, was placed, when cold, in a glass tube A fig. 1. of the capacity of $1\frac{1}{2}$ dram, connected with the negative side of a battery of fifty pairs of 2-inch plates; and distilled water in a tube B of similar capacity connected with the positive side, a bunch of asbestus of about $\frac{1}{4}$th inch thick, and moistened with alcohol, being interposed between the tubes. In one or two minutes gas began to be evolved from both poles. In five minutes a little brown matter like iodine began to be deposited in the positive water, in the immediate neighbourhood of the positive pole; and at the same moment an acid reaction was observed on the asbestus at its extremity on the positive side, and an alkaline at the negative pole, as well as on the positive side of the negative liquid. Both the browning and the acid reaction went on encreasing; as well as the effervescence at the negative pole, that at the positive continuing but not encreasing.* In about half an hour, in which time the positive liquid had become pretty brown, whilst the negative was not at all discoloured, the battery was reversed as before described, and without renewing the charge. The positive pole now in the alcoholic solution was immediately covered with reddish brown matter, and a red liquid continued to fall from it, there being no evolution of gas from that pole, but an effervescence from the negative, and in a few minutes alkali was detected at the negative pole.

In some previous experiments, with a similar arrangement and the same voltaic power, the principal differences being, that the connecting bunch of asbestus was not so thick, and the positive pole perhaps a little farther from the asbestus, the acid reaction could not be detected, although the browning appeared after a certain time, and these experiments, as well as the circumstance that iodine usually appeared at an earlier period with an alcoholic than with an aqueous solution, at first led me to think that iodide of potassium in solution in alcohol was really directly decomposed under voltaic agency; but the detecting acid

* It was necessary from time to time to add a little alcohol to the negative liquid, to prevent its level getting too low, an observation which applies to all the subsequently detailed experiments with alcoholic solutions.

as above stated, as well as more decidedly when stronger powers were employed,—that acid also appearing proportionally sooner in the positive liquid when alcoholic than when aqueous solutions were employed—soon shewed me the true nature of the action, and that the acid was not detected in the instances alluded to, merely from its having been formed in smaller quantity, and having suffered that decomposition to which it is so subject, as soon as it passed to the positive side. *Whence* the acid comes I shall explain presently.

When a power of seventy-two pairs of 4-inch plates was employed, all the other arrangements being exactly the same as in the beginning of the experiment described in the preceding page, acid was detected in five minutes not only at the positive end of the asbestus, but on the positive side of the negative alcoholic solution, with alkali at the negative pole; browning having begun to appear about a minute before, and effervescence at both poles still earlier. The browning went on increasing, without any discoloration having occurred in the negative tube in a quarter of an hour. The battery was then reversed, when the usual instant discoloration ensued at the positive pole.

In these experiments, therefore, the appearance of iodine at the positive pole before reversal of the battery, is really dependent on hydriodic acid being drawn to that side, and decomposed by nascent oxygen, as in the case of aqueous solutions of iodide of potassium. The hydriodic acid comes, I conceive, principally from the point where the alcoholic solution is *in contact with water*, and where it becomes an aqueous one of hydriodate of potash, which salt is resolved into its constituent acid and alkali by the voltaic agency.

Another source of the acid seems to be the secondary action of hydrogen at the negative pole, in virtue of which acid and alkali appear to be there formed as formerly stated, the acid being immediately afterwards carried towards the positive pole; and accordingly, in one of the preceding experiments, where the stronger power was employed, acid was detected in the alcoholic liquid,—the affinity of potassium for the oxygen of the water of the alcohol of course forming this secondary action.

There is another arrangement which shows, I think, still more clearly the secondary nature of the action, in virtue of which iodine appears.

The usual alcoholic solution of iodide of potassium was placed in a tube A, fig. 2. of the same size as before, and connected on the one side with the negative side of seventy-two pairs of 4-inch plates, and on the other by asbestus, with a glass cup B of the capacity of ¼th of an ounce containing water, which, in its turn, was connected by the same means with another glass cup C, also containing water, which was made positive. Within the first quarter of an hour, acid was detected at various places on the intermediate asbestuses, and at the positive pole;

with alkali at the negative, and effervescence at both poles; but in that time no discoloration of any of the liquids was visible. In twenty-five minutes the water in C had acquired a uniform although slight brown tint; and in forty minutes this brown colour had become much more decided. The liquid in B had been acid for some time, and was quite colourless, with the exception of the slightest possible yellow tint which its upper layer had acquired, and which was not only confined entirely to the upper part, but was far less deep than the brown colour which the whole of C had acquired. The liquid in A was not at all discoloured.

This experiment was repeated with the same arrangement and same power, the only difference being, that the three tubes were all of the size of $1\frac{1}{2}$ dram. In fifteen minutes a slight browning commenced near the positive pole in C, with acid on both sides of the liquid in B, and on the asbestus between B and C. In forty minutes C was brown throughout, and neither A nor B at all discoloured; and when in fifty minutes the process was stopped, the liquid in C smelt like a strong solution of iodine, whilst that in A and B was still without the least change of tint.

It seems clear, that in these two experiments iodine appeared in the positive tube C, only in virtue of hydriodic acid having been drawn through the water in B into that in C, and there decomposed by nascent oxygen. The very trivial discoloration of the hydriodic liquid in the upper layer of B in the former of these experiments, was, I conceive, merely accidental, and arose from some subordinate secondary action, caused probably by the evolution of a few bubbles of oxygen, at some of the intermediate points on the asbestus. Some similar instances will afterwards occur, and in these experiments we ought always to bear in mind how extremely susceptible of decomposition hydriodic acid is. The *main* secondary action was plainly that in C. This was made still clearer by an examination of the aqueous liquids in B and C, after the close of the experiments. When the liquid in B was examined, both before and after concentration by heat, it was found to be a weak solution of hydriodic acid. On the other hand, when the liquid in C was concentrated by heat till the free iodine had been all expelled, it was found to be a weak solution of *iodic* acid; in other words, the hydriodic acid passing from B to C, as shown by the acid reaction on the asbestus between them, had not only been decomposed, and its iodine set free, but *a part* of that iodine had been oxidated by the excess of oxygen at the positive pole. If any doubt remained as to the existence of a secondary action, these facts, I think, would suffice to remove them.

I have mentioned, that in these experiments iodine appeared sooner on the positive side than with aqueous solutions. This circumstance arises from acid appearing sooner on the positive side of the asbestus in the former case than in the latter, apparently owing to an action of the nature of endosmose; in which way a little of the salt itself may also be carried over, so as to augment the secondary action.

Alcoholic solutions of chlorides are less well adapted for such experiments than those of iodides, because if chlorine were evolved in an alcoholic liquid, we know that it would immediately react on the alcohol, giving rise to muriatic acid and other products. Still, if such a solution were connected in the usual way with water, some portion of the chlorine, if directly produced, might perhaps escape this reaction. It is at least proper to state the result actually observed, when it will appear that nothing contrary to the idea of a secondary action was noticed.

Absolute alcohol containing $\frac{1}{12}$th of recently ignited chloride of calcium was placed in a tube A of one and a half dram capacity, and connected by asbestus with water in a similar tube B, as in Fig. 2, the former liquid being made negative and the latter positive, by a power of 72 pairs of 4-inch plates. In four minutes acid appeared at the positive pole, and an alkaline reaction at the negative, with effervescence from both poles, but no smell of chlorine was perceived. After half an hour's action, there was still no smell of chlorine in either tube, nor any bleaching action, whilst the positive liquid was acid, and the negative showed an alkaline reaction, and the negative foil was coated with lime. On reversal, no chlorine was disengaged in the positive liquid, because it immediately reacted on the alcohol, which in consequence soon became strongly acid. In short, this experiment, if it affords no positive evidence in favour of a secondary action, is at least perfectly explicable on that idea.

Before proceeding to draw these general conclusions as to the nature of the voltaic decomposition of solutions in different solvents, to which I alluded in the outset of this paper, I think it better to describe those experiments which appear to illustrate the states in which haloid salts exist in solution in alcohol and in water, because additional evidence will be, in the course of them, afforded of the secondary origin of the electro-negative constituent of such salts in the electric decomposition of their solutions, and because we shall be better able to draw the conclusions referred to when the nature of such solutions has been examined.

IV.—*Voltaic Experiments illustrative of the state in which Haloid Salts are dissolved by water.*

The question, whether chlorides and other analogous salts are dissolved as such by water, or decompose it, and exist in solution as muriates, &c. remained after the old theory of the nature of chlorine had been abandoned, and the simple nature of that substance had been universally acknowledged. At the present day this remnant of that celebrated controversy still, in some measure, divides the chemical world, although undoubtedly the view that chlorides exist as such in solution has latterly gained ground very considerably, and numbers amongst its supporters many of the most distinguished British and foreign chemists. Before, therefore, venturing to state those galvanic experiments which appear to me to lead to the opposite opinion, I would wish, first, to advert very briefly to some of those arguments which have been adduced within the last few years in support of the existence of chlorides in aqueous solutions, for the purpose of inquiring whether any of them is of such force as to admit of no answer, and consequently may induce us to presume some fallacy in views leading to a contrary conclusion.

M. Dumas has argued, that because ether can separate the chlorides of iodine, of gold, of mercury, &c. from water, they must therefore exist in water in the same state in which they are dissolved in ether, *i. e.* as chlorides*. To this argument, the answer which Berzelius suggests for the use of the advocates of the opposite doctrine, although not himself a supporter of it, seems sufficient; the affinity of ether for the chloride may determine the decomposition of the muriate and the formation of water†.

Matteucci supposed, because he found that a weak voltaic power, which was incapable of decomposing acidulated water, produced, in aqueous solutions of metallic chlorides and iodides, metal at the negative pole, and chlorine or iodine at the positive, that, therefore, the chlorides and iodides had existed as such in solution.‡ This result is easily explained, on the view of a secondary action consistently with the solution of a muriate and hydriodate. The affinity of hydrogen for the oxygen of the oxide, and of oxygen for the hydrogen of the acid, leads to the voltaic decomposition of water in these circumstances, although, in the ordinary case, it might not occur with the power used; and a secondary production of metal and chlorine or iodine ensues.

An argument much more effective than either of the preceding, is one brought forward by Berzelius, in noticing that of Dumas.§ A solution of chloride of sodium evaporates at common temperatures, and leaves dry chloride of sodium. If a muriate was dissolved, then the tension of the water formed from the oxygen of the base, and hydrogen of the acid of the muriate, has come into play before its elements were united as such; or if it be said that this view involves no impossibility, still such tension must be admitted to be weaker than that of ready formed water; and yet chloride of sodium begins to be deposited by a saturated solution whilst water still remains. Considering the ingenuity of this argument, as well as the authority from which it comes, it is with hesitation that I at-

* An. Ch. et Phy. xliv. 271.

† Jahrsbericht, xi. 56.

‡ An. de Ch. et Phy. xlv. 324.

§ Jahrsbericht, xi. 57.

tempt to answer it; but I confess it appears to me to admit of a reply. The influence which water has on a variety of chemical phenomena has been shewn by M. Pelouze, and that of quantity and cohesive force was long ago pointed out by Berthollet; and in regard to the phenomena of solution, the operation of quantity is generally acknowledged. By the aid of such principles, the appearances under consideration seem to admit of explanation. Since every substance is dissolved, in consequence of a chemical affinity between it and the solvent, it follows, that if a chloride is dissolved as a muriate, the affinity of water for the muriate has a considerable share in determining the union of the metal with oxygen, and of the chlorine with hydrogen. Now, when the water of such a solution has evaporated away to such an extent as to leave a completely saturated solution, the affinity of the oxygen and hydrogen of the muriate for one another is resisted by the attraction of a much smaller quantity of water for the muriate than formerly, and is now aided by the incipient cohesive attraction of the salt in the act of crystallization, joined to the powerful affinity of chlorine and sodium for one another. The affinity of the elements of water, therefore, *under the circumstances in which they now come to be placed,* and not its tension before its existence, I should humbly think is the cause of the union of these elements; and this affinity may very well come into play *under the circumstances mentioned,* before the vaporizing tendency of the remaining actual water takes full effect; and it farther seems probable, that the affinity of the remaining muriate for water will aid this union of the oxygen and hydrogen of the salt in the act of crystallization, the water formed by the union of the oxygen and hydrogen of the salt probably uniting with the dissolved muriate before its dissipation by evaporation.

In so far, therefore, as I am able to judge, it does not appear that any of these arguments foreclose the inquiry; and we may still be at liberty to bring forward illustrations on the other side.

Let us assume for a moment, that chlorides and iodides are dissolved as such in absolute alcohol, and as muriates and hydriodates in water. Let us next suppose these alcoholic and aqueous solutions exposed to voltaic agency, *under circumstances in which no secondary action can take place at the poles from evolved oxygen and hydrogen.* What ought to happen in the two cases of alcoholic solution and of aqueous solution? Surely this. In the alcoholic solution, since it has been shewn, I hope successfully, that water only is directly decomposed, then neither constituent of the chloride or iodide, nor any acid, ought to be produced in that solution; whilst in the aqueous solution, the salt composed, ex hypothesi, of acid and alkali, ought to be resolved into its elements, conformably to the general law of the electric decomposition of ordinary salts, and acid and alkali, without chlorine or iodine, should appear in the solution, each on its proper side travelling towards the poles, which, by the supposition, are placed beyond the bounds of that solution. Let us see how far this result, which we thus predict, is supported by experiment.

Absolute alcohol, with about $\frac{1}{40}$ of iodide of potassium in solution, was placed in a glass tube of $1\frac{1}{2}$ dram measure, and on each side a glass cup containing a quarter of an ounce of water was connected with it by means of asbestus moistened with alcohol, one of these cups being made negative and the other positive by fifty pairs of 2-inch plates. The arrangement is represented in Fig. 3, B containing the alcoholic solution and A and C the water. In eight minutes iodine began to appear in C round the positive pole, with effervescence at both poles. After forty minutes' action, not a trace of acid or of alkali could be detected in the alcoholic solution in B, nor had either been observed all along in that liquid. On the other hand, alkali had been noticed soon after the commencement, and all along at the negative pole in A; but no acid reaction was any where observed.* At the conclusion of the experiment, in forty minutes, free iodine was very manifest in C both by the colour and smell; but not a trace of it was observed either in B or in A.

This experiment was repeated with a power of seventy-two pairs of 4-inch plates, and three tubes, each of the size of $1\frac{1}{2}$ dram, all other circumstances being the same. Iodine began to appear in C in about four minutes, with bubbles at both poles. In eighteen minutes neither acid nor alkali could be detected in the alcoholic solution in B; but a trace of acid was noticed on the asbestus above the surface of the positive water in C, and alkali had appeared before this at the negative pole. In three quarters of an hour there was still no acid nor alkali in B, whilst the acid reaction was strong at the place where it had previously appeared. There was then much free iodine in C; and in B only an insignificant trace of that partial discoloration to which I formerly alluded as probably proceeding from some subordinate and trifling secondary action.

Let us now contrast these results with those obtained with an aqueous solution.

Water containing $\frac{1}{40}$ of iodide of potassium was substituted in B for the alcoholic solution, all other circumstances, including the size of the vessels and voltaic power, being exactly the same as in the former of the two preceding experiments. The first decided acid reaction which was now observed was on the positive side of the solution in B, with alkali on the asbestus between A and B, and these in about fifteen minutes, effervescence having been in the mean time going on at both poles. During forty minutes only slight traces of acid appeared at the positive pole and on the asbestus between B and C, whilst a strong acid reaction continued in the liquid in B, with alkali at the negative pole, and on the asbestus between A and B. A slight discoloration of the water in C had just commenced at the end of this time, without any change of tint in A or B.

* In explanation of the non-appearance of acid in the water in this experiment, see p. 15, 16.

This experiment was repeated with a strong aqueous solution of iodide of potassium, the water containing one-third of its weight of the salt, and being placed in B, all other circumstances being the same as before. In five minutes there was a trace of acid on the asbestus between B and C, with alkali at the negative pole. In fifteen minutes there was also a trace of acid at the positive side of the solution in B, and this was quite decided in twenty minutes, and more so than that on the asbestus, whilst at the positive pole there was still no acid. In forty minutes the acid reaction at the positive side of B was powerful, whilst all the other acid reactions were slight or doubtful. On the negative side of B, and from that to the negative pole, alkali was observed. A slight discoloration appeared in C, and none in A or B.

With a power of seventy-two pairs of 4-inch plates, and the strong solution of iodide in B, and water in A and C, the three vessels being glass cups, each of the capacity of one-fourth of an ounce, there was slight acid at the positive pole in five minutes, and strong acid at the positive side of B in fifteen, with alkali at the negative pole and on the adjoining asbestus. During the forty minutes which the experiment lasted, the acid reaction in B continued to increase and became very powerful, whilst that at all other places where it was noticed continued slight. A discoloration of the water in C had been noticed in fifteen minutes, with none in B or A then or for half-an-hour, and in forty minutes the liquid in C had assumed a pretty deep red throughout, and smelt strongly of iodine, whilst in B there was only a slight yellow tint confined to a single spot on the positive side of its upper layer, and no smell of iodine at all.

When the liquid in C was concentrated by heat till it was colourless, it was found, when the larger voltaic power had been used, to contain a trace of iodic acid; but where the smaller had been employed the nature of the acid in C was rather ambiguous. As the iodic acid was, to all appearance, referable to a secondary action, and it was of some consequence, with a view to the true explanation of the phenomena, to ascertain with certainty that the acid formed in B under the voltaic influence was hydriodic acid and not iodic acid, the following experiment was made.

Water containing $\frac{1}{3}$d of iodide of potassium was placed in the tube B, and pure water in the tubes A, C, and D, Fig. 4, the whole being connected by asbestus, and A made negative and D positive by 72 pairs of 4 inch plates. In ten minutes there was slight acid at the positive side of B, and on the asbestus between C and D, and at the positive pole in D, but none in the liquid in C. In twenty minutes the acid reaction on the positive side of B was strong, with a less marked on the asbestus between C and D, and alkali on the negative side of B. In fifty minutes the liquid in D had acquired a uniform pretty deep brown, whilst those in C and A were not at all discoloured; and in B there was merely a pale yellow tint confined to the positive side of the surface. The liquids in C and D were then concentrated by heat, when the former was found to contain a trace of hydriodic acid, whilst the latter contained a trace of iodic acid. Hydriodic acid had thus been drawn, first, into C, and then into D, where it was decomposed by nascent oxygen, and a part of the liberated iodine oxidated.

A comparison was next instituted between alcoholic and aqueous solutions of the chlorides of calcium and of zinc.

Absolute alcohol containing $\frac{1}{16}$th of recently ignited chloride of calcium was placed in B, Fig. 3, and water in A and C, A being made positive and B negative by a power of 72 pair of 4-inch plates. In a quarter of an hour acid was detected at the positive pole, and on the adjoining asbestus, immediately above the surface of the liquid in C, but not a trace of it in B. In half an hour there was still no trace of acid in the liquid in B, but there was a trace of it on the asbestus at the positive side of B, having evidently spread from the other side of the same asbestus, where it was before observed, and where as well as at the positive pole it was now strong. In three quarters of an hour there was a just perceptible trace of acid in the liquid in B on the positive side, and strong acid all along the asbestus to the positive pole. This very trifling trace of acid in B had thus evidently spread from the asbestus, as just mentioned, and had not been *produced* in B. An alcaline reaction was observed at the negative pole. No smell of chlorine or bleaching action was any where noticed. When the experiment was concluded, the liquid in C was found to be a weak solution of muriatic acid, with a just perceptible trace of the smell of chlorine; this acid being too strong a combination to be so readily decomposed by nascent oxygen as hydriodic acid.

When a moderately strong aqueous solution of chloride of calcium was substituted in B for the alcoholic solution, and a power of fifty pairs of 2-inch plates employed, there was decided acid on the positive side of B in less than twenty minutes, with a slight trace at the positive pole, and none on the intermediate asbestus. In half an hour the acid in B was powerful, whilst there were still merely traces at the positive pole and on the asbestus, with alcaline reaction at the negative pole. No trace of chlorine was observed, although the action was continued above two hours.

With a pretty strong solution of recently ignited chloride of zinc in absolute alcohol in B, and water in A and C, the same voltaic power being employed as in last experiment, there was no acid reaction in B after forty minutes' action, whilst a trace of acid had appeared at an early period on the asbestus above the surface of the positive liquid in C. On the other hand, when an aqueous solution of chloride of zinc was substituted in B for the alcoholic, all other circumstances being the same, there was strong acid in B in ten minutes, with none at the positive pole, and only a trace on the intermediate asbestus. In neither experiment was any smell of chlorine observed after action, for three quarters of an hour. In the latter experiment a little metallic zinc was deposited on the negative foil, from a small quantity of the solution having passed into A by capillary action, as was ascertained by reactives.

With aqueous solutions of chloride of potassium of various strengths in B, and water in A and C, there was, after a time, strong acid reaction in the solution in B, whilst at the positive pole, and on the intermediate asbestus, the reaction continued slight, and there was no bleaching action or decided smell of chlorine any where after an hour's action.

In all cases in which the acid which had passed into C, where solutions of chlorides were acted on, was examined, it was found to be muriatic.

The leading *facts* which have thus been observed are these: *First,* When alcoholic solutions of chlorides and iodides are acted on in the circumstances mentioned, no acid is observed to be produced in the alcoholic solutions. *Secondly,* When aqueous solutions of these substances are employed, the corresponding hydracids are produced in the solutions. *Thirdly,* When alcoholic solutions are used, the corresponding hydracids are produced at the point of contact between the alcoholic solutions and the water with which they are connected. *Fourthly,* The hydracids arising in both these ways, are carried to the positive pole situated in the water.

Such apparently being the facts, let us see whether they are capable of explanation on the idea of chlorides and iodides being dissolved as such in water; and let us take the different cases which may be assumed. Let us first suppose that water only is directly decomposed in the aqueous solution, and that, according to the usual and most approved view of electric action, a series of decompositions and recompositions of the directly decomposed body ensues, until its elements arrive at their respective poles. On this view we evidently cannot explain the production of acid in the aqueous solution under the described circumstances of the experiment. Let us next suppose that the chloride or iodide alone is directly decomposed, and that its elements proceed in the above way to their proper poles. Precisely the same objection applies to this view. Lastly, Let us suppose that both water and chloride or iodide suffer decomposition, and that either the elements going to the same pole unite on their journey, or, by a mutual interchange, the electro-negative constituent of the water unites with the electro-positive of the salt, and the electro-positive of the water with the electro-negative of the salt. The former of these alternatives is contradicted by the fact, that the acid passing to the positive pole is the hydracid and not the oxyacid; and the second is not only at variance with the usual view, that the elements of substances under voltaic decomposition follow the road *of themselves* to their proper poles, by a series of decompositions and recompositions; but is not in accordance with the circumstance that no acid is observed to be produced under the described arrangement in the alcoholic solution; for alcohol contains water, and that water, as I trust it has been sufficiently proved, suffers voltaic decomposition, and thus an alcoholic as well as an aqueous solution presents the condition for a double decomposition, if such a double decomposition really can occur.

The observed appearances thus seem to be at variance on any reasonable mode of interpretation, with the idea that chlorides and iodides are dissolved as such in water. Let us take the other view, that they are dissolved as muriates and hydriodates, and what a contrast is observed. Not only are the phenomena easily explained, but they appear to be the necessary consequences of the supposition adopted. For if a salt composed of acid and alkali is dissolved in water, its constituents ought to go to their proper poles under voltaic agency; and in the experiments detailed, acid ought to be produced in the solution at its positive side, and to accumulate in that solution, if faster produced than carried over into the positive water, which experiment shews to be the case.*

The observation lately made by Dr Mohr of Coblentz, that no electric stream is produced by the union of a hydracid and an alkali,† finds its readiest explanation in the views above advocated; because such a union is thus placed in the same case with that of an oxyacid and an alkali, which, according to Mr Faraday, produces no voltaic current.

Although I have thus stated the conclusion on this point which appears to follow from the phenomena as observed, yet I am too well aware of the great subtlety of the subject, and have too much deference for the opinions of the many eminent men who have held different views, to wish to be understood as speaking dogmatically upon it. If any errors of observation, or mistakes in point of reasoning, affecting the conclusion which has been drawn, can be pointed out, I shall always be happy to acknowledge them, if they cannot be explained.

V.—*General Conclusions respecting the Voltaic Decomposition of Solutions in Water, Alcohol, and Ether.*

It is to Mr Faraday that we are indebted for experimental evidence in numerous cases of aqueous solutions, that the direct agency of the electric current is exerted upon the water of the solution only, and that the other appearances of decomposition in these instances are due to secondary actions.‡ Amongst the

* It will be readily understood, that when the poles are actually *in the solution,* the acid should first appear at the positive pole, and *thence* spread into the liquid when it has accumulated, as was shewn by M. de la Rive; but when the poles are *beyond* the solution, the acid must make its way from the solution to the pole through the interposed water, and unless carried through the water as fast as it is produced, it must accumulate in the solution.

† Pog. Annal. xxxix. 134.

‡ Experimental Researches, seventh series.

most important of these cases are aqueous solutions of the oxyacids, as that of sulphuric acid; and embracing, as I fully do, his opinion, that the appearance of sulphur at the negative pole in such a case is a secondary result due to nascent hydrogen, I may be allowed to add, that this view seems well illustrated and confirmed by the experiment which I formerly described, in which the iodine of an aqueous solution of iodic acid appeared at the negative pole, under circumstances in which the secondary nature of the action was quite obvious.

Mr Faraday, however, made an important exception in the case of aqueous solutions of the hydracids; but I shall here merely refer to the evidence so fully detailed now* and formerly, which has led me to infer, that in solutions of the hydracids, as well as in those of the oxyacids, water only, and not the dissolved acid, is directly decomposed. I have also to refer to the experiment with an aqueous solution of bromide of iodine, from which it appeared that that compound was not directly decomposed when in solution, but only the water.† It would appear, therefore, that we at present know of no combination of two elementary substances with one another, which, when in solution in water, is directly decomposed by the electric current; but have every reason to believe, that in such solutions the water only suffers direct decomposition.

Farther, I have endeavoured to shew that in alcoholic solution of acids, alkalies, and haloid salts, the water of the alcohol alone is directly decomposed.

The conformity between these views and the results obtained with etherial solutions is remarkable. I formerly stated that no evidence whatever was obtained from electric phenomena that ether contained water; and what was the farther observed result? When etherial solutions of potash, of chromic acid, of chloride of platinum, and of corrosive sublimate, were acted on by fifty pairs of 2-inch plates, there neither were any symptoms of decomposition, nor was the galvanometer affected.‡ Thus, whilst in aqueous and alcoholic solutions, water and not the dissolved body is decomposed, in etherial solutions, where there is no water present, no decomposition takes place at all.

In this way we are arrived within a few steps of the following general conclusion, which I cannot help thinking, if it shall be fully supported, is one of considerable interest, and not I believe hitherto anticipated: "That when solutions of binary combinations of elementary substances, in water, alcohol, or ether, are submitted to voltaic agency, the dissolved substance is not directly decomposed; but only the solvent, if itself an electrolyte."

In laying down any general law, which must, if well founded, comprehend a vast multitude of facts, one of course feels the necessity of having proceeded on an extensive induction; or at least of having established the leading analogies

* P. 7, et seq.

† P. 10.

‡ Edinburgh Transactions, vol. xiii. p. 331.

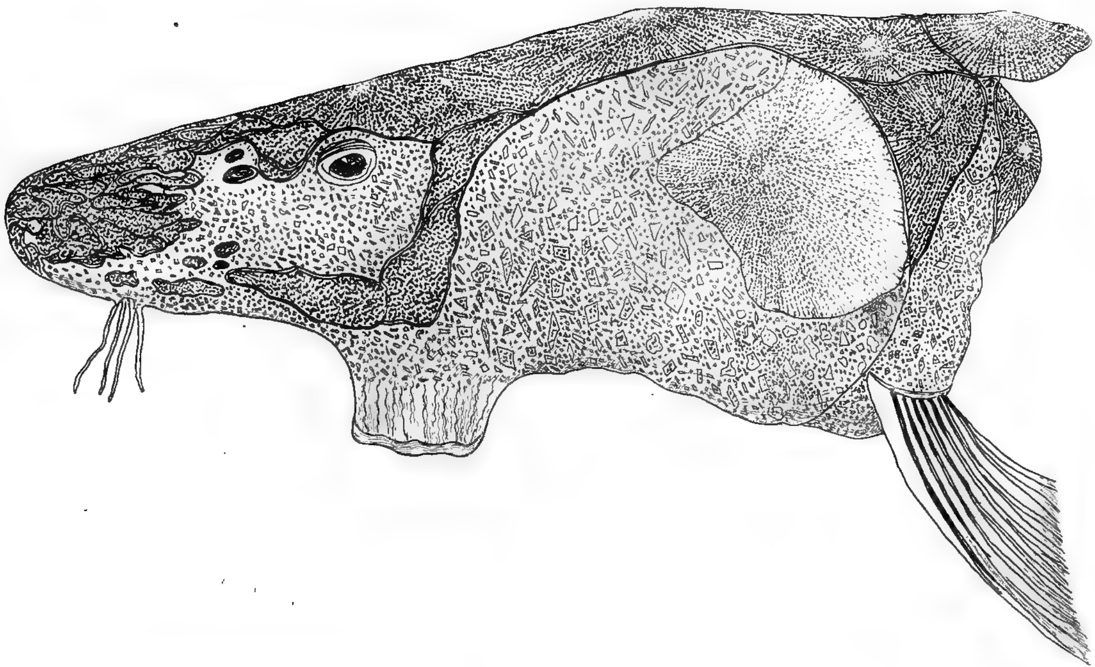
comprehended within the bounds of the proposed law, for it rarely happens that the inductive process actually embraces every particular, its imperfection in point of logic being usually supplied by the necessary connections of the individual cases. All I can say is, that I am at present acquainted with no exception to the proposed law, and that all the experiments which I have yet made, go to support it; but still as there are some cases comprehended in it, which may and ought to be experimentally investigated, I shall not yet take upon me to give it as established in its utmost generality, but shall probably in a future communication state the farther results obtained.

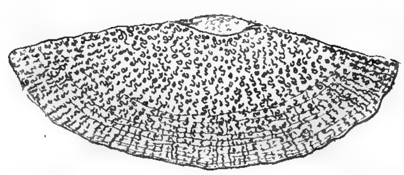
This rule is of course entirely confined to compounds of elementary bodies. Every one knows that an ordinary salt dissolved in water, is resolved into its constituent acid and alkali under voltaic agency. The same observation I have found to apply to alcoholic solutions of such salts. With respect to their etherial solutions, it would seem that it does not hold, in so far as reliance can be placed on a single experiment with a moderate voltaic power. A solution of nitrate of uranium in rectified ether was submitted in a close tube to the action of fifty pairs of 2-inch plates, without any appearance of the constituents of the salt at their respective poles, or action on the galvanometer formerly described.

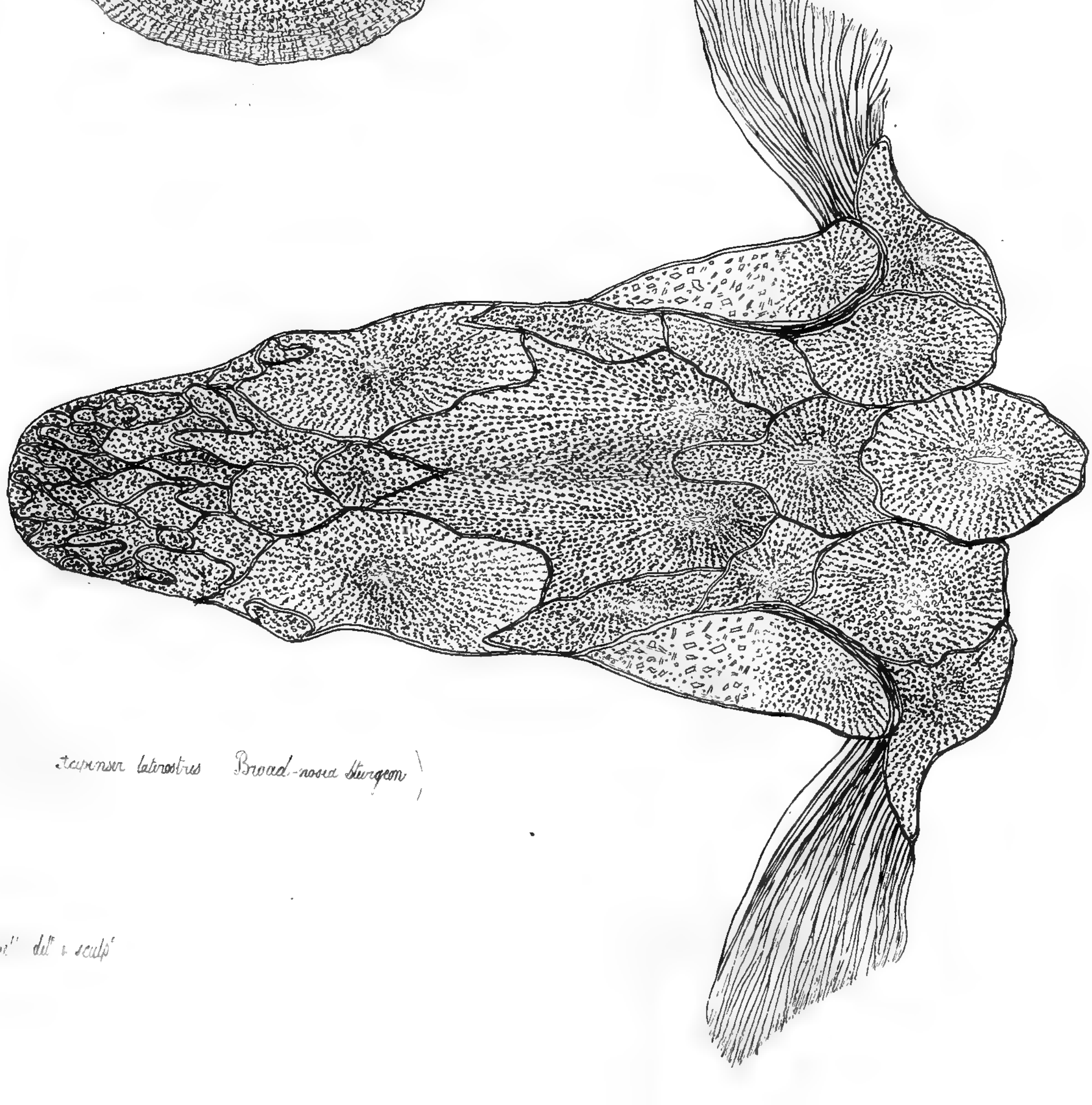
ERRATA in former Memoir in Vol. XIII. of Edinburgh Transactions.

Page 316 (p. 2 of separate Memoir), line 20, for *liquid* read *solution*
— 334 (p. 20 — — —), — 8, — *ether* — *alcohol*
— 337 (p. 23 — — —), note, line 4, for *effects* read *quantities*
— 346 (p. 32 — — —), lines 8 and 9 for *positive* read *negative*, and for *negative* read *positive*

Royal Soc. Trans. Vol. XIV



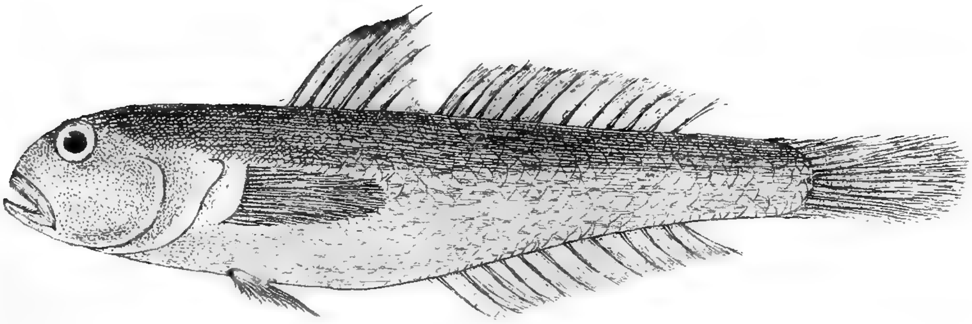




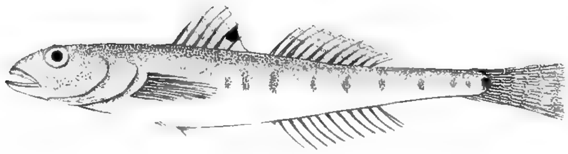
Acipenser latirostris Broad-nosed Sturgeon

T. Tarrell del. & sculp.

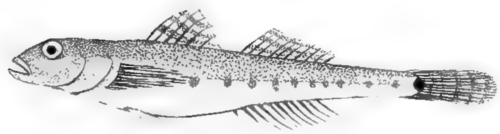
PLATE V.



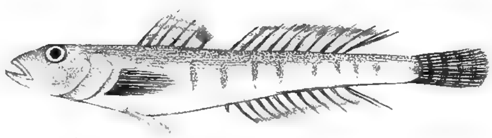
Gobius niger Black Goby.



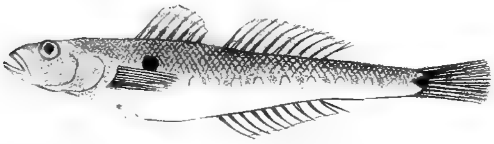
Gobius unipunctatus (One-spotted Goby



Gobius minutus (Freckled Goby)



Gobius gracilis (Slender Goby)



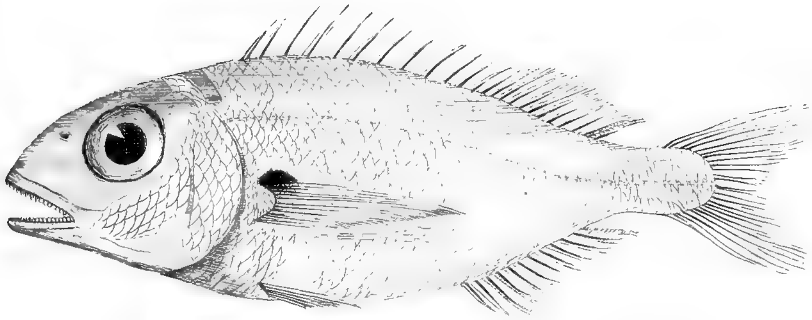
Gobius bipunctatus (Two-spotted Goby)

Gobius albus (White Goby

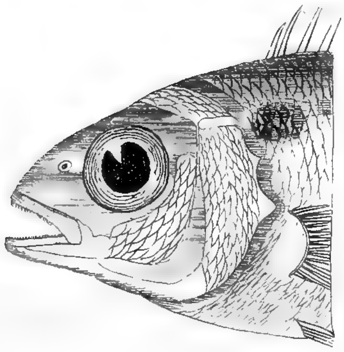
R. Parnell del. & sculp.

PLATE VI.

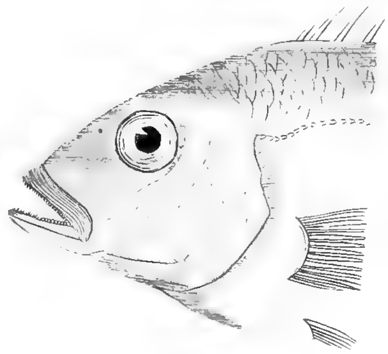
Royal Soc. Trans. Vol. XIV.



Pagellus acarne. Cuv. VI. p. 191
Axillary Bream



Pagellus centrodontus. Yarrell vol. i. p. 107.
Sea Bream.

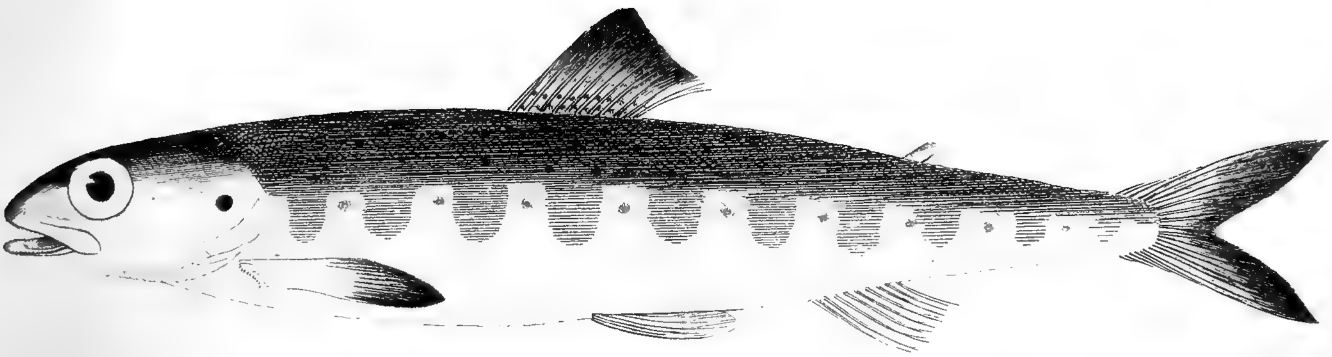


Pagellus erythrinus.. Cuv et Val t. VI. p. 170.
Spanish Bream.

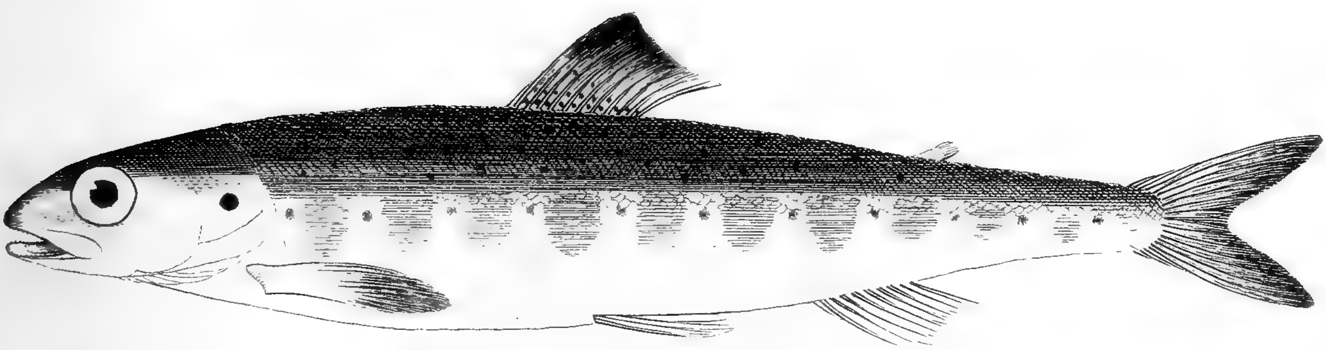
Raia intermedia. Parnell
Flapper Skate

R. Parnell del. & sculp.

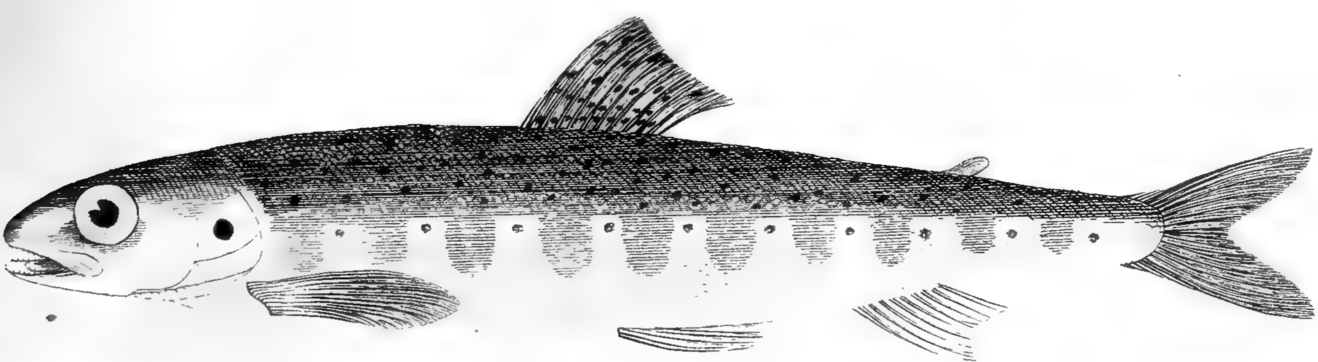
Royal Soc. Trans. Vol. XIV.



Salmo Salar Yarrell Vol. II.
Young Salmon or Smolt. Natural size. Captured May 13d.

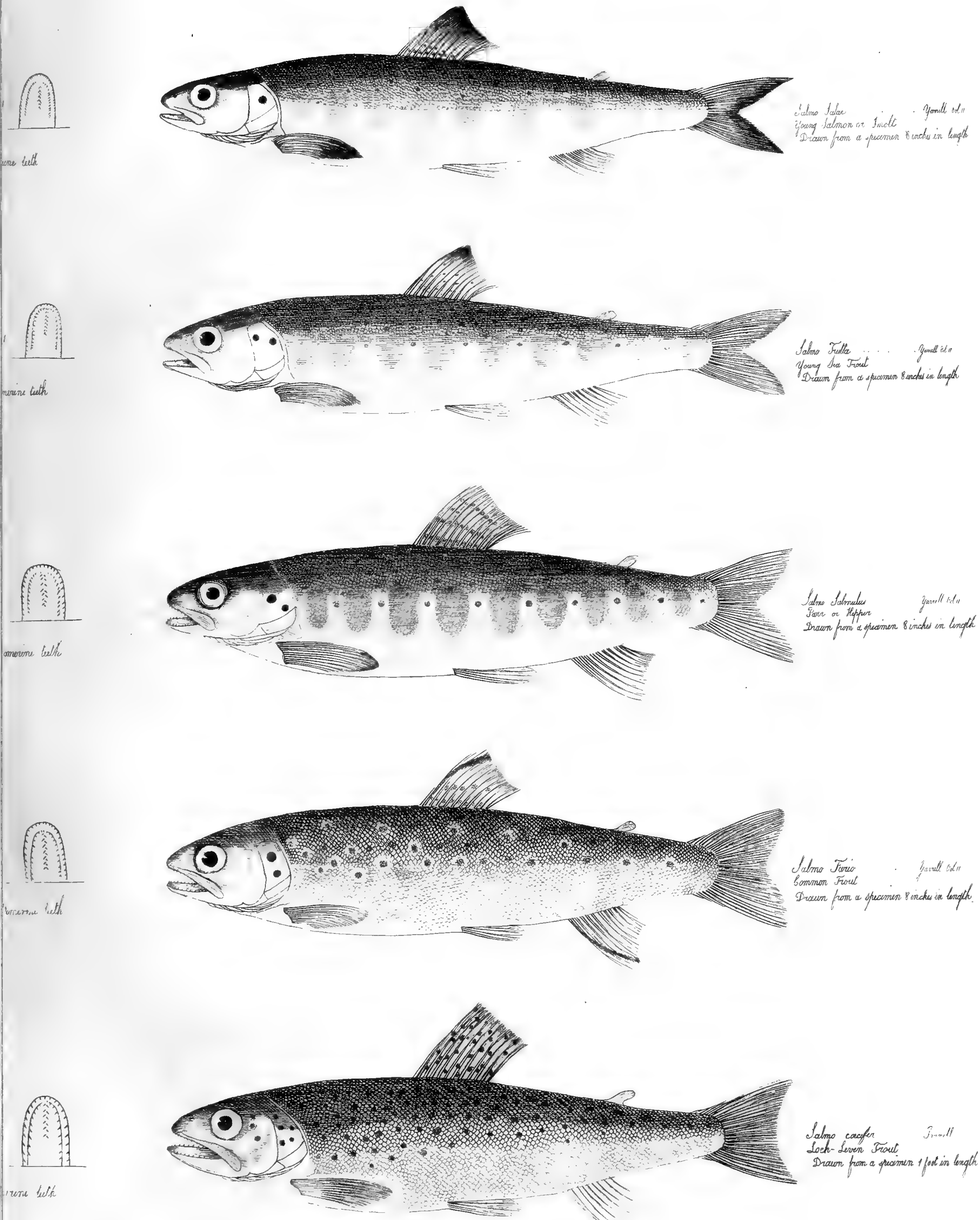


Salmo Eriox . . Yarrell Vol. II.
Salmon Trout. Young. Natural size. Captured May 13th.



Salmo Salmulus ...... Yarrell Vol. II.
Parr or Hepper. Natural size. Captured May 14d.

R. Parnell del. & sculpt.



R. Parnell del. et sculp.

V. *An Account of Three New Species of British Fishes, with some Remarks on Twenty others new to the Coast of Scotland.* By Richard Parnell, M.D., F.R.S.Ed. &c.

Read 20th February 1837.

Although a great number of Fishes have been described by naturalists as inhabiting the coast of Scotland, and though much has been done to increase its Fauna in other respects, yet in a field so extensive much must still remain to be done. Many lakes and estuaries in Scotland are still unexplored, and many fishes being merely local, a more thorough and accurate examination of these lakes and estuaries must take place, before the Ichthyology of Scotland can be fully ascertained.

Acipenser latirostris, *Parnell.*

In the works of Pennant, Donovan, Fleming, Yarrell, and other writers on Ichthyology, is mentioned but one species of British sturgeon (*Acipenser sturio*); but from the observations of practical fishermen as well as my own, I think there is little doubt but that two species at least will in future be recognised as inhabiting the British coast.

It has long been noticed by the fishermen of the Solway Frith, that two species of sturgeon are occasionally entangled in their salmon-nets, the one with a blunt nose, and the other with a sharp one; the latter species being the most common of the two.

A fine specimen of the blunt-nosed sturgeon was taken in the Frith of Forth in the month of July 1835, and brought to the Edinburgh market for sale, the head of which I preserved. (See Plate IV.) A few weeks after, another was taken in the Tay, which differed in no respect from the former except in sexual distinction.

Description.—Length 7 feet 9 inches; weight 8 stones. The colour of the back and sides is of a light grey, with a shade of olive; the belly dirty white. The body is armed with five rows of osseous shields, running from the head to the tail. The first row commences behind the head, and runs down the central ridge of the back; the two next rows arise one on each side of the former. Immediately on the lower margin of the pectorals the other two rows commence. The skin is rough, with a number of small angular osseous plates intermixed with very minute spicula. The first free shield on the dorsal ridge is nearly circular, and very slightly carinated; all the rest in that row are of an oval form. The snout is wide and depressed, much broader than the diameter of the mouth. On the under surface, placed nearer to the tip of the snout than to the mouth, are four cirri arranged in an irregular line. The summit of the head is rough, with the central plates beautifully radiated and of a fibrous appearance. The position of the fins is the same as in other sturgeons.

This fish differs from the common sturgeon (*Acipenser sturio*) in having the tip of the snout much broader than the mouth, in the keel of the dorsal plates being but slightly elevated, and having the cirri placed nearer to the tip of the snout than to the mouth.

The sturgeons are all much allied to each other; and not being able as yet to find the right synonym for the present one, I have proposed, in the mean time, the name *latirostris*, as characteristic of the species.

The genus GOBIUS is recognised by having the ventral fins united together so as to form a disk, incapable of adhering to surfaces.

The species of this genus have received but little attention, perhaps on account of their small size, and the great facility with which they may be overlooked while inhabiting their natural element.

Before the appearance of Mr YARRELL'S excellent work on the British Fishes, great confusion prevailed as to the discriminating characters of the species of Gobius found inhabiting the British coast.

By PENNANT, DONOVAN, and FLEMING, these fish were all confounded under two species, the *Gobius niger* and the *Gobius minutus*. JENYNS, in his Manual of the British Vertebrate Animals, makes three species; and YARRELL, in his work before mentioned, has added another, making the number up to four British species, viz. *Gobius niger*, *G. minutus*, *G. gracilis*, and *G. bipunctatus*. I hope I shall not be considered as multiplying the species beyond their due limits by adding two more to the list of the British Gobies.

GOBIUS UNIPUNCTATUS, *Parnell*.—One-spotted Goby. (See Plate V.)

The species for which I have proposed the name of *unipunctatus*, is perhaps more nearly allied to the *Gobius minutus* than to any other, differing from it in having the intervening membrane of the fifth and sixth ray of the first dorsal fin marked by a large conspicuous black spot, and in having the tail even at the end; whereas the *minutus* has no black spot at this place, and the tail is rounded at the end. The *unipunctatus* grows to a much larger size, and is seldom found associating with the *minutus*. The largest specimen measures three and a half inches in length; the back is of a light reddish-brown, slightly tinged with yellow, and marked with a few dark lines of a deeper colour. The first dorsal fin commences in a vertical line over the upper third of the pectorals, and ends in a line with the termination of the ventral rays; the second dorsal fin commences over the vent, and ends opposite to the base of the last anal ray, leaving a wide space between it and the base of the tail. The head is rather long; the eyes are situated high on the forehead, and nearly approximating. Each jaw is furnished with a number of small sharp teeth in two rows. The cheeks are tumid; the margin of the operculum is rounded. The lateral line is straight, marked with six or seven dark spots, the one at the base of the tail being the most conspicuous. The numbers of the fin rays are,—first dorsal 6; second dorsal 11; caudal 12; ventral 13; anal 11; pectoral 20. The upper part of the membrane between the fifth and sixth ray of the first dorsal fin is marked with a large black spot, which is always constant; the second dorsal fin is mottled with reddish-brown, as well as the tail, which is even at the end. The belly, ventral, and anal fins are white.

This fish, I first noticed in the Frith of Forth, in the neighbourhood of Queensferry, where it may be found throughout the whole summer in water from two to three feet deep. It seldom reaches the shore as the *minutus* is observed to do, but keeps more in the deep water.

In the Solway Frith I found it rare, but the *minutus* abounds there in great numbers. At Exmouth on the coast of Devon, I have taken it in many situations where I could not find a single specimen of the *minutus*, and I have also found the *minutus* where the *unipunctatus* was never observed. If we compare this fish with the rest of the British Gobies, we shall find it to differ from them in other respects, besides having a black spot on the first dorsal fin, and the tail even at the end.

G. unipunctatus,	comp. with	*G. niger*.
Dorsal fins widely separate.		Dorsal fins closely approximate.
G. unipunctatus,	... ...	*G. bipunctatus*.
First Dorsal fin with six rays.		First Dorsal fin with seven rays.
G. unipunctatus,	... ...	*G. gracilis*,
Anterior rays of second dorsal fin, longest.		Anterior ray of second dorsal fin, shortest.
G. unipunctatus,	... ...	*G. albus*.
First Dorsal fin with six rays		First Dorsal fin with five rays.

Gobius albus, *Parnell*.—White Goby.

This species of goby holds such a conspicuous place in the genus, that it cannot well be mistaken for any other. I first noticed it in the Solway Frith, in

June last, where I obtained in one day after the recess of the tide, fifty specimens. They are evidently the fry of a large species. When first taken from the water, they are soft and transparent; the eyes are large and prominent; the scales which cover their body, are large and thin, and very deciduous. The length is about two inches; the head is large; the gape is wide; the teeth are long and sharp, placed in one row in each jaw. The first dorsal fin commences over the upper third of the pectorals, and terminates at a point a little behind its rays; the second dorsal fin commences over the vent, and ends opposite to the base of the last anal rays. The cheeks are tumid, the border of the operculum rounded; the body is transparent and marked by a number of fine depressed lines, placed in an oblique direction; the lateral line is straight throughout its length.

The numbers of the fin rays are: First dorsal 5; second dorsal 13; caudal 12; ventral 13; anal 13. The last ray of the anal and second dorsal fin is longer than the first, and reaches, when folded down, to the base of the tail. The tail is rounded at the end. These fishes are supposed by the fishermen to be the young of the sting-fish *(Trachinus vipera)*, and are consequently destroyed whenever they come within their reach. On transferring them to a bottle of alcohol they lose their transparent aspect, and become hard and opaque.

In the month of July when I had occasion to revisit the Solway Frith, I endeavoured to obtain additional specimens, presuming that by this time they would have somewhat increased in size, but not a single specimen could be found, nor has the parent fish ever come within the observation of the fishermen.

The first dorsal fin of this fish, as possessing but five rays, is sufficient to distinguish it from every other British species of the same genus.

Observations on Twenty New Species of Scottish Fishes.

Trigla hirundo, *Yarrell*, vol. i.—Tub-fish.

Specific Character.—Pectoral fins dark blue, reaching beyond the vent; lateral line and body perfectly plain and smooth.

This fish on the coast of Scotland is undoubtedly rare, compared to the numbers that are taken on the coast of Devon. In young specimens the dorsal ridges are found to be sharply serrated, but when the fish increases to the weight of nine pounds, these ridges of the back are no longer serrated, but crenated, as is observed in the *T. gurnardus* when full grown.

Trigla blochii, *Yarrell*, vol. i.—Trigla cuculus, *Bloch*.

Specific Character.—Pectorals not reaching to the vent, lateral line and dorsal ridge strongly serrated; first dorsal fin with a black spot.

In the Frith of Forth this fish is frequently met with, from five to six inches in length, in the month of July. I may add that I can see no difference between this fish and the young of the grey gurnard (*T. gurnardus*) of equal size. The grey gurnard, when less than six inches in length, is always marked with a black spot on the first dorsal fin, and the lateral line and dorsal ridge are strongly serrated; but as the fish increases in size, so the black dorsal spot and serratures become obliterated, and at length crenated, as in the grey gurnard a foot in length. In Mr YARRELL'S work on the British Fishes, the first spine of the first dorsal fin of *T. Blochii,* is represented as being longer than the second spine, which it ought not to be, as the first dorsal spine in the whole of the gurnards is much the shorter of the two.

COTTUS BUBALIS, *Yarrell,* vol. i.—Sea bull-head.

Specific Character.—Præoperculum with four spines.

To Mr YARRELL we owe the first discovery of this fish as British. From the great similarity which exists between it and the *C. scorpius,* there is no doubt but that they have often been taken for the same species, as they both inhabit the same places, and are found equally common. The difference which exists between these two fish is evident when placed side by side. In the *bubalis* the first gill-cover has four spines, and the lateral line is rough; whereas in the *scorpius,* the same gill-cover has but three short spines, and the lateral line is smooth. They are both common in the Solway Frith, as well as in the Frith of Forth. Their flesh is coarse and disagreeable to the taste.

GASTEROSTEUS TRACHURUS, *Yarrell,* vol. i.—Full-armed Stickleback.

Specific Character.—Plates extending the whole length of the sides.

Not common on the east coast of Scotland; more frequently met with on the west coast.

GASTEROSTEUS SEMIARMATUS, *Yarrell,* vol. i.—Half-armed Stickleback.

Specific Character.—Plates extending as far as the vent.

Not so frequently met with as the last species; found to inhabit fresh as well as salt water pools.

PAGELLUS ERYTHRINUS, *Cuvier.*—Spanish Bream.

Specific Character.—Origin of the lateral line and base of the pectorals without a black spot.

The Spanish bream is one of the rarest of our British fishes. It has been noticed, though not often, on the coast of Cornwall, and has been taken once in the Frith of Forth. The flesh is of superior quality for the table. It much resembles the sea bream, but its eye is smaller, its snout is longer, and the origin of the lateral line is slightly bent. (See Plate VI.)

PAGELLUS CENTRODONTUS, *Yarrell*, vol. i.—Sea Bream.

Specific Character.—Origin of the lateral line with a large dark spot.

This is a common fish on the coast of Devon. It is frequently met with on the coast of Essex; but, as we approach the eastern shores of Scotland, it becomes scarce. A few specimens are annually taken in the Frith of Forth, about the month of July, sometimes with the hook, but more frequently in the salmon nets.

The principal distinguishing characters of this species are, the largeness of its eye, and a conspicuous dark spot at the origin of the lateral line, over the base of the pectorals. (See Plate VI.)

MUGIL CHELO, *Yarrell*, vol. i.—Thick-lipped Grey Mullet.

Specific Character.—Upper lip thick and fleshy.

Mr COUCH of Cornwall is the only naturalist who has hitherto noticed the thick-lipped grey mullet on the British coast. This species of mullet is common on the east coast of Scotland, where the *M. Capito* is of rare occurrence. It is occasionally taken the length of twenty-two inches. The flesh is held in low estimation for the table.

GOBIUS BIPUNCTATUS, *Yarrell*, vol. i.—Two-spotted Goby.

Specific Character.—First dorsal fin with seven rays.

This fish is frequently met with at the mouth of the Frith of Forth, swimming about among fuci, particularly in rocky situations. (See Plate V.)

GOBIUS GRACILIS, *Jenyns.*—Slender Goby.

Specific Character.—Last rays of the second dorsal fin longer than the first. First dorsal fin with six rays.

More common than the last species; found inhabiting sandy situations. The ventral and anal fins are always tinged with black; the markings on the lateral line are long and narrow; the middle mark often extending the width of the body. (See Plate V.)

CRENILABRUS TINCA, *Yarrell*, vol. i.—Ancient Wrasse.

Specific Character.—Base of the tail without a black spot.

This species is common throughout the coast, frequenting rocky places. Its flesh is white, soft, and insipid, seldom made use of as an article of food. It feeds on crustacea, and spawns about the end of April.

CRENILABRUS CORNUBICUS, *Yarrell*, vol. i.—Goldsinny.

Specific Character.—A black spot at the base of the tail, under the caudal extremity of the lateral line.

This fish is occasionally met with in rocky situations. It is seldom found to exceed the length of six inches. RONDELETIUS has figured this fish, page 179, in his work De Piscibus Marinis. It has been confounded by YARRELL with the goldsinny of JAGO, which is the *Lutjanus rupestris* of BLOCH.

LEUCISCUS DOBULA, *Yarrell*, vol. ii.—Skellie, or Dobule Roach.

Specific Character.—Dorsal and anal fins even at the end.

A single individual of this species was obtained by Mr YARRELL, in the Thames, in August 1831. No other instance of its capture in Britain has hitherto been recorded. In July last, I was surprized in finding the Dobule roach a common fish in many of the burns falling into the Solway Frith. It exists in the River Annan in great numbers all the year round; and a few are occasionally found entangled in the salmon-nets in the Solway Frith. In the county of Dumfries, these fish are named *skellies*, and have been mistaken by naturalists for the common roach (*Leuciscus rutilus*). They do not appear shy, or in the least choice as to their food; they take eagerly the minnow, the worm, and the fly, and afford excellent amusement to the angler. In the month of April they are in the best condition for the table, but are seldom eaten, their flesh being white and insipid. Skellies are sometimes taken of the weight of five pounds or more, although Mr JENYNS, in his work on the British Vertebrate Animals, states, that they seldom exceed the weight of half a pound. According to Mr YARRELL, this species inhabits the Oder, the Elbe, and the Rhine; it frequents large lakes, and is observed to enter rivers, from March till May, for the purpose of depositing its spawn.

The Dobule roach is very much allied to the common roach and to the dace, differing from them, however, in the following characters:—

Leuciscus dobula.	*Leuciscus rutilus.*
Dorsal and anal fins even at the end.	Dorsal and anal fins concaved at the end.
The middle ray of the tail more than half as long as the longest ray of the same fin.	The middle ray of the tail not half as long as the longest ray of the same fin.
Lateral line with fifty scales.	Lateral line with forty-three scales.
Dorsal fin with nine rays.	Dorsal fin with eleven rays.

Leuciscus dobula.	*Leuciscus vulgaris.*
Seven scales and a half in an oblique row between the dorsal fin and the lateral line.	Eight scales and a half in an oblique row between the dorsal fin and the lateral line.

Leuciscus dobula.	*Leuciscus vulgaris.*
Dorsal and anal fins even at the end.	Dorsal and anal fins concaved at the end.
The middle ray of the tail more than half as long as the longest ray of the same fin.	The middle ray of the tail not half as long as the longest ray of the same fin.

ALOSA VULGARIS, *Yarrell*, vol. ii.—Alis Shad.

Specific Character.—Jaws without teeth; sides without spots.

This species is said to abound in the Severn, and is also occasionally taken in the Thames. On the coast of Scotland it is rare, only appearing in the winter months.

ALOSA FINTA, *Yarrell*, vol. ii.—Shad.

Specific Character.—Jaws with teeth; sides with spots. In the months of July and August this fish is common on the east coast of Scotland. In the winter months it is seldom met with.

RHOMBUS HIRTUS, *Yarrell*, vol. ii.—Top-knot.

Specific Character.—First ray of the dorsal fin not longer than the second; under surface smooth.

This species of fish is rare, both on the English and Scotch coasts. It is seldom known to take a bait, but is occasionally taken in the crab-pots.

RAIA CHAGRINEA, *Montagu.*—Shagreen Ray.

Specific Character.—Tail with two rows of recurved spines; the middle ridge without spines.

Few naturalists appear to have met with this fish. PENNANT obtained a specimen from Scarborough, and Colonel MONTAGU, in the Wernerian Transactions, mentions its occurrence on the coast of Devon.

In the Frith of Forth, in the early part of spring, I have seen specimens taken, both male and female; they inhabit deep water. Their flesh is considered inferior as food to that of the grey skate. I may here mention, to prevent confusion hereafter, that the fish figured and described by Mr YARRELL, in his work on the British Fishes, under the name of ***Raia chagrinea***, appears to be a new species; it certainly is not the Shagreen Ray of MONTAGU or of PENNANT, nor does his figure or description agree with the specimens that I have examined from the Frith of Forth. Mr YARRELL, in his figure of this fish, has given too great a length of snout, and the spines on the tail, which ought to be very much curved, are represented as perfectly straight, which latter character is peculiar to the ***Raia Batis*** or grey skate.

In his description, he observes, the snout is very much produced, narrow and sharp; the upper surface of the body slightly roughened; the second fin on the tail about its own length from the end, and the under surface of the body dirty greyish-white.

The specimens of the shagreen skate that I have examined possess characters differing widely from those above mentioned. The back is very much roughened; the under surface of the body is pure white; the second fin on the tail is not quarter its own length from the end; and the snout is but moderately produced, not so long as is observed in the ***Raia Batis***. If we compare a specimen of the R. ***Batis*** with a specimen of ***R. chagrinea***, both of three feet in length, we shall find that the ***Batis***, from the tip of the nose to the eye measures seven inches, whereas, in the *chagrinea*, the distance between these points measures but five inches.

Raia radiata, *Yarrell*, vol. ii.—Starry Ray.

Specific Character.—Spines on the middle ridge of the tail, conical, three times as large as the lateral spines.

This fish was first made known to naturalists by Mr Donovan, who received a specimen from the north coast of Britain. It has since been found in Berwick Bay, and in the Frith of Forth, but in no other locality has it yet been discovered. This species inhabits deep water in rocky situations, and is taken with the hook in the months of March, April, and May, after which time it is seldom met with until the following spring. I have seen these fish occasionally common in the Edinburgh market. They are well known to the fish-women, who consider them equal to the maiden skate as food.

This species may be distinguished from the rest of the rays, by having three rows of spines on the tail, extending up as far as the transverse cartilage of the back; the spines forming the lateral rows being three times as small, and four times as numerous, as those of the middle row.

Trigon pastinaca, *Yarrell*, vol. ii.—Sting Ray.

Specific Character.—Middle of the tail armed with a long serrated spine.

This fish is more frequently taken on the southern coast than elsewhere. A single specimen was taken in the Frith of Forth in 1835: no other instance of its capture on the coast of Scotland has hitherto been recorded.

Ammocœtes branchialis, *Yarrell*, vol. ii.—Pride.

Specific Character.—Mouth without teeth; under lip transverse.

This fish is not uncommon in the river Teith, inhabiting muddy situations. It is of a light grey colour, seldom exceeding five inches in length.

VI. *Account of a New Species of British Bream, and of an Undescribed Species of Skate: To which is added a List of the Fishes of the Frith of Forth, and its Tributary Streams, with Observations.* By Richard Parnell, M.D., F.R.S.E., &c.

Read 17th April 1837.

In the beginning of July last, I obtained, from the Frith of Forth, a species of bream, which has not been mentioned by naturalists as inhabiting the British seas. A few days after, I procured, from the same quarter, a second specimen of the same species, each exhibiting a conspicuous dark violet-coloured spot at the base of the upper part of the pectoral fins. (See Plate VI.)

On consulting the continental works on Ichthyology, I find this bream to agree best with the description Baron Cuvier has given of the *Pagellus acarne*, an inhabitant of the Mediterranean; but, as no figure of the fish accompanies his description, the discrimination of the species is rendered somewhat uncertain.

From the great similarity the breams bear to each other in their external form, it is not to be wondered at if naturalists have occasionally noticed two species under one synonym, for without accurate figures, or the specimens themselves before us, the closely allied species are with difficulty discriminated.

I think it not improbable, judging from the description Mr Yarrell has given of the *Pagrus vulgaris*, that a specimen of the *acarne* has fallen under his observation, and been mistaken for a variety of the *Pagrus vulgaris*, which it greatly resembles; for, in his description of that fish (vol. i. page 103), he says, "the pectoral fins have *occasionally* a violet-coloured spot at their origin;" a character which is constant in the *acarne*, and which has not been noticed by any other author as occurring in the *Pagrus vulgaris*.

Generic Characters.—Pagellus.—Front teeth conical, sharp, and numerous. Molars rounded.

Specific Character.—On the base of the pectoral fins, a large dark spot.

Description.—Length 13 inches; depth, in the region of the pectorals, 4 inches. *Head* one-third the length of the body, exclusive of the caudal rays. *Eye* placed half way between the tip of the upper jaw and the posterior margin of the operculum; its diameter one-fourth the length of the head. *Pectorals* reaching as far as the first ray of the anal fin. *Dorsal* fin commencing over the posterior margin of the operculum, and ending in a line with the last ray of the anal fin. Pectorals and ventrals commencing on the same line. Middle ray of the tail not half so long as the longest ray of the same fin. First flexible rays longer than the terminating spiny rays. General form resembling that of the sea-bream, but not so deep in proportion to its length. Dorsal line rounded, descending obliquely from the nape to the nostrils, from thence more suddenly to the lips. *Scales* large—seventy forming the lateral line. Six and a-half in an oblique row between it and the first ray of the dorsal fin. *Lateral line* strongly marked, commencing at the upper part of the operculum, and taking its course parallel to the curvature of the back. Spiny rays of the dorsal fin sharp and stout; the first spine short, about half the length of the second; the fourth the longest; the remainder gradually decreasing in height to the commencement of the flexible rays. Jaws nearly equal, the under rather the shorter. Anterior teeth small and numerous, disposed in many rows; the outer row, composed of thirty teeth, longer and more bent than those within. Molars large, disposed in three rows in each jaw. (In one of the specimens but two rows were perceptible, and these irregularly placed.) The number of the fin rays are—

D. 12 spinous, 12 soft. P. 16. V. 8. A. 3 spinous, 11 soft. C. 20.

The intervening membranes of the caudal, and of the last two rays of the dorsal and anal fins, are covered with small, thin scales, diminishing in size as they approach the summits of the rays. *Colours:* Body pale silvery-red. Dorsal and caudal fins rose-red. Ventral and anal fins paler. Space between the eyes reddish-brown. In front of the eyes, and on the lower half of the præoperculum, metallic gray. On the upper part of the base of the pectorals is a dark violet-coloured spot, very conspicuous even in the dried specimen.

The British breams with which the *Pagellus acarne* is most likely to be confounded, are the *Pagellus centrodontus*, *Pagellus erythrinus*, and the *Pagrus vulgaris*.

It differs from *P. centrodontus*, in the eye being smaller, the molars larger, and in having a dark spot on the base of each pectoral fin, which the *P. centrodontus* never exhibits.

It differs from the *P. erythrinus* in having the origin of the lateral line straight, whereas in the *P. erythrinus* it is suddenly bent; in the eye being larger; and in the pectoral spot, which is never found in the *P. erythrinus.*

The *Pagellus acarne* is distinguished from the *Pagrus vulgaris*, by the form and arrangement of the anterior teeth: the teeth of the *P. acarne* being about thirty in number in the first row on the upper jaw, and nearly of equal size. The *Pagrus* has never more than six teeth in front, which are much longer than those within, besides shewing no pectoral spot.

Raia intermedia, *Parnell,*—Flapper Skate. (See Plate VI.)

The species of Rays are but imperfectly understood; for perhaps there is no genus of fishes which has received so little attention from naturalists.

From the great numbers of skate captured in the Frith of Forth during the summer months, and from the great ease with which they can be examined in the market-place, where scores of them are daily to be seen, ample opportunity is afforded to the naturalist of judging for himself, and becoming acquainted with those more common varieties, which have created so much confusion among writers on Ichthyology. It is through the opportunities thus afforded me, that I have been enabled to add another species to the British Fauna.

This species of ray is not unfrequently met with in the Frith of Forth, and it belongs to the division of the sharp-nosed rays.

Specific Characters.—Body on the upper surface perfectly smooth; under surface dark green.

Description.—Length two feet. *Body* thin. Flesh hard. *Snout* sharp and prominent, from the tip to the middle of the eye one-third the length of the body. *Tail* rather short, being no longer than the space from the base of the anal fin, to the anterior margin of the eye. *Eyes* rather small, with a sharp spine placed in front of each. *Tail* with a row of spines placed on the mesial line only, not extending farther up than to the base of the anal fin. First dorsal fin rather remote from the second; second dorsal fin placed near the rudimentary caudal fin. Body perfectly smooth on both surfaces. *Teeth* small, not so sharp as those in *Raia batis. Colours: Body* above dark olive-green, on the under surface dark grey, with minute specks of a deeper colour. In one of the specimens, the upper surface was mottled with large white spots, thickly placed on each pectoral fin.

This species of skate appears to be the connecting link between the *Raia batis* and the *Raia oxyrhynchus,* to both of which it is closely allied: it is from this circumstance, that I suggest the specific name *intermedia.*

It is distinguished from the *R. batis* or grey skate, by the nose being longer, by the first dorsal fin being more remote from the second; and by the skin on the back being perfectly smooth, which in the *R. batis,* is covered with small spicula, and rough to the touch.

It is at once removed from the *R. oxyrhynchus* of Montagu, by the under surface of the body being of a dark grey colour; which part in the *R. oxyrhynchus* is perfectly white.

It has occurred to me, that it might be useful to add to this notice a List of the Fishes of the Frith of Forth, and its tributary streams; as no catalogue of the fishes of this district of Scotland, has been drawn up since that published by Dr NEILL, in the first volume of the Wernerian Memoirs in 1805. In that catalogue seventy-six species are enumerated. In the list now presented to the Society, one hundred and twenty-three species are mentioned, about forty of which have been added by myself from personal observation: three of these have not been described before as fishes of Scotland, and two are new to the British Fauna. Three species are noticed by Dr NEILL as being found in the Frith of Forth, viz. *Mugil cephalus*,* *Squalus maximus*, and *Raia oxyrhynchus*, which I have not yet met with; besides one noticed by Mr CHARLES STEWART, the *Scomber Pelamys*, and one, the *Syngnathus æquoreus*, mentioned by Sir ROBERT SIBBALD; and I have no doubt but that other species would be found, were attention directed to this pursuit in the proper seasons at the different fishing-stations of the Frith.

The following list therefore is not given as complete; yet it may serve to aid future inquirers as supplying a record of the observed species up to the present period.

The observations I intend to offer on the great family of the SALMONIDÆ, a tribe of fishes important in so many respects, I postpone for the present, till further investigation enable me to give more definite information in regard to the species and their habits.

A List of the Fishes found in the Frith of Forth and its Tributary Streams, with Observations.

Perca fluviatilis, *Yarrell*, vol. i.—Perch. Frequent in lochs in the neighbourhood of Edinburgh; occasionally found in the Forth above Alloa.

Labrax lupus, *Yarr*. vol. i.—Basse or Sea Perch. Occasionally taken in the salmon-nets along with the thick-lipped grey mullet. Not common.

Trachinus vipera, *Yarr*.—Sting-fish or Adder-pike. Taken on Musselburgh Sands. Rare. Found by Mr STARK at Portobello in 1831. Some authors state that "they grow to the length of a foot." The oldest fisherman in the Solway Frith (where these fish are in great abundance) never saw or heard of one more than six inches long.

Trigla cuculus, *Yarr*.—Red Gurnard or Crooner. Rare.

Trigla hirundo, *Yarr*.—Tub Gurnard. Rare.

Trigla gurnardus, *Yarr*. Grey Gurnard or Crooner. Common; taken principally with the hook.

Trigla Blochii, *Yarr*.—Bloch's Gurnard. Common in the month of August. May not this species prove to be the young of the Grey Gurnard?)

Cottus scorpius, *Yarr*.—Short-spined Bull-head. Common in pools left by the receding of the tide.

* I am informed by Dr NEILL that it is the *Mugil cephalis* of DONOVAN, now supposed to be the *Mugil capito* of CUVIER.

Cottus bubalis, *Yarr.*—Long-spined Bull-head. As common as the last; inhabiting the same situations.

Aspidophorus cataphractus, *Yarr.*—Shell-backed Bull-head. Common.

Gasterosteus trachurus, *Yarr.*—Full-armed Stickleback. Not common. Found in saltwater pools at Guillon.

Gasterosteus semiarmatus, *Yarr.*—Half-armed Stickleback. Occasionally found in saltwater marshes. Not common.

Gasterosteus leiurus, *Yarr.*—Quarter-armed Stickleback. Common in fresh and salt water ditches.

Gasterosteus spinulosus, *Yarr.*—Four-spined Stickleback. First made known by Mr Stark, who found a number of specimens in the Hope Park Meadow ditches. Specimens have also been found in Duddingston Loch, at Queensferry, and at Berwick-upon-Tweed. In the latter locality I have found a variety of this fish with the third spine one-half as short as the fourth. Not common.

Gasterosteus pungitius, *Yarr.*—Ten-spined Stickleback. Not common.

Gasterosteus spinachia, *Yarr.*—Fifteen-spined Stickleback. Not common. Found in deep pools among fuci.

Sciæna aquila, *Yarr.*—Maigre. Rare.

Pagellus erythrinus, *Yarr.*—Spanish Bream. Rare.

Pagellus acarne, *Parnell, Trans. Roy. Soc. of Ed.* 1837.—Axillary Bream. Rare.

Pagellus centrodontus, *Yarr.*—Sea Bream. Occasionally taken in the salmon-nets. Not common.

Brama Raii, *Yarr.*—Ray's Bream. Rare.

Scomber scomber, *Yarr.*—Mackarel. Common.

Thynnus pelamys, *Yarr.*—Bonito. Rare. On the authority of Mr Charles Stewart. *Elem. Nat. Hist.* vol. i. p. 363.

Xiphias gladius, *Yarr.*—Sword-fish. Rare.

Caranx trachurus, *Yarr.*—Horse mackarel. Not common.

Zeus faber, *Yarr.*—John Dory. Rare.

Lampris luna, *Yarr.*—Opah. Rare. A specimen was obtained from the Frith of Forth, in 1835, the head of which is in my collection.

Mugil cephalus, on the authority of Dr Neill. Wern. Trans. i. 544. Mugil capito of Cuvier.

Mugil chelo, *Yarr.*—Thick-lipped Gray Mullet. Frequently taken in the salmon-nets at Musselburgh and Queensferry.

Atherina presbyter, *Yarr.*—Atherine. Rare. Sometimes taken in the cruives at Kincardine.

Blennius pholis, *Yarr.*—Smooth Blenny, or Stone-fish. In pools, under stones. Common.

Muraenoides guttata, *Yarr.*—Spotted Gunnel, or Stone-checker. Common in pools left by the receding of the tide, under sea-weed.

Zoarces viviparus, *Yarr.*—Viviparous Blenny, or Green-bone. Common.

Anarrhichas lupus, *Yarr.*—Wolf or Cat-fish. Common.

Gobius niger, *Yarr.*—Black Goby. Not common. Frequenting rocky situations.

Gobius minutus, *Yarr.*—Freckled Goby. Common in sandy places.

Gobius unipunctatus, *Parnell.*—One-spotted Goby. Frequently met with at Queensferry.

Gobius bipunctatus, *Yarr.*—Two-spotted Goby. Not common. Inhabiting deep pools, among sea-weed.

Gobius gracilis, *Yarr.*—Slender Goby. Not common. Occasionally found at Queensferry.

Callionymus Lyra, *Yarr.*—Gemmeous Dragonet. Not Common.

Callionymus dracunculus, *Yarr.*—Sordid Dragonet. More frequently met with than the last species, inhabiting deep water.

Lophius piscatorius, *Yarr.*—Angler, or Merring. Common.

Labius maculatus, *Yarr.*—Ballan Wrasse. Not frequent.

Labius trimaculatus, *Yarr.*—Trimaculated Wrasse. Rare.

Crenilabrus tinca, *Yarr.*—Ancient Wrasse. Not common. Met with occasionally in the salmon-nets.

Crenilabrus cornubicus, *Yarr.*—Goldsinny. Not common.

Leuciscus rutilus, *Yarr.*—Roach. Common in the Union Canal. Found by Mr J. Wilson.

Leuciscus phoxinus, *Yarr.*—Minnow. Common. Water of Leith, &c.

Cobitis barbatula, *Yarr.*—Bearded Loach. Common in fresh-water streams.

Esox lucius, *Yarr.*—Pike. Frequently met with a few miles above Stirling, in Lochend, and Duddingston Loch.

Belone vulgaris, *Yarr.*—Sea Needle. Not uncommon in the month of August. Some authors consider the teeth as wanting in the vomer. In the dried state, teeth are found in a small cluster on the roof of the mouth, as well as in a single row in each jaw.

Scomberesox saurus, *Yarr.*—Skipper. Common in some seasons.

Salmo salar, *Yarr.*—Salmon. Salmon are found in the Frith of Forth in the greatest abundance towards the end of July. They ascend the Forth, the Teith, and the Allan, to deposit their spawn, and after this is accomplished return again to the sea. The spawn which is thus shed during the months of November, December, and the early part of January, begins to vivify in March, when the fry are seen nearly an inch in length issuing from the gravel beds with the ovum still attached. About the end of April or the beginning of May, they are seen from three to four inches long sporting about in the shallows. Towards the end of May, when they perform their first migration to the sea, they are observed a few miles below Stirling in brackish water from five to seven inches in length. (See Plate VII.) From the time they reach the sea to a month or six weeks after, they are not seen; and we can only infer their growth from the fact, that after the lapse of that period we find them returning to the rivers in which they were bred, having acquired a weight between a pound and three-quarters and two pounds and a-half; they are now named *Grilses*. These fish rapidly increase in size; and by the end of July are taken from five to six pounds in weight, when they have assumed the appearance of perfectly-formed salmon.

The characters by which the salmon is distinguished from the sea trout are by naturalists ill-defined, and this seems owing to the fish not having been examined at the several stages of its growth.

Mr Yarrell (in his excellent work on the British Fishes, vol. ii.) speaks of "the teeth of the vomer in the salmon, seldom exceeding two in number, and no other teeth extending along the vomer as in the sea trout." Now, if we examine a young salmon of eight inches in length, we shall find the teeth as numerous as in a trout of equal size, as it has from twelve to fourteen in number running back the whole length of the vomer. (See Plate VIII.) In a young salmon of eighteen inches in length, the teeth are from five to seven in number, placed on the anterior part of the vomer; and in a salmon three feet long the teeth on the vomer are often entirely wanting.

The principal character, which at once removes the salmon from the migratory trout, is derived from the anatomical structure of the internal organs.

The cæcal appendages in the salmon I have never found less than fifty-eight in number, the average number being sixty-two; whereas in no instance have I ever found the cæcal appendages in the migratory trout more than fifty-seven in number, the average number being fifty-four.

By combining a number of external characters together, the experienced Ichthyologist finds no difficulty in distinguishing the salmon from the trout at all ages.

The salmon has never more than six spots below the lateral line, and often is without any. The lower third of the pectorals is always black, as well as the intervening membrane between the first three rays of the ventral fin. The middle ray of the tail is never more than half as long as the longest ray in that fin.

Mr Yarrell, in his extensive collection of prepared fishes, possesses a young salmon about a pound weight. Dr Johnston of Berwick has a specimen in his possession a pound and a-half in weight; and the young salmon which I have now the honour of exhibiting to the Society, measures eighteen inches in length, the weight being a pound and a-quarter.

Salmo eriox, *Yarr.*—Bull Trout. Common.

Salmo trutta, *Yarr.*—Salmon Trout. Common. Most naturalists have confounded many species of migratory trout under the names of *Salmo eriox* and *Salmo trutta*; and I hope, at a future meeting of the Society, to shew, that, instead of two species only inhabiting the British waters, there are in reality seven; the characters of which depend on the structure of their internal organs, the number of scales, and the form and arrangement of the lateral spots. The teeth in the vomer of the migratory trout cannot be depended on as a character excepting in fishes of a certain size, for when they are eight inches in length the whole vomer is armed with teeth, and when they exceed nine pounds in weight, the vomer has never more than three teeth, and frequently has only one; the number of teeth depending on the age of the fish.

Salmo albus, *Dr Fleming*.—Hirling or Whitling. Common. In the Frith of Forth never taken, owing to the meshes of the salmon-nets being too large for their capture. Many naturalists, as well as most practical fishermen, consider the hirling as a distinct species of trout. It is said never to exceed a foot or fifteen inches in length, having a dark back, silvery sides, and a forked tail. Last summer, with the view of examining these fish more minutely than had hitherto been done, I remained several weeks on the banks of the Solway Frith, where I had an opportunity of inspecting several hundred specimens as soon as they were taken from the nets. After carefully dissecting two hundred specimens, and finding them to differ exceedingly from one another, in their anatomical structure, in the number of scales, in the colour of the flesh, and in the form and arrangement of the lateral spots, I came to the conclusion, that they are not a distinct species, but the young of different species of trout, which, if allowed to remain uncaught, would increase to six or even eight pounds in weight.

Every British species of migratory trout less than fourteen inches in length has the tail deeply forked, and, as the fish increases in size, the middle rays become elongated, so that, by the time the fish reaches the weight of nine pounds, the tail is even at the end. Colour in trout cannot be depended on as a constant character, being liable to vary with accidental circumstances.*

The natural history of the migratory trout is somewhat similar to that of the salmon, but the growth of the latter fish seems to be more rapid than that of the former.

From the beginning of June to the middle of July, trout are observed to leave the salt-water and ascend rivers, in search of a suitable situation to deposit their spawn; this they shed in the months of October, November, and December, and when this law of their nature is fulfilled, they, like the salmon, return again to the sea. In March and April the fry make their first appearance, from an inch to an inch and a half in length; in June they are found from two to three inches long; in August, September, and October, they are taken by anglers, under the name of *Parrs*, from four to five inches in length. In December they are somewhat larger, and in April and May the following year, they make their first

* Stark in Edin. New Phil. Jour. Oct. 1830, p. 327.

migration to the sea, when they are observed six and a half, seven, and even eight inches in length.

After they reach the sea they are lost sight of until the middle of the following July, when they are seen in their return to the rivers, from ten to twelve inches in length, under the name of Herlings or Whitlings.

The herlings, so far as I have observed, remain in the rivers until the end of December or beginning of January, and return again to the sea. In June and July they reappear, being from a pound and a half to two pounds in weight, when they are named *Sea-trout*, and are now of sufficient size to reproduce their species in the following October.

Salmo cæcifer,* *Parnell.* Salmo levenensis, *Walker.*—Lochleven Trout. Common in Lochleven.

This species of trout, which is well known to many persons as a delicate article of food, is considered by most naturalists as a variety of the *Salmo fario* or common fresh-water trout, the redness of its flesh depending on the nature of its food.

I consider it, however, not only as distinct from the *Salmo fario*, but as one of the best defined and most constant in its characters of all the species hitherto described. It is at once distinguished from the common fresh-water trout by the number of its cæcal appendages, which varies from sixty to eighty, whereas in the *S. fario* they are never more than forty-five or forty-six in number. Its tail is crescent-shaped at all ages, and its body has never a vestige of a red spot. The tail of the *common trout* is sinuous, and at length even at the end, and its body is almost always marked with red spots, besides its flesh being always of a white appearance. (See Plate VIII.)

I have no doubt but that more than two species of trout are to be met with in our freshwater streams, which at present receive the name of *Salmo fario.*

Salmo Salmulus, *Yarr.*—Parr. Common in the river Forth. Though the parr is stated by Ichthyologists as a distinct species of trout, yet characters have not hitherto been given, by which it is to be distinguished either from the young of the sea-trout or from the young of the salmon, and it is from the want of some constant specific character that the parr has been so often mistaken for the young of the salmon.

If a young salmon of eight inches in length be compared with a parr of equal size, they will be found to differ in the following respects (See Plate VIII.):

The pectoral fin of the parr is large and dusky at the end, measuring one-fifth of the length of the body, exclusive of the caudal rays. The same fin in the

* Bearing cæca,—the cæca being more numerous than in any of its congeners.

young salmon is black at the end, and measures one-sixth of the length of the body.

The dorsal fin of the parr is situated nearer to the base of the middle caudal rays than to the tip of the upper jaw. In the young salmon the dorsal fin is placed exactly half way between the point of the upper jaw, and the base of the middle caudal rays.

In the parr the cæcal appendages are never more than forty-five in number. In the salmon they always exceed fifty-seven.

It is supposed by most fishermen, that the blue bands which are observed on the sides of the parr, and the black spot on the operculum, are peculiar to that fish; but the same kind of mark is also observed in the young salmon, the sea-trout, the bull-trout, and the common fresh-water trout.

The parr of eight inches in length differs from the sea-trout of equal size, in the same respects as it does from the young salmon, excepting in the number of cæcal appendages and the colour of the pectoral fin. (See Plate VIII.)

The parr is distinguished from the common fresh-water trout (*Salmo fario*) by the middle ray of the tail being less than half the length of the longest ray of the same fin; the middle ray of the tail in the trout being more than half as long as the longest ray of that fin. (See Plate VIII.)

The parr is considered by some authors to be a migratory species, and "as soon as they have spawned, they retire, like the salmon, to the sea, where they remain till the autumn, when they again return to the rivers."*

As the parr has never yet been found in salt-water, I am inclined to suppose it to be a fresh-water species, remaining, like the common trout (*Salmo fario*), in our rivers throughout the year.

The natural history of the parr is still involved in great obscurity; nor is this difficulty, any more than the multitude of unsettled points in science, to be cleared up by mere conjecture or hypotheses, but by the slow accumulation of facts, and the unsparing correction of error.

Salmo fario, *Yarrell.*—Common Fresh-water Trout. Tail sinuous, and at length even at the end, its middle ray more than half as long as the longest ray of the same fin. The summit of the four anterior dorsal and anal rays, white, with a black band beneath. Common in the neighbouring streams. (See Plate VIII.)

Salmo umbla, *Yarr.*—Charr. Occasionally taken in Lochleven.

Osmerus eperlanus, *Yarr.*—Smelt. Common at Alloa.

Clupea harengus, *Parnell*, Zool. and Bot. Mag. vol. i. p. 54.—Herring. Common.

* YARRELL's British Fishes, vol. ii.

Clupea sprattus, *Parnell*, Zool. and Bot. Mag. vol. i. p. 52.—Sprat. Common.

Clupea alba, *Parnell*, Zool. and Bot. Mag. vol. i. p. 50.—Whitebait. Common.

Clupea pilchardus, *Yarr.*—Pilchard. Rare.

Alosa finta, *Yarr.*—Shad. Common.

Alosa vulgaris, *Yarr.*—Rock-Herring. Rare.

Gadus morrhua, *Yarr.*—Cod. Common.

Gadus æglefinus, *Yarr.*—Haddock. Common.

Merlangus vulgaris, *Yarr.*—Whiting. Common.

Merlangus pollachius, *Yarr.*—Pollock. Rare.

Merlangus carbonarius, *Yarr.*—Coal-fish. Common. The young are named Podlies.

Merlangus virens, *Yarr.*—Green-Cod. Not Common.

Merlucius vulgaris, *Yarr.*—Hake. Rare.

Lota molva, *Yarr.*—Ling. Common.

Motella vulgaris, *Yarr.*—Three-bearded Rock-Ling. Not common.

Motella quinquecirrata, *Yarr.*—Five-bearded Rock-Ling. Common.

Raniceps trifurcatus, *Parnell*, Zool. and Bot. Mag. vol. i.—Tadpole Fish. Not Common.

Platessa vulgaris, *Yarr.*—Plaice. Common.

Platessa flesus, *Yarr.*—Mud-Flounder. Common.

Platessa limanda, *Yarr.*—Sand-Flounder or Saltie. Common.

Platessa microcephala, *Yarr.*—French Sole or Lemon Dab. Not common.

Platessa Pola, *Yarr.*—Craig-Fluke or Pole. Not common.

Platessa limandoides, *Parnell*, Phil. Journ. No. 37, 1835.—Sand-sucker. Not unfrequently met with in the month of May.

Hippoglossus vulgaris, *Yarr.*—Holibut. Common.

Rhombus maximus, *Yarr.*—Turbot. Common.

Rhombus vulgaris, *Yarr.*—Brill. Not so common as the turbot.

Rhombus hirtus, *Yarr.*—Black-Fluke. Rare.

Solea vulgaris, *Yarr.*—Sole. Not Common.

Cyclopterus lumpus, *Yarr.*—Padle. Frequently taken in the salmon-nets.

Liparis vulgaris, *Yarr.*—Sea-Snail. Rare.

Anguilla acutirostris, *Yarr.*—Sharp-nosed Eel. Common.

Anguilla latirostris, *Yarr.*—Broad-nosed Eel. Common.

Conger vulgaris, *Yarr.*—Conger Eel. Not common.

Ammodytes tobianus, *Yarr.*—Wide-mouthed Sand-eel. Not common.

Ammodytes lancea, *Yarr.*—Small-mouthed Sand-eel. Common.

Syngnathus acus, *Yarr.*—Great Pipe-fish. Frequently met with under sea-weed.

Syngnathus typhle, *Yarr.*—Lesser Pipe-fish. Not common.

Syngnathus æquoreus.—Æquoreal Pipe-fish. Rare; on the authority of Sir ROBERT SIBBALD, *Prod.* part ii. sect. ii. p. 24. tab. 19.

Syngnathus ophidion, *Yarr.*—Snake Pipe-fish. Rare.

Orthagoriscus mola, *Yarr.*—Short Sun-fish. Occasionally met with.

Acipenser sturio, *Yarr.*—Sturgeon. Not common.

Acipenser latirostis, *Parnell, Trans. Roy. Soc. Edin.* 1837.—Broad-nosed Sturgeon. Rare.

Scyllium canicula, *Yarr.*—Small Spotted Dog-fish. Not common.

Scyllium catulus, *Yarr.*—Large Spotted Dog-fish. Not common.

Lamna cornubica, *Yarr.*—Porbeagle-shark. Occasionally found.

Galeus vulgaris, *Yarr.*—Tope-Shark. Occasionally met with.

Mustelus lævis, *Yarr.*—Smooth-Hound. Not common.

Selachus maximus.—Basking-Shark. Rare; on the authority of Dr NEILL, *Wern. Trans.* i. 550.

Spinax acanthias, *Yar*—Dog-fish. Common.

Squatina angelus, *Yarr.*—Angel-Fish. Rare.

Raia chagrinea, *Montagu, Wern. Trans.* vol. ii.—Shagreen Ray. Not common.

Raia oxyrhynchus.—Sharp-nosed Ray. On the authority of Dr NEILL, *Wern. Trans.* i. 553.

Raia batis, *Yarr.*—Grey-Skate. Common.

Raia intermedia, *Parnell, Trans. Roy. Soc. Edin.* 1837.—Flapper Skate. Occasionally met with.

Raia maculata, *Yarr.*—Spotted-Ray. Not common.

Raia clavata, *Yarr.*—Thornback. Common.

Raia radiata, *Yarr.*—Starry-Ray. Frequently met with in the month of April.

Trygon pastinaca, *Yarr.*—Sting-Ray. Rare.

Petromyzon marinus, *Yarr.*—Sea-Lamprey. Not common.

Petromyzon fluviatilis, *Yarr.*—River-Lamprey. Common.

Petromyzon Planeri, *Yarr.*—Planer's Lamprey. River Forth, rare.

Ammocœtes branchialis, *Yarr.*—Pride. Frequently met with in the River Teith.

CORRIGENDUM.

Page 141, line 7, *for* and at length crenated, as in the grey gurnard a foot in length. *read* and at length the lateral line and dorsal ridge become crenated, as is seen in the grey gurnard when a foot in length.

On the Power of the Periosteum to form New Bone. By James Syme, Esq., Professor of Clinical Surgery in the University of Edinburgh.

Read 6th March 1837.

The object of the following paper is to put at rest a question which has been long agitated in Surgical Pathology, and which is intimately connected with some important points of Practical Surgery. An apology may seem due to the Society for bringing under its consideration a subject, which, though not exclusively professional, is still little studied except by those physiologists whose views are directed to surgery; but as the inquiry into which I propose to enter is neither long nor tedious, while it is quite intelligible without any previous knowledge of its details, I trust the patience of the members will not be exhausted; and if the question shall, as I hope, be decided to the conviction of those members who are conversant with surgical discussions, the prevailing diversity of sentiment relative to the point at issue will be more effectually composed than if I attempted to combat it through any other channel.

The question which I propose to consider is, "Whether the Periosteum, or membrane that covers the surface of the bones, possesses the power of forming new osseous substance independently of any assistance from the bone itself?"

This property was first attributed to the periosteum by Duhamel, just 100 years ago. Having been engaged in the study of vegetable physiology, and more particularly the formation of wood, he imagined that there might be an analogy between the inner layer of the bark and the periosteum, and that as the former hardens in successive layers so as to constitute the wood, the latter might suffer a corresponding conversion into bone. He supported this opinion by the following arguments: 1. That when bones are burned in the fire or exposed to the weather, they separate into a number of thin plates. 2. That in consequence of disease arising from external violence, the bones frequently throw off thin scales, or exfoliations as they are called. 3. That when animals are fed alternately with madder and without it, their bones exhibit alternate layers of a red and white colour; and, 4. That when bones are fractured, they unite by means of an osseous capsule formed externally to, and embracing the broken extremities, just as the branch of a tree acquires strength after being grafted, or simply broken across.

This theory of Duhamel was strenuously opposed by Haller, who urged, as altogether inconsistent with it, the mode in which bones are originally formed.

He carefully investigated the process of ossification during incubation, and detailed the steps of its progress in the chick as well as other young animals. The rudiment of the future bone being traced from its earliest distinguishable appearance, was found first to present the characters of a jelly; then to acquire the consistence of cartilage or gristle; and finally to reach the perfect osseous state: whence it was contended, that a structure which thus originated in a distinct form, and independently of any other, could not owe its increase afterwards to a different source. HALLER also engaged his pupils DETLEF and BOEHMER in extensive series of experiments, by breaking the bones of animals, and feeding them with madder during their recovery; from the results of which he inferred, that DUHAMEL had been mistaken in supposing that fractures are reunited by ossification of the periosteum.

Notwithstanding these objections, and the authority of the physiologist from whom they proceeded, the doctrine of DUHAMEL still maintained its ground; and not long afterwards, viz. in the year 1780, derived a great accession of strength from the experiments of TROJA, who, by destroying the marrow of bones, caused their death, and the formation of new shells surrounding them, apparently from ossification of the periosteum. This experiment, which TROJA himself performed some hundreds of times, when repeated and varied by the pathologists of almost every country, seemed to confirm the ossific power attributed to the periosteum beyond question, until SCARPA, the late distinguished Professor of Pavia, again investigated the grounds on which it was originally founded by DUHAMEL. In SCARPA'S treatise "De Penitiori Ossium Structura," which was published in 1799, he explained, that the foliated appearance presented by bones that had been burnt did not depend upon the development of a structure naturally belonging to them, but was an effect produced by the unequal action of the fire; and that the separation of scales from diseased bones was no stronger proof of their possessing a laminated structure, since thin and broad portions of dead substance are wont to be detached from the skin and other soft textures, in which it was never supposed that layers existed naturally. He recalled attention to the synthetic experiments of HALLER, who, by investigating the formation of bone from the earliest stage to its perfect state, had established the reticulated nature of its texture; and by an opposite process of an analytic kind, which consisted in depriving bones of their earthy constituent by means of diluted acids, and then macerating them for a long while in water, he unravelled the texture so as to shew that it really was reticular. As a consequence of these observations, SCARPA denied that bone could be formed by the periosteum; and this opinion was keenly embraced by several pathologists of the present century, and particularly by the French surgeon LEVEILLE.

At present, professional opinion is divided in regard to the ossific power of the periosteum, and different sides of the question are maintained by teachers and writers in this as well as other schools of medicine. As the point in dispute is not merely a matter of curiosity, but one of great practical importance, it is very desirable that the truth should be ascertained. It would detain the Society too long were I to shew how the different opinions on this subject may influence the practice of surgery; and I shall, therefore, proceed to state the considerations which have completely satisfied my own mind, and are, I think, sufficient to satisfy any one who is open to conviction, that though DUHAMEL was misled into many errors by the false analogy which he supposed to exist between wood and bone—in regard to the mode of their natural formation—the periosteum nevertheless does possess the power of producing new osseous substance in certain conditions of disease.

The well-known and often-repeated experiment of TROJA, which consisted in perforating the cavity and destroying the marrow of a bone, so as to kill it and cause the formation of a substitute in the form of a shell surrounding the old one, was devised in imitation of a process which not unfrequently occurs spontaneously in the human body. In this disease, which has been named Necrosis, a portion of the old bone dies, and becomes surrounded by a new one. There is an example of this on the table, in which the tibia, or principal bone of the leg, has been thus affected. The new shell is of a larger size and more irregular form than the old one, which may be seen through a number of circular apertures lying a prisoner within this structure, intended by Nature to serve as a substitute for it. Those who deny the ossific power of the periosteum, maintain that in all such cases, whether resulting from injury purposely inflicted with the view of experiment, or proceeding from diseased action, a portion of the old bone remains alive, and serves as the germ of a new one; that, in short, the formation of the new bone is simply an expansion or growth from the remnant of the old one, and that if merely the extremities of the bone affected remain alive, they will prove sufficient for generating the substitute shell.

It is difficult to reconcile this explanation with the rapid growth and uniform thickness of the new bone; since, if its formation proceeded from the extremities, the process should be slow and progressive towards the centre; but there is another objection still more conclusive against it. If the new bone is formed by a portion of the old one that remains alive, then the removal of a part by mechanical means should be supplied from the same source. But in all the cases where this has been done, either in the way of experiment or for the cure of disease, the loss of substance, unless of small extent, has been found imperfectly repaired. For instance, after the operation of trepanning the skull, the aperture in the bone, though it becomes diminished in extent, is not altogether obliterated, and the newly-formed bone is not only smaller than the portion removed, but also thinner, as may be seen from the specimens before me.

In the fore-legs of dogs and rabbits, there are two bones of nearly equal size, and so connected, that a large portion of one may be taken away without destroying the rigidity of the limb. There is here, therefore, a convenient opportunity of trying what can be done by the extremities of a bone for restoring losses of substance in its shaft. Experiments of this kind have accordingly been frequently performed on these animals, and the result has uniformly been, that when the portion removed exceeded an inch in length, there was a permanent deficiency of osseous substance, the ends of the bone being merely produced towards each other in a conical form, and connected together by a tough ligamentous texture. Sir A. Cooper has given representations of the results he met with; and on the table there is a specimen of my own experience (see Fig. 1, Plate IX).

Some of those pathologists who deny the ossific power of the periosteum, and claim the whole production of new osseous substance for the bone itself, have attempted to explain away the difficulties which have just been stated, by supposing, that in cases of necrosis where a new bone is formed, the old one, in consequence of the increased action preceding its death, may determine the effusion of organizable matter into the surrounding soft textures, which will serve as a matrix or foundation for the new shell, and be ready to take up the ossifying process so soon as it is communicated from the surviving extremities of the bone. That the process of reproduction may be accomplished in this way I am not prepared to deny, but that it is not necessarily, or always so performed, will, I think, appear from the following case.

A girl twelve years of age strained her ankle in the month of March 1835. Inflammation followed, extending up to the knee, and attended with violent fever. She was brought to the hospital, and placed under my care. Incisions were soon afterwards made to evacuate a large collection of matter which had formed in the leg. And the bone being found dead, while the patient's strength was rapidly giving way, I amputated the limb above the knee five weeks after the injury had been received. The girl recovered, and is now well. In examining the limb to ascertain the extent to which the bone had died, I found that it was partially surrounded by the commencement of a new one. This shell had already acquired considerable firmness at some parts, but was not equally thick throughout, and did not seem fixed to the ends of the old shaft. This observation led to a very careful dissection of the parts concerned; and they are now before the Society. It will be seen that the tibia had died very nearly from end to end, and that the new shell inclosing it has been formed in the periosteum. The new osseous substance may be observed at some parts in the form of small distinct scales. At other parts it looked as if it had originally consisted of separate portions, and been composed by their union. The periosteum connecting these portions to each other and to the extremities of the bone was not thickened beyond its natural condition; and where it covered the posterior surface of the tibia, though quite detached from the old bone, had not suffered any farther change.

There is here, then, an instance of a bone dying suddenly in consequence of acute inflammation, without any thickening having previously formed in its neighbourhood, and nevertheless succeeded by the production of a new osseous shell, which evidently could not proceed from the old bone, and no less evidently depended upon an ossific process resident in the periosteum.

As Nature is not capricious or variable in her proceedings, I regard this case as sufficient of itself, without any farther evidence, to establish the ossific power of the periosteum. But, with the view of making the matter still more clear, I performed the following experiments. I exposed the radius of a dog, and removed an inch and three-quarters of it together with the periosteum. At the same time I exposed the radius of the other leg, and removed a corresponding portion *without* the periosteum, which was carefully detached from it and left quite entire, except where slit open in front. Six weeks afterwards the dog was killed, and the bones examined. In the one from which a portion had been taken together with the periosteum, the extremities were found extended towards each other in a conical form, with a great deficiency of bone between them, and in its place merely a small band of tough ligamentous texture. In the other, where the periosteum had been allowed to remain, there was a compact mass of bone not only occupying the space left by the portion removed, but rather exceeding it (see Fig. II.). This experiment was repeated, and afforded the same results.

I next exposed the radius of another dog, and separated the periosteum from the bone as in the former experiment; but then, instead of cutting out the denuded bone, inserted a thin plate of metal between it and the periosteum. The edges of the membrane, as well as those of the skin, were sewed together, and the wound healed kindly. At the end of six weeks I dissected the limb, and found a deposition of osseous substance in the periosteum, forming a bony plate exterior to the metal, and not connected with the old bone except by the membrane.

I lastly exposed the radius of a dog, and cut away the periosteum to the same extent that it had been merely detached in the experiment just mentioned, and surrounded the denuded bone with a piece of metal. At the end of six weeks, I found a thick tough capsule formed, enclosing the metallic plate, but having no osseous substance in it.

The evidence which has now been adduced seems to me sufficient for putting beyond all question the power of the periosteum to form new bone, independently of any assistance from the old one. I submit it, with deference, to the Society, in the hope, that those members who have directed their attention to the subject, will give it their dispassionate consideration, and either admit the opinion which it supports, or shew the fallacy by which it has misled.

EXPLANATION OF PLATE IX.

Fig. I. Radius of a dog from which a piece of bone was taken along with the Periosteum.

Fig. II. Radius of the same dog, from which a similar piece of bone was taken without the Periosteum.

Fig. III. Longitudinal Section of the last mentioned bone, to shew the solidity of the newly formed portion.

Fig. IV. Portion of bone which was removed in this experiment.

On the Optical Figures produced by the Disintegrated Surfaces of Crystals.

By Sir DAVID BREWSTER, K.H., D.C.L., F.R.S.

Read 6th February 1837.

THERE is no branch of natural science about which we know so little as that which relates to the structure of crystalline bodies. By assuming the form of an integrant molecule, crystallographers have found no difficulty in building those geometrical solids which minerals and artificial crystals present to our observation. They conceive that these molecules unite by their homologous sides in the formation of the primitive crystal, and by supposing that they arrange themselves in plates on the faces of that crystal, each plate successively diminishing in size by the abstraction of a certain number of these molecules in lines of a given direction,—all the secondary forms of the crystal may be easily deduced.

In place of employing, as HAUY has done, integrant molecules having the form of a tetrahedron, a triangular prism, and a parallelopiped, others have suggested the more philosophical idea of constructing crystals out of spheroidal elements, including, of course, the sphere by which the oblate passes into the prolate solid. But in whatever way crystallographers shall succeed in accounting for the various secondary forms of crystals, they are then only on the threshold of their subject. The real constitution of crystals would be still unknown; and though the examination of these bodies has been pretty diligently pursued, we can at this moment form no adequate idea of the complex and beautiful organisation of these apparently simple structures. The double refraction and pyro-electricity of crystals related to certain fixed points of their primitive forms; and the phenomena of circular polarisation in quartz and amethyst, connected with the plagiedral faces of the crystal, indicate remarkable peculiarities of structure; and I have had occasion to shew that all the properties comprehended under Double Refraction and Polarisation do not exist in the ultimate molecules of the body, but are wholly the result of those forces by which these molecules are combined. Structures still more complicated have been discovered by the analysis of polarised light, and in the complex formations of *apophyllite* and *analcime,* we witness the operation of laws resembling more those which regulate the structures of animal life, than those which had previously been observed in crystalline formations.

The doubly refracting structure of crystals, or to use the language of the undulatory theory, the law according to which this structure permits the ether to be distributed in their interior, relative to one or more axes, becomes the index as well as the measure of certain changes of structure which in some cases arise during the process of crystallisation. When the atoms approach each other in a pure and undisturbed solution, the crystal which they form will be a correct type of the species; but if the solution has been exposed to agitation,—if its electrical condition has been changed,—if foreign matter, crystallised or uncrystallised, opaque or transparent, coloured or uncoloured, amorphous or isomorphous with the crystal;—if any such matter has been introduced into the solution, we may expect a crystal deviating from the type of perfect crystallisation, in transparency, or colour, or density, or hardness, or refractive power, or in doubly refracting and polarising structure. A very remarkable example of such changes I discovered long ago in *chabasie*. When the crystal had begun to form, it possessed the structure of the perfect mineral, but the force of *positive* double refraction of each successive layer began to diminish till it wholly disappeared. The changes, however, did not stop here; a *negative* doubly refracting structure commenced at the neutral line, and gradually increased till the crystal was completed. This singular effect I ascribed to the introduction of foreign matter between the integrant molecules of chabasie, which weakened their force of aggregation, and consequently the double refraction produced by the mutual compression which arises from that force. By pursuing the same idea, I have been recently led to discover the cause of the beautiful but perplexing phenomena of *dichroism*, and I hope to be able to lay before the Society an artificial combination in which the actual phenomena are reproduced.

Having thus briefly adverted to the present state of our knowledge of the interior constitution of crystals, I shall now proceed to the proper subject of this paper, which is to describe the optical figures produced by the disintegrated surfaces of minerals and artificial crystals. The disintegration by which these figures are developed, is produced by three causes:—

I. By the natural action of solvents on the mineral, either at the time of its formation or at some subsequent period in the bowels of the earth.

II. By the action of acids and other solvents upon the surfaces of perfect crystals; and,

III. By mechanical abrasion.

I. The first examples of *Natural* disintegration which I met with, were in *Brazil Topaz*. In a great number of these topazes, I observed cavities filled with a white pulverulent substance, which BERZELIUS, who analyzed it at my request, found to be a sort of *marl*, consisting of *silex*, *alumina*, *lime*, and *water*, and which, as he remarks, would have formed a *zeolite* had it been crystallised. Upon examining the sides of the cavities which contained this substance, I found that they were rough and irregular, as if they had been disintegrated by a solvent; and I observed the very same effect on the flat summits and pyramidal faces, but never on the faces of the prism. As it was impracticable to apply the goniometer to the mensuration of the angles of the minute facets which the microscope rendered visible on these disintegrated surfaces, I thought of obtaining a general idea of their position by examining the manner in which they arranged the reflected images of a luminous point placed at a distance. Upon making this experiment, I was surprised to see a beautiful optical figure, consisting of the most elegant curves of contrary flexure, studded with tufts of light, and arranged with the most perfect symmetry round the central image of the luminous point which is formed by those portions of the summit of the crystal which had escaped from the action of the solvent. This remarkable arrangement of the reflected light is shewn in Plate X, *Fig.* 1. where it consists of *three* curves of contrary flexure of the general form of *lemniscates*, having at the extremities of their greater axis two semicircular tufts of light, and at the extremities of their lesser axis two triangular tufts of light. These figures undergo considerable changes on different specimens, depending, as will afterwards be seen, either on the time during which the solvent has acted upon it, or upon its dissolving power:—but they never deviate from the general type, and in the most imperfect and rough specimens, of which I have examined more than an hundred, it is easy to recognise the elements of the perfect figure. One of these variations in the figure is shewn in Fig. 2, where the light of the inner curve is diffused over a nebulous figure with a crescent at each end, and an elliptical space in the centre, from which the image of the candle, or luminous point, has wholly disappeared. Hence it appears that the whole of the original surface of the flat summit of the crystal has been removed by the action of the solvent, an effect which may be imitated, as we shall presently see, in artificial crystals. The nebulous expansion of which we have been treating has sometimes rectilineal branches at its extremities, and is sometimes filled up in the middle, where the image of the candle is distinctly seen. In other specimens, this nebulous portion is the only part that is visible. The angular magnitude of the figure varies greatly in different specimens, and also its distinctness and continuity. When the elementary facets are large, the outline of the figure is marked by separate images of the candle, and when these facets are very small, the luminous tracery is soft and nebulous, and sometimes shading off into coloured tints, like the fringes produced by the interference of common or polarized light.

The optical figures produced by the faces of the pyramid are less distinct and beautiful, but not less remarkable, than those which we have been describing. Upon faces inclined about 145° to the summit plane, and which seem to be those marked *s* by HAUY,* the strange figure shewn at A, Fig. 3, is seen; on the ad-

* Plate 44, Fig. 1, &c., first edition.

joining face the same figure reversed is seen as shewn at B; on the next face is seen the figure C, the same as A; and on the next face again the figure D, the same as B. I have observed other figures on faces differently inclined to the axis, but they are not of sufficient distinctness to merit delineation.

Optical figures analogous to those seen in topaz, may be observed in various other minerals; but it is very difficult to find specimens that have undergone disintegration on their surfaces.

On the cubical faces of a specimen of *White Fluor-spar* from Shaionce town, Illinois, U. S., sent to me by Professor SILLIMAN, I have observed a figure consisting of *four* radiations, inclined 90° to each other, and having the bright central image entirely obliterated. On the octohedral surfaces of the common fluor-spar, the figure consists of *three* radiations, inclined 120° to each other.

In a crystal of *Hornblende*, the four summit planes at each end of the prism give the figure of a small luminous circle, as shewn in Fig. 4, the central image being wholly obliterated. On the faces of the prism, which are not those of cleavage, the figure is a luminous rhomboid, as shewn in Fig. 5, with a nebulous image at each angle, and one in the centre, the shorter axis of the rhomboid coinciding with the axis of the prism. In some specimens the luminous lines uniting the four images at the angles are not developed.

In a specimen of *Axinite*, I observed the remarkable geometrical figure shewn in Fig. 6. It consisted of two images *a, b*, joined by a line of light, and each of them sending out, in opposite but parallel directions, luminous rectilineal branches *a c, b d*. The line *a b* is perpendicular to the edges of the prism, and *a c, b d* parallel to the sides of the reflecting face. On the opposite side of the prism the figure is reversed.

On the faces of the primitive cube of *Boracite*, the optical figure seen by reflection is a rectangular luminous cross, with a central image, the radiations being perpendicular to the edges of the square faces. *Muriate of soda* that had begun to deliquesce in a humid atmosphere exhibits the same figure.

The faces of the octohedron of *oxidulated iron* gives six luminous radii, inclined 60° to each other; but each alternate image is stronger than the one adjacent to it.

On the rhomboidal faces of the dodecahedron, *Garnet* gives an optical figure like a St Andrew's cross, the line bisecting the arms of the cross being perpendicular to the longer diagonal of the rhomboidal face.

The natural faces of a fine octohedral *Diamond* gave *three* luminous radiations, inclined 120° to each other; and the same figure was exhibited by the faces of a rough pyramid of *Amethyst*, and by some of the cleavage planes of *Oligist Iron-ore*.

As minerals with disintegrated surfaces are not to be found in mineralogical cabinets, owing to their being in general bad specimens, I have not been able to pursue this branch of the subject any farther; but I have no doubt that if I had such a copious supply of other minerals as I had of topaz, I should be able to find among them specimens of equal interest.

II. We come now to the second and the principal branch of the subject,—to describe the optical figures produced by the action of water, acids, and other solvents, upon the surfaces of perfect crystals, both natural and artificial.

The crystals which I have found to be best adapted for exhibiting the action of solvents in producing optical figures by reflection, are *Alum*, *Fluor-spar*, and *Calcareous Spar*.

If we take a fine crystal of *Alum*, and look at the image of a candle reflected as perpendicularly as possible from one of the faces of the octohedron, it will appear perfectly distinct, and without any luminous appendages. If we now immerse it for an instant in water, and dry it quickly with a soft cloth, the reflected image will send out three luminous radiations, as shewn in Fig. 7. By a second immersion in the water, three small images of the candle will be developed at 1, 2, 3; and by a little farther action of the solvent, these images connect themselves with the central image S, by the radial lines 1 S, 2 S, 3 S, inclined 110° to each other, and 30° to the principal radiations from S. By continuing the action other three images start up at 4, 5, 6, but apparently without any radial connection with S. The principal radiations *a* S, *b* S, *c* S begin at this period to grow faint between 4 and 1, 5 and 2, and 6 and 3. Another immersion of the crystal developes the images 7, 8, 9; and by continuing the action, the images 1, 2, 3 become the brightest, and the branches A, B, C become more like images at *m*, *n*, *o*. The central image S has now transferred almost all its light to the new images, and another immersion will make it disappear altogether, leaving the central part of the figure perfectly dark, as in Fig. 8.

It is now obvious, that by repeated actions of the solvent we have removed the whole of the original surface of the crystal by which the central image S was formed, and have replaced it by a great number of facets, which reflect, in consequence of their various inclinations, the different portions of the geometrical image shewn in Fig. 8. If we carry the process of solution farther, the figure will undergo successive changes, becoming larger and more discontinuous in its outline.

The beauty and regular development of these phenomena, depend in some measure on the perfection of the original surface of the crystal, and greatly on the uniform temperature of the water, and the shortness of the period during which the crystal is immersed in it. The successive development of the figure may be pretty well seen upon an artificial surface of the octohedron of alum, provided it is nearly parallel to the original surface. When the inclination of the artificial face is considerable, the optical figure loses its symmetry, and gradually passes into other figures produced by the other faces of the crystal towards which the artificial face is inclined.

On some occasions I have found the principal radiations united by a beautiful nebulous web of a triangular form (as shewn in Fig. 9).

All the figures above described, may be seen by reflection from the smallest portion of the face of the octahedron, and they are often more beautiful on one part than another. The principal radiations are shewn in the figures as if they were seen from the centre of the triangular face, in which case they point to the angles; but in all other cases, the radiations are perpendicular to the opposite sides of the triangle.

If we expose any of the SIX square faces perpendicular to the three axes of the octohedron to the action of water in the manner already described, and examine the optical figure which it produces by reflection, we shall see four rectangular radiations as in Fig. 10, each radiation being perpendicular to a side of the square, and consequently passing into one of the three radiations formed by each face of the primitive octohedron. By successive actions, these *four* radiations become shorter towards the central image, which gradually grows fainter and sometimes disappears.

If the same experiment is made with the TWELVE faces formed on the *twelve* edges of the octohedron, we shall obtain a figure with *two* radiations, forming an oval line with the image of the candle in the middle of it. This image becomes gradually nebulous and finally disappears, leaving a kind of elongated oval nebula, with a dark oval centre, as shewn in Fig. 11, where the line AB is perpendicular to the replaced edge, and parallel to an axis of the octahedron. The *two* radiations A, B, obviously pass into one of the *three* radiations given by the adjacent faces of the octohedron; and if we were to cut a great number of artificial faces variously inclined from that which gives the *two* radiations in Fig. 11, to that which gives the *four* in Fig. 10, we should observe Fig. 11. gradually passing into Fig. 7, and acquiring a third radiation, and Fig. 7. passing into Fig. 10, and acquiring a fourth radiation.

From the phenomena exhibited by *alum*, I proceed to those produced by *fluor-spar*, a mineral having the same primitive form and cleavage. Having immersed one of the faces of the octohedron for a few days in sulphuric acid, I obtained by reflection the beautiful figure shewn in Fig. 12. The three principal radiations A, B, C, with the luminous triangular centre, are first developed, and by continuing the action of the acid, *six* new images are produced at *ef*, *g h*, and *i k*, connected by lines of light with the other part of the figure. A continuance of the action developes six luminous curves proceeding from the images *ef*, *g h*, *i k*, as in Fig. 13, having each a new image within their concavity. Three insulated images appear also at *l*, *m*, *n*, distant 120° from each other, and 60° from the principal radiations.

When the faces of the cube formed by planes replacing the angles of the octohedron are acted upon by the acid, the beautiful figure shewn in Fig. 14. is produced, the half moons at the four angles being more distinctly brought out in some cases than in others.

The mutual connection of these figures will be seen in Fig. 15, where the triangles represent the faces of the octohedral pyramid unfolded as it were, and the quadrangular figure the square base of the pyramid.

Among the variations of figure produced by the strength of the acid, or the duration of its action, one of the most interesting is the one represented in Fig. 16, where the three principal radiations are inclosed in a luminous equilateral triangle, having a bright image at each of its angular points. If we grind and polish the opposite surface of the octohedron, so as to have a parallel plate, we shall see Fig. 16. much more brilliantly by transmitted light.* If we now expose this second surface to the action of the acid, we shall see the optical figure shewn in Fig. 17, which is Fig. 16. inverted. The cause of this inversion is, that this second face is parallel to a face in the opposite pyramid of the octohedron, whose apex lies in an opposite direction to that of the face which gives Fig. 16. If we now look through the two faces that have been acted upon by the acid, we shall see the beautiful luminous figure shewn in Fig. 18, each image produced by the one surface being converted into an optical figure by the second. When the figure produced by the first surface of the plate of spar has its simplest form of three radiations, the multiplied figure seen by transmission contains only the twelve bright images and the central image of Fig. 18.; but when it exhibits the more compound form of Fig. 12. or 13, the transmitted figure becomes exceedingly complex. It is obvious that the figure shewn in Fig. 14. will not be altered by transmission through two surfaces. Its brilliancy, however, and distinctness will be increased.

In some specimens I have observed three beautiful luminous arches, *m n*, *mo*, *no*, as shewn in Fig. 12.

Upon the face of a cube of fluor-spar, which had been ground and smoothed, but not polished, before it was acted upon by dilute *muriatic acid*, I observed the appearance in Fig. 19. The original image had entirely disappeared from the centre of the rounded square of light, and the interior of the cube was filled up with a faint nebulous light of uniform intensity. The *eight* round images were equidistant and equally bright, and the perimeter of the square was brightest at its angles and the middle points of its sides.

* This is the case with all the optical figures previously described.

After the specimen had been exposed for some time to the action of boiling muriatic acid, the face which had given Fig. 19, now exhibited the remarkable phenomena shewn in Fig. 20. The nebulosity had almost disappeared from the interior of the square, and collected, as it were, in its centre. The brightest parts of the figure were the curved masses at the angles, the middle parts of the sides of the figure being exceedingly faint.

From the *tessular* I proceeded to the *rhombohedral* system of crystallization, and I employed calcareous spar and sulphate of potash in the inquiry.

Having immersed a rhomb of calcareous spar in dilute *nitric acid*, four parts of water being added to one of acid, I observed the reflected figure from all the faces of the rhombohedron, to have the form shewn in Fig. 21. The obtuse angle of the crystal was in the direction CE, and the angle ACB was greater than 120°. As the obtuse angle of the opposite face has an opposite direction, the figure which it gives by reflexion is the inverse of Fig. 21, so that, by looking through the parallel faces, we obtain a figure with six luminous radiations. By varying the strength of the acid, the time of its action, and taking the surfaces of different crystals, the figure undergoes remarkable changes; but though two individual figures often occur between which no similarity exists, yet, by observing the transitions of a considerable number, we may trace the family likeness through them all.

The thin web of light AEB, BD, and DA, appear at an early stage of the action, but it is often wanting between A and B, and by continuing the action, a radiation often appears at F, sometimes united, and sometimes not, with the centre C. The radiation CD sometimes expands suddenly below C, into a diverging brush of light, and in other cases it is often wholly wanting, as well as the triangular luminous centre C. In this case we have only two luminous brushes, A, B, with a small central image at C, A and B being sometimes joined by bright light, and sometimes by a small arch of nebulous light, the centre of which was at C.

On the faces of three different crystals, a figure with *five* radiations, diverging at unequal angles, was produced. Two of these were the radiations A, B, the third was the brush developed at E, and the *fourth* and *fifth* were formed by the division of CD into two branches. Sometimes the whole of the central part of this five-rayed figure was wanting, leaving the expanded part of the radiation in the circumference of a sort of oval ring, which was sometimes luminous throughout, but studded with the five brushes of stronger light.

When the solvent was pretty strong muriatic acid and water, the figures have often a great similarity to those already described, but in some cases they have the form of luminous shields of a triangular form, as shewn in Fig. 22. The place of the central image is at C; the brightest part of the figure is at E, with a reddish margin, and the next brightest at A and B. In other crystals the lights

A, B, E, were connected with C by the radiations, and in one case, where a weaker acid was used, E was elevated farther above C, and a horizontal band of light passed below C to the sides AD, BD.

When strong vinegar was used as the solvent, I obtained the figure shewn in Fig. 23, the letters having the same indications as in the preceding figures.

I now proceeded to apply the solvents to the summit planes of the prism. My first experiment was made on an artificial face, perpendicular to the axis. By the action of vinegar it gave the figure shewn in Fig. 24, which consists of three radiations, inclined 120° to each other, having its centre sometimes dark, and sometimes occupied with a small image. The rudiments of other three radiations, inclined 60° to the former, are distinctly visible, and beside a luminous circle circumscribing the whole, there are three non-concentric circular arches, similar to those seen in fluor-spar.

The very same figure, with the exception of the circle and the circular arches, was obtained from the action of dilute muriatic acid on the natural faces of the *chaux carbonatée basée* of HAUY.

Having ground and repolished the artificial summit which exhibited Fig. 24, I exposed it to the action of dilute muriatic acid, when I was surprised to see it produce the strange figure shewn in Fig. 25. Although the symmetry of the figure is hostile to the idea that its shape might have been partly the effect of accident, yet I found it unaltered by repolishing, and again disintegrating the surface, and what is still more decisive, I obtained the very same effect twice from another crystal of calcareous spar.*

By placing the crystal which gave this remarkable figure in a stronger acid solution, it gave on both its faces the figure in Fig. 26, the light of which is strongest in the circular arches. By continuing the action of the same acid, the three inclosed radiations disappear entirely, and what is still more singular, they reappeared by a farther continuance of the action. The action being prolonged, they again disappeared, the circular arches grew wider and more confused, till they filled up the space which they at first inclosed.

Another crystal of spar exhibited the very same series of successive changes which I have now described.

I now reground and polished the faces of both these specimens. When they were plunged into strong dilute acid, their disintegrated surfaces produced no figure, but by increasing the strength of the solution the figures were developed as formerly.

In order to observe the effect produced upon faces that were not coincident either with the primary or secondary faces of the crystals, I ground down one of

* The brightest part of the figure was *a b*, the part above *a* being faint.

the acute solid angles, and replaced it by a plane inclined 71° to a face of the rhomb, the common section being parallel to the long diagonal of this last face. After being immersed in dilute nitric acid, it gave the strange branching figure shewn in Fig. 27, where *a b c* forms the brightest portion. I obtained the same figure with another crystal, but the parts *x y* were wanting, and *b* and *c* were continued through *a* to *m* and *n*. The side *a* was directed to the obtuse angle of the rhomb. With another crystal, in which the artificial face was inclined 104° instead of 71°, the figure shewn in Fig. 28 was produced.

My next experiments were made with *sulphate* of *potash*, a crystal which belongs to the rhombohedral system. By the slightest action of water upon the flat summit of a hexagonal prism, it produced six luminous images, symmetrically arranged round the central image, each image being opposite a side of the hexagon. All these images were connected with the central image by a halo of fainter light. The faces of the hexagonal prism produce the figure shewn in Fig. 29, the line AB being coincident with the axis of the prism. By continuing the action the branches C, D vanished, and the figure appeared as in Fig. 30, the images being connected with a haze of light.

A more remarkable effect was produced with the faces of the truncated pyramid. Three of the six faces produced the effect shewn in Fig. 31, while the other three alternate faces produced the same figures, but without the wings E, E. Sometimes two images are seen below B.

My attention was now directed to the system of crystallization in which the base of the primitive crystal is a square. Having immersed a fine crystal of *Faroe apophyllite* in dilute nitric acid, the summits of the prism were alone acted upon. They produced a figure with *four* rectangular radiations directed to the angles of the summit, and four much shorter ones pointing to the sides of the square summit. The four large rays appeared first connected with a luminous web, and the four small ones were subsequently developed.

The very same figure, but with some modifications, was produced by the action of water upon the summit of the square prism of sulphate of potash and copper. The small radiations were produced last, as in apophyllite, but what is remarkable, they are directed to the angles, and not to the sides of the square face. The extreme solubility of this salt renders it difficult to develope the figure distinctly.

From another crystal of the same class, *superacetate of copper and lime*, I obtained the beautiful figure shewn in Fig. 32, where the eight radiations are of equal length, and the images at their extremities connected by beautiful curves of light concave outwards.

I have made a great number of experiments with crystals belonging to the prismatic system, such as *sulphate of magnesia*, *borax*, *tartrate of potash and soda*, *sulphate of iron*, and *sulphate of copper*, but, though I have delineated many of the figures which they produce, and though some of them have considerable interest, I am not able to present the details in the form which I could wish. I expected to have been able to obtain interesting and definite results by subjecting the faces of a large class of minerals to the action of fluoric acid; but, in so far as my experiments went, I was disappointed. Dr Fyfe, many years ago, exposed several crystals of quartz and amethyst, which I sent him for this purpose, to the action of fluoric acid, but the disintegration of the surfaces was such that they would not reflect any light at all. I have no doubt, however, that, by weakening the action and carrying it on very slowly, the desired effect will be produced.

During the preceding experiments I was led to observe, that different solvents had a tendency to produce different figures, and I confirmed the truth of the observation by many experiments. When muriatic acid, for example, acts upon *alum*, it produces a figure with six radiations, not unlike those of sulphate of potash, and, by continuing the action, the central image vanishes. If in this state we immerse it in water, three of the radiations vanish, and it assumes the usual form. When again immersed in muriatic acid the six images reappear. Diluted nitric acid has the same effect as muriatic acid; but diluted sulphuric acid gives such a form to the radiations, that their extremities are included within an equilateral triangle, the larger radiations pointing to the three angles, and the shorter ones to the three sides.

Diluted *alcohol*, though it acts feebly upon *alum*, produces a figure different from water and the acids. It gives a figure with three short radiations; and, by farther dilution, the figure undergoes changes which give it a greater resemblance to the aqueous figure.

In order to retard or diminish the action of solvents upon highly soluble crystals, I conceived the idea of immersing them in solutions of the crystal of different degrees of strength. In making this experiment on alum, I took a crystal which gave the figure shewn in Fig. 8, and, having immersed it in a saturated solution of alum for a single instant, I found that it had, as it were, seized the particles of alum in the solution, and replaced them in their proper position on the disintegrated face. By subsequent immersion the face repassed through all the stages at which it produced the phenomena shewn in Fig. 7, and finally became perfect, reflecting a single image of the candle. *The singular fact in this experiment is, the inconceivable rapidity with which the particles in the solution fly into their proper places upon the disintegrated surface, and become a permanent portion of the solid crystal.*

In repeating and varying these experiments, I observed a number of curious facts, which it would be out of place here to describe. I immersed crystals of alum in saturated solutions of nitre and other salts, and observed many remark-

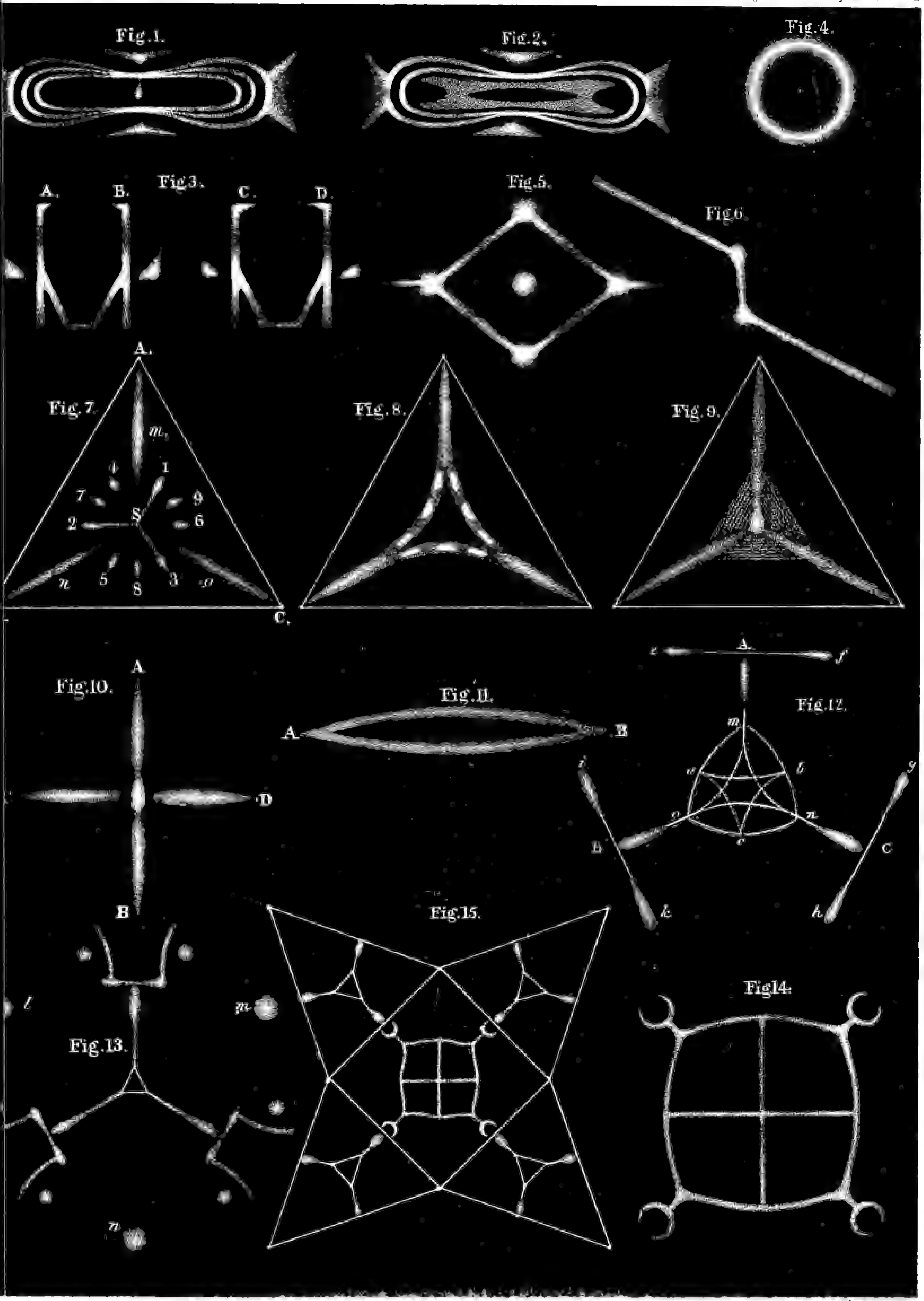


Fig. 1.
Fig. 2.
Fig. 4.
Fig. 3.
A.
B.
C.
D.
Fig. 5.
Fig. 6.
Fig. 7.
A.
C.
m
n
S
1
2
3
4
5
6
7
8
9
Fig. 8.
Fig. 9.
Fig. 10.
A
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D
Fig. 11.
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Fig. 12.
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Fig. 13.
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Fig. 15.
Fig. 14.

Fig. 18.

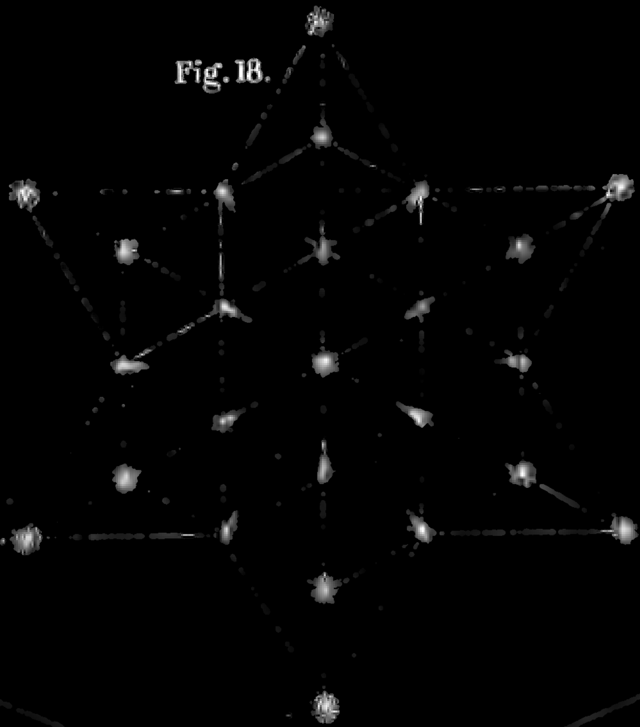


Fig. 16.

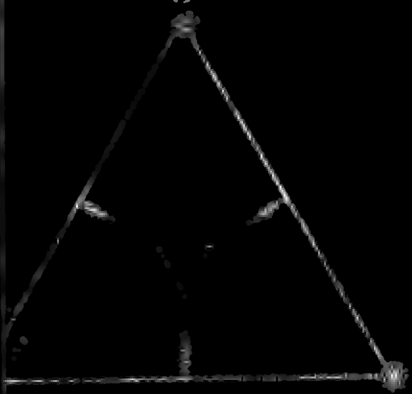


Fig. 17.

Fig. 20.

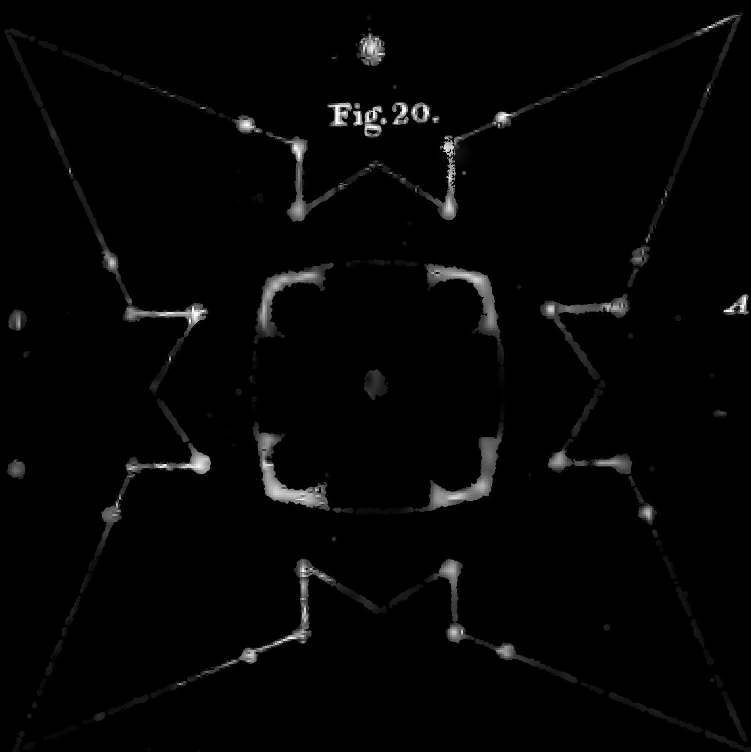


Fig. 19.

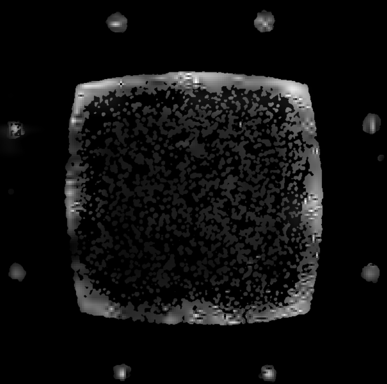


Fig. 22.

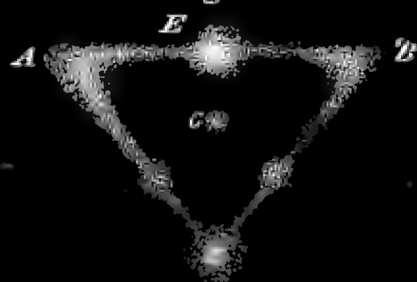


Fig. 21.

Fig. 23.

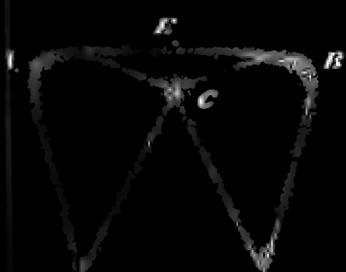
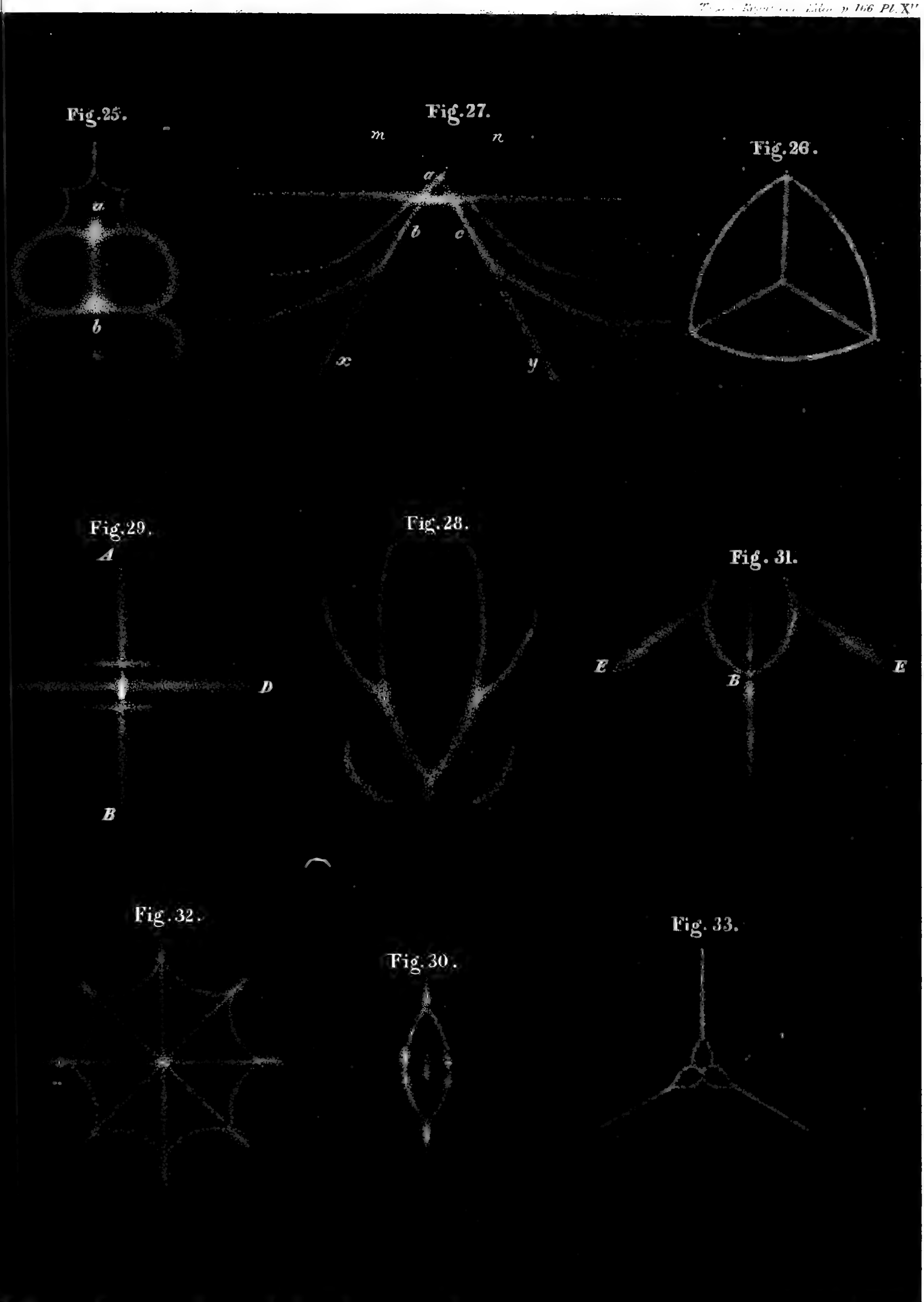


Fig. 24.



Forrester & Nichol lithog.

able changes upon the figures which they produced. The changes take place principally upon the central parts of the figure, as shewn in Fig. 33, which represents one of the forms which a solution of *nitre* gave to the figure produced by *alum*; but in other cases the whole figure suffers a change. A crystal of *sulphate of potash*, which gave the hexangular radiations already described, produced the same figure, with twice the angular magnitude, when dipped for a few seconds in a saturated solution of nitre.

In consequence of having observed that the natural cleavage planes of crystals gave indications of regular optical figures, similar to those produced by solution, I was led to make some experiments on the effects of *mechanical abrasion*, as produced by coarse sandstone, or by the action of a rasp or large-toothed file. Surfaces thus torn up produced, in a rude manner, the optical figure given by solution; but what was very remarkable, *the figure had a different position, or had the position which solution would have developed on the opposite face.* This is also true of the figures produced by natural cleavage planes, in which the separating surfaces have been slightly torn up.

It is scarcely necessary to observe, that the power of producing the optical figures described in this paper may be communicated to wax or isinglass, &c. The impressions on isinglass enable us to see the figure by transmitted light, and to observe its form and dimensions with greater accuracy.

ALLERLY, *February* 1. 1837.

IX. *Researches on Heat. Third Series. § 1. On the unequally Polarizable Nature of different kinds of Heat. § 2. On the Depolarization of Heat. § 3. On the Refrangibility of Heat.* By James D. Forbes, *Esq. F. R. SS. L. & E., Professor of Natural Philosophy in the University of Edinburgh.*

Read 16th April 1838.

Introductory.

1. The following paper is divided into three sections, containing three distinct yet intimately connected investigations. The two first on the Polarizability and Depolarization of Heat have arisen immediately out of the train of investigation contained in my two former papers, and the researches of others to which they gave rise. The third is on the Refrangibility of Heat, a point of the highest importance for theory.

2. The experiments on which these investigations are based have been performed almost exclusively during the past winter. Part of the experiments on Depolarization were, however, made in the winter 1836–7. The mode adopted for trying Refractive Indices I had long ago contemplated. It was not, however, put in practice until January last.

3. The methods of measuring heat, &c. are exactly those fully detailed in the Second Series, § 1. The only modification of importance was attaching a lens in front of the pile, as described in Art. 56 of this paper.

4. During the two years which have elapsed since the publication of the Second Series, I have not discovered any correction which I have to make upon the statements of my former papers, excepting as to the measure of the polarizing power of a pile of plates of rock-salt (Second Series, art. 25), which I find to be inexact.

§ 1. *On the unequally Polarizable Nature of different kinds of Heat.*

5. It has been my anxious wish to preserve these papers pure from even the appearance of controversy, and such members of the Society as have paid attention to the recent history of our present subject must be aware that, without making direct allusion to the doubts which have at different times been thrown upon my experiments, I have contented myself with adducing new facts and more convincing reasonings; and I have had the satisfaction to see that the general result

of this course has been the gradual abandonment of such doubts, and the entire adoption of my conclusions.

6. I believe that only a single exception remains to this statement. I expressed my belief in my *first* paper that heat was differently polarizable, according to the source whence it was derived. M. MELLONI* failed to verify this result, and the opposite conclusion, namely, that all kinds of heat are equally polarized by a given pile of mica, was prominently put forth by himself and M. BIOT as an important discovery.† Without any undue confidence in my first, confessedly imperfect, researches, I proceeded in my second paper (art. 22, *et seq.*) to give what I considered ample proofs of the correctness of the statement, though the great dissimilarity of the numbers arrived at from those of my first paper, shewed that the latter were worthy of very little confidence on the ground of numerical exactness, which, indeed, I never claimed for them. The later experiments, however, were made with a view to accurate results, and I stated certain forms of the experiment which I had devised on purpose to meet the objections of M. MELLONI, although I avoided mentioning his name.

7. It seems, however, that M. MELLONI, returning to the subject with his accustomed diligence, after receiving my second paper, still confirmed his former results, and he has attempted to shew, in a very long paper, published in the *Annales de Chimie* for May 1837 (which only appeared in October), that his results must be exact, and the probable source of my errors. I contented myself with giving a very brief answer to this paper in the Philosophical Magazine for December 1837, admitting the improbability that so experienced an operator as M. MELLONI should be wrong in his numerical results, but stating convincing grounds for believing that his explanation of my conclusion, founded on experimental errors, was inapplicable. The inquiry which I have since been led to make, and the entirely satisfactory explanation at which I have arrived of a difference so puzzling, terminating in a confirmation of my original statement, I now proceed to lay before the Society.

8. I have not the remotest intention of examining and criticising M. MELLONI's paper in the *Annales de Chimie* for May 1837, as respects trifling or personal matters, which I readily confide to the impartiality of those best qualified to judge: but it is quite necessary to state the facts which I had observed, and M. MELLONI's mode of accounting for them.

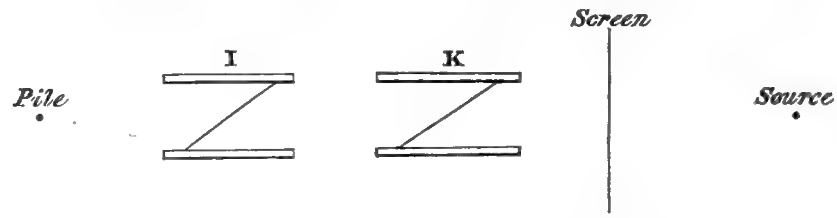
9. With two polarizing mica bundles of great tenuity, prepared in the method described, (II.‡ art. 20) marked I and K, I found that, with heat from an Argand-lamp, 72 to 74 per cent. of the incident rays were polarized, that is,—of 100

* Comptes Rendus de l'Académie des Sciences, ii. 140. † Ibid. p. 194.

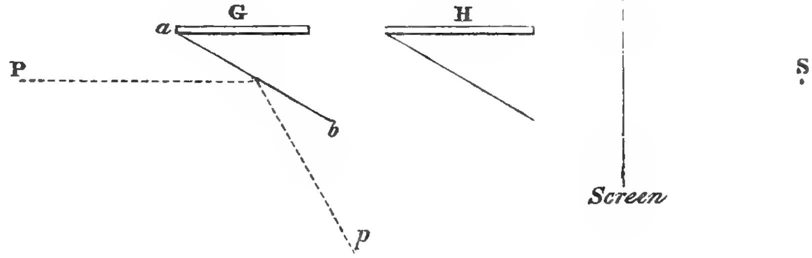
‡ To avoid circumlocution, I shall denote by I. II., &c. the First, Second, &c. Series of Researches, and by the succeeding Arabic numeral the Article referred to.

rays transmitted when the plates were parallel, 72 to 74 were stopped when one was *crossed* or its plane of refraction turned through 90°. With heat from boiling water, but 44 per cent. were polarized, and heat from sources of intermediate intensities gave intermediate results.

10. M. Melloni ingeniously argued that this appearance might arise from the circumstance that the mica bundles becoming most heated by those kinds of heat which they absorbed most readily, or transmitted least easily (viz. heat of low temperature), the pile was continually receiving a supply of heat by secondary radiation from the mica, which, having no relation to the *Parallel* or *Crossed* positions of the plates I and K, of course tended to diminish the apparent polarization of the heat, or to equalize the effect in the two positions.



11. The supposed effect of secondary radiation from plates had been so often urged against my experiments, that, though as often proved to be insignificant or insensible, it gave me no surprise to see it started afresh, and in so plausible a manner. M. Melloni was probably not aware that the screen for intercepting the heat was placed *between* the source of heat and the polarizing plate K (as shewn in the figure above), so that the mica plates were only absorbing heat during the exceedingly short time (10 seconds) of one swing or dynamical impulse of the needle, otherwise I do not think he would have urged so infinitesmal an objection.* I endeavoured, however, to meet it directly in this way. I took two mica bundles, G and H (see II. 22), and placed them parallel, as shewn in the figure below. But instead of placing the pile at P, where it receives at once the directly transmitted heat from S (the screen being removed), and the supposed secondary radiation of the surface *ab* of the mica plate, I placed it at *p*, identically situated with respect to the surface *a b*, but wholly removed from the influence of direct radiation from S.

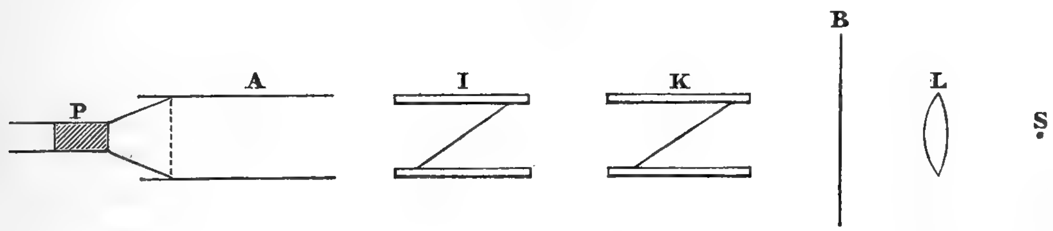


* I might add, too, that, had he been aware of the extreme tenuity of the mica plates employed (of which more hereafter), he must have been led as a necessary consequence of his own reasonings to admit that the effect must be insignificant.—*Ann. de Chimie, Mai* 1837, *p*. 13, *note*.

When this experiment was performed with dark heat (which, according to Melloni, ought to give the greatest effect), not the slightest movement of the Galvanometer-needle was observable on removing the screen, during a far longer space of time than is ever in practice allowed for the absorption of heat. This experiment ought to be considered quite conclusive.

12. M. Melloni had hinted that the different dimensions of the Sources of Heat, and the various angles under which the rays fell on the mica plates must materially affect the results; and, as I was quite convinced that operating with parallel rays was the most correct method, I proceeded to repeat my experiments on his plan, with a salt lens placed in front of the source of heat so as to render the rays parallel; I also removed the polarizing and analyzing plates to a considerable distance from the pile, and afterwards varied their distance in order to see whether any adequate explanation of the discrepancy could thus be obtained.

13. The apparatus was arranged in the following way. P, the pile; A, a square pasteboard tube to protect it from currents of air; I and K, the analyzing and polarizing plates; B, a moveable screen; L, a rock-salt lens, in the focus of which is placed S, the source of heat. In these experiments the distance from the centre of the pile to the centre of the first mica plate or P I was 12 inches; P S = 24 inches. With this apparatus I found my former conclusions fully confirmed.



The apparent polarization was somewhat increased, as I had anticipated from the rays falling more nearly at a constant angle when previously rendered parallel; but the different polarizability of the different kinds of heat was even more distinctly marked than ever; whilst the distance of the mica plates from the pile was now such as to reduce to insignificance any effect of secondary radiation, had such before been sensible.

14. In prosecuting these experiments, most of which were repeated many times under various circumstances, I remarked more distinctly than formerly the influence of particular states of combustion of the source of heat upon the index of polarization, and the accidental variations to which this gives rise on different days, and even during the progress of an experiment. Heat from brass about 700° I have generally found the most uniform on different days, though there occasionally occurs in a series of experiments, considerable deviations from the mean. The Locatelli Lamp seems subject to greater variations, and the Argand still more; indeed, I have found it so impossible to maintain an Argand-lamp in a uniform state of combustion, even for a quarter of an hour, that I have lately abandoned the

use of it. But the quality of the heat from incandescent platinum varies between the widest limits. Nor is this wonderful; it is composed of heat from two very different sources combined in uncertain proportions, that from the incandescent coil of wire, and that from the alcohol flame which heats it. The intensity of incandescence, too, varies exceedingly. On one occasion when the incandescence was unusually bright, and the alcohol flame very low, I obtained a higher degree of polarization than I have ever done before or since. The ordinary proportion between the indications with I and K *parallel* and *crossed*, is with incandescent platinum 100 to 26 or 27. In this case it was 100 : 20; and when the heat was lifted by an interposed plate of thin glass, it rose as high as 100 : 13.

15. The general results obtained in the way above described are stated in the following Table, in which I have included the numbers for mercury heated to 410°, and for boiling water taken from the Second Series, art. 22; those experiments not having been repeated because the use of a lens is in those cases of little avail.

Polarizing Plates I and K.

Source of Heat.	Rays out of 100 polarized.
Argand-Lamp,	78
Locatelli-Lamp,	75 to 77
Incandescent Platinum (usually)	74 to 76
Incandescent Platinum, with Glass .06 inch thick, interposed, 6 to 7 per cent. more, or	80 to 82
Alcohol Flame,	78
Brass heated to about 700°,	66.6
Ditto, with a plate of Mica .016 inch thick interposed, (between K and B)	80
Mercury in a Crucible at 410°,	48
Boiling Water,	44

16. I next tried whether the closest possible approximation of the mica plate I to the pile would produce any effect. The pasteboard A was removed, and the mica plate I was brought up until it touched the funnel-shaped reflector of the pile. In this extreme case the apparent polarization was found to be diminished about 2 per cent., whether in the case of incandescent platinum or of dark heat. I shall not inquire whether any or how much of this effect was due to the heating of the mica surface, and how much to the reflection of heat from the interior of the cylindrical tubes containing the mica plates, since it is evident that this could not have produced the variation of effect shewn in the above experiments.

17. I presume that it will be conceded, that the experiments now cited, incontrovertibly establish the unequal polarizability of heat from different sources. Yet, I confess, I should have felt uneasy, could I have thrown no light upon the cause of the discrepancy between M. Melloni's results and my own. This I believe, that I am able completely and satisfactorily to do, allowing him every credit for the perfect exactitude of his experiments. For the sake of clearness, I will state the course by which I myself arrived at this result.

18. It occurred to me, that it would be satisfactory for the farther and independent confirmation of the conclusions just given (which were then only partially obtained), to examine the index of polarization (by which I mean the percentage of the heat stopped in the crossed position of the polarizing and analyzing plates) deducible for different sorts of heat, from a series of experiments made wholly without reference to this question, I mean those on Depolarization, considered in another section of this paper, and which, it will be seen by a reference to the mode of reduction there employed, required to be recomputed in order to give the index of polarization.

19. I at first imagined, that the experiments made with each of the three kinds of heat then employed (Argand-lamp, incandescent platinum, and dark hot brass) would give throughout the same result. This was far from being the case; the interposition of the depolarizing plate of mica between the polarizing and analyzing plate, acting simply by transmitting only certain rays of heat, had modified the index of polarization, and that more or less, as the thickness of the interposed mica was more or less considerable. Such a result might have been anticipated, as in exact conformity with the discovery I had formerly made; but I was misled by a false notion, which I had heedlessly adopted, and suffered to remain unquestioned, that, in order to affect the index of polarization, the heat must have been modified by transmission *previous* to its falling upon the first or polarizing plate, whilst, in the experiments referred to, the modification took place between polarization and analyzation.* Of course, when I perceived this oversight, the confirmation of my views was greater, because it was unforeseen.

20. But the most material result of the examination of those experiments was this. By a reference to the section on Depolarization, it will be seen that five different thicknesses of mica (varying from three to sixteen thousandths of an inch) were interposed successively, and the index of polarization determined for each of the three kinds of heat. Now, upon examining the result of these fifteen experiments, I clearly perceived (amongst occasional irregularities) this law to prevail,—*that, whilst a film of mica* .003 *inch thick scarcely altered the characteristic properties of heat from different sources, as shewn by their variable indices of polarization, an increased thickness of mica had almost no sensible effect upon the heat from the Argand-lamp, but it increased the index of polarization of dark heat so fast, that, with a thickness of mica of* .016 *inch interposed, the apparent index of polarization for heat from the Argand-lamp, incandescent platinum, and dark hot brass, was almost the same.*

* Lest this confusion should, by possibility, occur to any one, as it did to myself, I will observe that the position of the sifting or modifying plate, absorbing the least refrangible rays, is quite immaterial, provided it occur between the *source* and the *indicator* of heat, for whether the rays in question are absorbed before or after polarization, those which ultimately escape and reach the pile are the only ones of which the index of polarization is measured.

21. When I had fully seized this conclusion, the explanation of M. MELLONI's results was easy and complete. It appears from the account of his experiments, that he still employs piles of mica of the form I at first used, consisting of distinct laminæ separated by a knife, then laid together and united at the edges, up to the number of 30, 60, and even more.* On the other hand, the piles I and K, which for two years and a half I have employed, are of a degree of tenuity really surprising. The mode of their construction I mentioned briefly in my last paper, art. 20, and it is so very superior to any other, that it is probably from inadvertence that it has not been generally employed. The piles laminated by the action of violent heat, afford a multiplicity of parallel surfaces in a given thickness of mica, which no mechanical method can approach. The actual thickness of mica which they contain, I am unable accurately to estimate. The plates marked G and H are much thicker, perhaps twice as thick as those marked I and K, which I commonly use; yet the former, as I roughly estimate by the tint they give in polarized light, are only about one-thousandth of an inch in thickness. At the utmost, the plates I and K can be but one fifteen-hundredth of an inch; and yet it appears that their polarizing power (depending solely on the number of surfaces they contain) is equal to M. MELLONI's pile of ten distinct plates placed at the same angle (35° to the incident rays). The mean thickness of the elementary plates can, therefore, be only one fifteen-thousandth of an inch; and they reflect abundantly the colours of NEWTON's rings.

22. Now, I have found by the depolarization experiments, that it requires a much greater thickness of mica than that traversed by the heat in passing through the plates I and K (even allowing for the obliquity) to affect materially the index of polarization of heat from different sources, such as from brass at 700°, and incandescent platinum. It is, therefore, a necessary consequence of the construction, that the heat passes through such piles as I use unaltered, or nearly unaltered, in its character, whilst in passing through bundles of detached plates laid together, the thickness of mica to be traversed is sufficient to modify the heat by absorption, in such a way that *the difference of quality has vanished, whatever be the source, in the very act of transmission.* It is hardly likely, considering the size

* *Annales de Chimie*, Mai 1837. At p. 17, &c., M. MELLONI has given a minute account of that method of constructing the piles, which, "amongst several different ways, he considers the preferable one." No one could doubt from his language that he is describing a new and improved form of the apparatus. I regret for a moment to descend to notice an apparent want of justice and courtesy towards myself; but it is impossible for me not to observe, that the procedure he so exactly details, is, to almost the minutest particular, identical with that which I myself used in June 1835, in constructing, in M. MELLONI's presence, the first pair of piles used for polarizing heat which existed in France, at a time when M. MELLONI expressed his unqualified scepticism as to the polarization of heat generally; which piles I left, at his desire, where I presume they now are,—in his own possession. This mode of construction I soon after abandoned, for the improved one alluded to in the text.

of M. MELLONI's mica plates (4 inches long and 2 wide), that they could be less than one fifteen-hundredth of an inch thick each; a pile of ten would then be ten times as thick as my pile of equal energy, and at an incidence of 55° the thickness traversed would not be much shorter than that of the mica plate alluded to in art. 20, which we have there seen to be sufficient to obliterate all distinctive character as to polarizability between an Argand-lamp and dark heat.

23. Being now fully aware of the importance of the construction of piles of mica which I had adopted, I thought it worth while to examine the proportions of heat from different sources, which these very delicate laminæ were capable of transmitting, which, I presumed,* would be found far less variable than when plates of the usual thickness are employed. My expectations were more than realized, as is seen in the following Table, the second column of which shews the proportion, to the whole incident heat, of that transmitted by the two mica piles I and K placed parallel to each other; by far the greater proportion of the loss being that due to the obliquity of reflection and the number of surfaces.† By way of contrast, I have placed in the third column the proportion of the whole incident heat transmitted at a vertical incidence by a plate of mica .016 inch thick.

Source of Heat.	Rays out of 100 transmitted by Plates I and K parallel.	Rays out of 100 transmitted by Mica Plate .016 inch thick.
Locatelli-lamp,	18.8	57
Ditto, with plate of glass .06 inch thick interposed,	16.2	72
Incandescent platinum,	17.6	50
Dark hot brass (700°),	15.5	15
Heat from boiling water,	10.	8

24. It is very evident that, for the first four sources of heat at least, the transmissive power of the plates I and K varied little, and in no sort of proportion to the characteristic action of mica even in moderate thicknesses. This will be more evident, if we compare the ratios of the heat from different sources transmitted in the two cases, taking the heat from the lamp sifted by glass as the standard for each column.

	Plates I and K.	Mica .016 inch.
Locatelli with glass,	100	100
Locatelli,	116	79
Incandescent platinum,	108	70
Brass at 700°,	96	21
Heat of 212°,	62	11

25. I need hardly add, that so remarkable a result as that the heat sifted by glass should be less readily transmitted by the thin mica laminæ, than the direct heat from a lamp, was carefully verified.

* I do not state this as a new idea; it has been repeatedly remarked by M. MELLONI, that, in proportion as substances are thinner, they possess a more equable diathermancy for heat of different qualities.

† The part of the effect due to reflection, I had previously established to be nearly the same for different kinds of heat.

26. Since, then, the first four kinds of heat are transmitted without any great difference of proportion, by the piles I and K, and since, especially, the heat from a lamp sifted by glass and that from dark brass possess almost exactly similar characters in this respect, it is very clear that we have a new ground for rejecting as untenable M. MELLONI'S supposition (mentioned in art. 10), that the apparent differences of polarization in my experiments, arose from the unequal proportions of heat absorbed by the mica piles when the source varied.

27. Admitting, then, the fact of the variable index of polarization exhibited by heat of different qualities similarly treated, we are tempted to inquire what explanation can be offered of it. This question, inferring for its answer a knowledge of the nature of heat, we are not prepared to answer with confidence. My former suspicion of its being due solely to the difference of the refractive index of mica for heat of different kinds (II. 24), I am disposed to retract as inadequate, or at least to suspend my judgment respecting it. I at one time thought, that, supposing the mica bundles unequally permeable to heat from different sources, a difference of ratio in the total heat reaching the pile with the plates I and K *parallel* and *crossed* might be accounted for. But a careful analysis of the circumstances convinced me, that the absorptive action, if assumed the same for common and polarized heat, could produce no such effect. One of the most plausible suppositions which occurred to me was this,—that, supposing the reflection of luminous heat to take place more copiously at the mica surface than that of dark heat, and supposing the angle of incidence to be that of total polarization, since the refracted ray contains as much heat (if heat be like light) polarized *perpendicular to* the plane of incidence, as is reflected and polarized *in* the plane of incidence, the ratio of the polarized to the total heat transmitted would be greatest in the heat of highest temperature. Unfortunately for this theory, careful experiments assured me that heat from different sources underwent the same, or nearly the same, intensity of reflection under the same circumstances.

28. We are, therefore, led to regard this character of unequal polarizability, as probably indicating a difference of character of a fundamental kind between heat and light; at least a superadded quality or peculiarity of vibration, which becomes more and more sensible as heat is removed in its character from light, or has (as we shall hereafter see), generally speaking, a lower degree of refrangibility. A sensible undulation, normal to the surface of the wave, would of course satisfy this condition. I am far from saying that my experiments warrant such a conclusion. I am aware that it is inconsistent with the ideas entertained by some ingenious speculators upon the nature of heat;* but this very circumstance has led me to bestow the greater pains upon establishing the phenomenon in an incontrovertible manner.

* KELLAND on Heat, Art. 166.

§ 2. *On the Depolarization of Heat.*

29. In the first series of these researches, § 4, I entered pretty fully into the subject of depolarization. The establishment of the fact was of the highest importance, since there is little probability of proving in any more direct manner the doubly refractive energy of crystals with respect to heat. But, besides the demonstration of the fact, I pointed out in that paper the important numerical determinations to which it might lead; determinations of the first consequence to the theory of heat, and the discrimination of heat from light. The measure of depolarization in the case of light, or the quantity of light which has become polarized in a new plane by passing through a doubly refracting plate, such as mica, depends, 1. upon the length of a wave of light; and, 2. upon the retardation which one of the doubly refracted pencils of light suffers, upon the other, in passing through the mica, which retardation differs with the material of the plate, varies directly as its thickness, and *may* also vary with the quality of the incident ray.

30. Hence, as a little reflection clearly shews, if the quantity of light (or, by analogy, of heat) depolarized by a plate of given thickness be numerically estimated, we may, if the length of the wave be given, determine the retardation, or energy of double refraction; or, if the latter be assumed or known, we may find the length of a wave. Considering the latter element as the more important, and not being then in possession of any more direct mode of determining it numerically, I proposed to *assume* the retardation due to double refraction as the same for heat as in the case of light, (considering heat as but less refrangible light), and to determine the length of a wave in the way which I fully explained in the First Series, art. 68–75.

31. Two circumstances require notice by way of precaution. The first is, that, for the very reason that we have periodical colours in the case of light, there are *different* thicknesses of mica and different measures of retardation, which, for the same length of a wave, will give the same measure of depolarization; these dubious cases (which the formula of depolarization completely expresses) must be distinguished. The second remark is, that all our sources of heat furnishing heterogeneous rays, each has its own period of maximum and minimum intensity, just as in the case of solar light, and since our means of numerical estimation embraces the sum of all the effects of heterogeneous rays, we cannot expect results which shall rigorously satisfy a formula, in which homogeneity (or constancy of λ, the length of a wave), is assumed, but consider the approximate result as representing the mean or predominating character of the heat employed.

32. Recalling, then, Fresnel's formula, quoted in art. 70 of the First Series, we have

$$\frac{E^2}{F^2} = \sin^2 180^\circ \left\{ \frac{o-e}{\lambda} \right\}$$

where F^2 is the intensity of the whole incident polarized ray; E^2 the intensity of that portion which, after transmission through the depolarizing plate, is capable of being analyzed in a perpendicular plane. These two quantities being determined from observation, the first side of this equation, or their ratio, becomes known. On the second side we have two quantities, either of which may be assumed, and the other becomes known, viz. $o-e$ the retardation of the one doubly refracted ray upon the other within the crystal, and λ the length of a wave. Now, it is obvious from the form of the expression, that an infinite number of values of $\frac{o-e}{\lambda}$ will satisfy the equation; in light there can be little ambiguity arising from this cause, because the phenomena of periodic colours at once afford the means of selecting the true solution. In the case of heat, we must proceed with more caution, the value of $\frac{o-e}{\lambda}$ being wholly unknown; we only assume (as we are entitled to do) that this quantity increases uniformly with the thickness of the plate, which it necessarily must, since the retardation is as the thickness, and λ is independent of it. By a very simple process, the true value was easily selected.

33. Five depolarizing mica plates, of different thicknesses, of exactly the same quality, and each as uniform as possible, were provided. They were cut to the same size, and of such a form that each could at once be placed with its neutral axis (a line in the plane passing through the two axes of double refraction) vertical, or inclined 45° at pleasure. Their thickness was next to be determined. The examination of the colours shewn by polarized light was the most obvious method, but not susceptible of the exactness which was required. It was, however, used as a check. These colours were:

	Retardation in Millionths of an inch.*
No. 1. White inclining to yellow,	12
No. 2. Rich blue,	28
No. 3. A blue purple,	43
No. 4. Between red and orange,	36
No. 5. Pink,	80

34. The relative thicknesses which these numbers afford, are tolerably verified (excepting the first) by the following results of actual measurement, by means

* These numbers are obtained by doubling those due to the corresponding tints of thin plates of air in NEWTON'S Table. In the case of the two last numbers, there might have been some doubt as to the order of colours to which they belonged, but this was removed by the measurements given farther on, which shewed that the pink of No. 5. is a colour of the fourth order.

of a pair of callipers constructed for such purposes by Troughton. These results are the mean of ten measures each, which were rendered difficult by the elastic and fissile nature of the substance.

	Thickness in parts of an inch.
No. 1.	.0026
No. 2.	.0044
No. 3.	.0074
No. 4.	.0060
No. 5.	.0157

35. With these mica plates in succession, employed for depolarizing, I proceeded to determine the ratio $\frac{E^2}{F^2}$ (art. 32) for the most part exactly in the way described and illustrated by an example in art. 71, *First Series,* which I found preferable to any other. This laborious investigation I performed for heat from three sources; (1), an Argand-lamp with glass chimney; (2), incandescent platinum; and, (3), brass heated (not to visible redness) by an alcohol flame. The thickness of the plates No. 3. and No. 4. being very nearly the same (and giving, as they ought to do, almost exactly the same measure of depolarization), I preferred using the united thickness of Nos. 2. and 3. as an interpolation between Nos. 3. and 5. The swings of the needle, or dynamical effects, (II. 8) were always observed, and are alone given. The polarizing and analyzing plates were the same, marked I and K, before fully described (II. 20), and a plate is said to be at 0° or at 90° as its plane of refraction is vertical or horizontal. With these explanations, and a reference to art. 71, First Series, the following specimens of observations will, it is hoped, be intelligible.

Argand-Lamp: 16 inches from centre of Pile, depolarizing Mica Plate No. 3.

Position of Polarizing Plate K (I being always at 0°).	Position of Neutral Section of Mica.	Galvanometer. Dynamical Effect.	Total Polarization F^2.	Depolarization E^2.
At 0°	At 0°	11.9°	8.45°	
At 90°	——	3.45		+ 5.35°
——	At 45°	8.8		
At 0°	——	6.75		— 5.35
——	At 0°	12.1	8.35	
At 90°	——	3.75		+ 5.05
——	At 45°	8.8		
At 0°	——	6.7		— 5.35
——	At 0°	12.05		
At 90°	——	3.7	8.35	
		Means	8.38	5.27

36. The following experiment was made with heat wholly unaccompanied by light, and with the same mica plate.

DARK HOT BRASS: 14 inches from Pile, depolarizing Mica No. 3.

Position of Polarizing Plate K (I being always at 0°).	Position of Neutral Section of Mica.	Galvanometer. Dynamical Effect.	Total Polarization F^2.	Depolarization E^2.
At 0°	At 0°	°5.25	[°3.25]*	
At 90°	—	2.0		+°3.75
—	At 45°	5.75		
At 0°	—	2.15		−3.8
—	At 0°	5.95	4.0	
At 90°	—	1.95		+3.8
—	At 45°	5.75		
At 0°	—	2.6		−3.3
—	At 0°	5.9	3.95	
At 90°	—	1.95		+3.7
—	At 45°	5.65		
At 0°	—	2.3		−3.5
—	At 0°	5.8	4.0	
At 90°	—	1.8		
		Mean	3.98	3.64

37. It now remains to explain how these observations have been discussed. The ratio $\frac{E^2}{F^2}$ is at once obtained by dividing the second mean result by the first, and I have purposely quoted these observations, to shew how very nearly the plane of polarization was thrown at right angles by the action of this particular thickness of mica, especially in the case of dark heat, which appears to be owing to its greater homogeneity, as we shall presently have reason to infer.

38. We have seen above (art. 32) that

$$\frac{E^2}{F^2} = \sin^2 180^\circ \left\{\frac{o-e}{\lambda}\right\}$$

And therefore,

$$\frac{o-e}{\lambda} = \frac{\sin^{-1}\sqrt{\frac{E^2}{F^2}}}{180^\circ}$$

Since the radical has an ambiguous sign, the equation will be satisfied by a value of $\frac{o-e}{\lambda}$ equal to a fractional number a, or by $1-a$, or $1+a$, or $2-a$, or

* Omitted in the mean as manifestly too small, arising from the lamp being just lighted, and the brass not fully heated.

$2+a$, or $3-a$, &c. In the case of the two examples given above, we have for the Argand

$$\frac{E^2}{F^2}=\frac{5.27}{8.38}=.629; \sqrt{\frac{E^2}{F^2}}=\pm .793$$

And $$\frac{o-e}{\lambda}= .29 \text{ or } .71 \text{ or } 1.29 \text{ or } 1.71, \&c.$$

For the dark heat, $$\frac{E^2}{F^2}=\frac{3.64}{3.98}=.915; \sqrt{\frac{E^2}{F^2}}=\pm .957$$

And $$\frac{o-e}{\lambda}= .41 \text{ or } .59 \text{ or } 1.41 \text{ or } 1.59, \&c.$$

The true value must be such, that, when a number of plates are employed, $\frac{o-e}{\lambda}$ must increase *uniformly* with the thickness of the plates.

39. Clearly to mark this, and at the same time to combine the results by graphical interpolation, I projected the numbers obtained as above in the way shewn in Plate XI. Figs. 1, 2, and 3. On a horizontal line spaces representing the thickness of the plates (art. 34) were set off as abscissæ, and a few of the ambiguous values of $\frac{o-e}{\lambda}$ as ordinates, which are marked by dots. It was then easy to select those points thus set off, through which a straight line could most nearly be drawn, representing the linear relation between the thickness of the plate and the quantity $\frac{o-e}{\lambda}$, (both vanishing when the thickness $=0$), and inspection of the figures will shew that no doubt can attach to the choice of the ambiguous numbers, and also that the straight line represents in general remarkably closely the course of those points.

40. There is one exception to this statement, and it is an important one. It will be observed that in all the three figures the interpolating line, instead of passing through any of the dots set off for the mica plate No. 3, bisects exactly *two* dots, which are nearest to one another in the case of dark heat,—wider apart with incandescent platinum, and widest of all in the case of the Argand-lamp. The explanation is complete and satisfactory. The interpolating line in all these cases gives a value of $\frac{o-e}{\lambda}=\frac{1}{2}$, which gives a value of $\frac{E^2}{F^2}=1$; in other words, infers a *total* polarization of the heat in the horizontal plane (or in the case of light total darkness, when the polarizing and analyzing plates are parallel) which we know can only occur when the heat is absolutely homogeneous. The want of mathematical coincidence in this case infers the admitted physical condition of want of homogeneity in the incident rays. Hence, we infer that dark heat is most homogeneous; next, that from incandescent platinum; and, least of all, that from the Argand.

41. The numbers from which these projections are derived are contained in the following Table.*

Source of Heat.	Depolarizing Plate.	$\frac{E^2}{F^2}$	Values of $\frac{o-e}{\lambda}$
ARGAND-LAMP, . . .	No. 1.	$\frac{2.15}{7.38} = .291$	.18, .82, 1.18, &c.
	repeated.	$\frac{1.91}{6.67} = .286$	
	No. 2.	$\frac{5.65}{8.53} = .662$	.30, .70, 1.30, &c.
	No. 3.	$\frac{5.27}{8.38} = .629$	.29, .71, 1.29, &c.
	No. 2. + No. 3.	$\frac{2.07}{5.55} = .373$	.21, .79, 1.21, &c.
	No. 5.	$\frac{1.31}{4.38} = .299$	.185, 815, 1.185, &c.
	repeated.	$\frac{1.44}{4.83} = .298$	
INCANDESCENT PLATINUM,	No. 1.	$\frac{2.00}{7.60} = .264$	.17, .83, 1.17, &c.
	repeated.	$\frac{1.90}{7.68} = .248$	.165, .835, 1.165, &c.
	No. 2.	$\frac{4.66}{7.31} = .638$	.30, .70, 1.30, &c.
	No. 3.	$\frac{5.02}{6.70} = .749$	.335, .665, 1.335, &c.
	No. 4.	$\frac{5.63}{7.08} = .795$	.35, .65, 1.35, &c.
	No. 2. + No. 3.	$\frac{1.48}{4.66} = .318$	.19, .81, 1.19, &c.
	No. 5.	$\frac{1.35}{6.36} = .212$	.15, .85, 1.15, &c.
DARK HOT BRASS, .	No. 1.	$\frac{1.94}{7.35} = .264$	.17, .83, 1.17, &c.
	No. 2.	$\frac{3.17}{4.15} = .764$	.34, .66, 1.34, &c.
	No. 3.	$\frac{3.64}{3.98} = .915$	.41, .59, 1.41, &c.
	No. 2. + No. 3.	$\frac{1.01}{3.38} = .299$	.185, .815, 1.185, &c.
	No. 5.	$\frac{0.62}{4.89} = .127$	.115, .885, 1.115, &c.

42. When we examine the projected interpolating lines of Plate XI, which an attentive inspection will shew to have been laid down with the greatest care,† we are struck by the remarkable coincidence which obtains between them; a result so far contrary to what I expected, that it shews that by this method we cannot hope

* Most of the experiments on incandescent platinum were made early in 1837, the remainder during the winter 1837–8.

† The interpolating line for *Incandescent Platinum* is in the engraving placed at rather too high an angle.

to discriminate the different lengths of waves of these kinds of heat, as I had formerly supposed, and shews that the variation of λ must be very small, or else (what is improbable) that it is constantly proportional to the variation of the retardation $o - e$.

43. All the three figures give as nearly as possible a value of 1.4 for $\frac{o-e}{\lambda}$ at a thickness of depolarizing mica, equal to .020 inch, or .07 for a thickness of .001 inch. Let us compare this with the case of light. The sum of the retardations for the various mica plates, as given in art. 33, amounts to .000199 inch; the sum of the thicknesses in the next article is .0361 inch, consequently the mean value of the retardation or $o - e$ is .0000055 for a thickness of mica of one-thousandth of an inch. But the length of λ for extreme red is .0000266, for extreme violet, .0000167 inch. Hence for a plate of mica .001 inch thick the values of $\frac{o-e}{\lambda}$ are

For extreme Red light, ... $\frac{55}{266} = .207$

For extreme Violet light, . . . $\frac{55}{167} = .329$

For Heat, ... $= .07$

44. If we assume the retardation, or $o - e$, to be the same for all lengths of waves, and for heat as for light, we immediately deduce the value of λ, or the length of a wave of heat. For since for a plate .001 inch thick, $\frac{o-e}{\lambda} = .07$, as above, $o - e = .0000055$, we have

$$\lambda = \frac{o-e}{.07} = \frac{.00055}{7} = .000079 \text{ inch},$$

about three times as long as a wave of red light, and four and a half times that of violet. But it is always to be remembered, that this proceeds on the supposition of the retardation being invariable.

45. I have taken the trouble to calculate and project in a similar manner my original observations on Depolarization given in the First Series of these researches, art. 74, in order that, though probably less accurate, they might form a check upon the results just given. The plates *then* employed, and marked No. 1 and No. 2, (which are not to be confounded with those so designated in this paper) had thicknesses (deduced from the retardations) of .0072 and .0036 inch. I have the gratification to find that the computed results agree almost precisely with those just obtained, although from the accidental thicknesses of the two plates employed the observations with *these alone* do not enable us to select the appropriate value of $\frac{o-e}{\lambda}$, there being at least two values which still remain ambiguous, but when taken in conjunction with the observations of art. 41, the ambiguity is at once removed, and the numerical value of λ comes out almost ex-

actly as stated above, for incandescent platinum and dark heat, and somewhat smaller for that of the Argand-lamp.

46. I desire it to be recollected, that, in speaking of these somewhat startling lengths of waves of heat, I am using the language of only one of the two hypotheses which serve to interpret the results of this section; for, if the variation be in $o-e$, or the difference of the velocities of the doubly reflected rays in mica, the result would be the same. The experiments in a subsequent part of this paper may serve to guide us in our choice. Meanwhile, I would observe, that, supposing the above results to be explained on the supposition that $o-e$ is smaller, instead of λ greater for heat than for light, it is equivalent to supposing the doubly refracting energy weaker, or a greater thickness of a crystal required to produce a given effect. Our suggestion respecting the existence of sensible vibrations normal to the wave surface (art. 28) will not avail us here. For, by the mode of reducing the experiments on Depolarization, the unpolarized part of the heat does not enter into consideration at all;* consequently those parts of the total effect which are due to transverse vibrations alone, are not modified by double refraction as so much light would be.

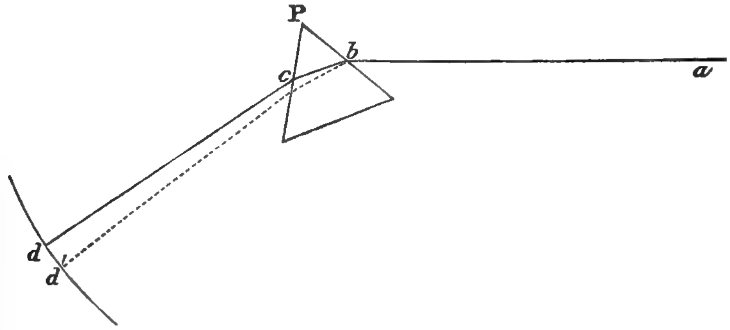
§ 3. *On the Refrangibility of Heat.*

47. Since the admirable discovery by M. Melloni of the power of rock-salt to transmit and refract heat of every kind, one of the most obvious and important questions (formerly intractable) of which it seemed to offer the means of solution, was the accurate determination of the refrangibility of heat from various sources, luminous or non-luminous. Such a determination is of the first consequence to the formation of a just theory of heat, and a detection of the subtle bond by which it is connected with the comparatively familiar modifications of light.

48. Such experiments have not been awanting. M. Melloni, in his Second Memoir on Radiant Heat, in the *Annales de Chimie* for April 1834, has described the apparatus which he employed, and which is figured in Plate III. of that volume. It consists of a thermo-electric pile, constructed of a single vertical row of elements, so as to be exposed to a very narrow beam of heat. It was made to move on a sector of a circle, at whose centre was placed a prism, by which the beam of heat was refracted from its primitive direction $a\,b$ into that $c\,d$, (see next page), and therefore produced a maximum effect on the galvanometer when

* I do not mean to offer any opinion on the nature of light in a partially polarized ray generally; but, as in the present case, the angle of incidence is that of complete polarization nearly, I presume that the transmitted ray is undoubtedly composed partly of light polarized perpendicularly to the plane of incidence, and partly of common light.

the pile was at *d*. The other parts maintaining the same positions, it is evident that the pile must be moved into the position *d′*, if the source of heat be now one



yielding rays of greater refrangibility. Although the radius of the circular arc was (if I understand the account rightly) eleven inches, but little deviation of position was required for heat from different sources; and M. Melloni admits that, whilst his experiment *indicates* the difference of refrangibility, it is inadequate to *measure* it.

49. There are many reasons why such a form of apparatus must be rejected for accurate observations. I will mention only the impossibility of obtaining a beam of heat which shall preserve the same breadth at different distances from its source (of course supposing the rays rendered as parallel as possible by refraction through a rock-salt lens), arising, 1. from the angular magnitude of the source; 2. from the scattered reflection and refraction at the surfaces of the lens and prism; 3. from the want of homogeneity of the ray. On all these accounts, the beam must have acquired a very sensible breadth at the distance of the pile, and consequently the effect of heat must be perceptible, and even nearly uniform, through a certain space. I may also add from experience, that the difficulty of varying the arrangement of an experiment, so as to get a maximum heating effect at the pile, is so considerable, that no delicate result can be deduced from the merely tentative procedure. Finally, the smallness of the variation of refrangibility, seems to require some more critical method of ascertaining its measure. On all these grounds, it seemed to me desirable to discover a method in some degree less open to objection.

50. The phenomenon of total reflection, successfully employed by Dr Wollaston in the measurement of refractive indices in the case of light,* presents the advantage of being (theoretically at least) abrupt in its action, the transition from partial to total reflection being (with the necessary exception arising from the want of homogeneity) an instantaneous change, amounting in the case of light to many times the intensity of the smaller effect. It seemed reasonable to expect, that an apparatus constructed on the principle of determining the critical angle

* *Phil. Trans.* 1802.

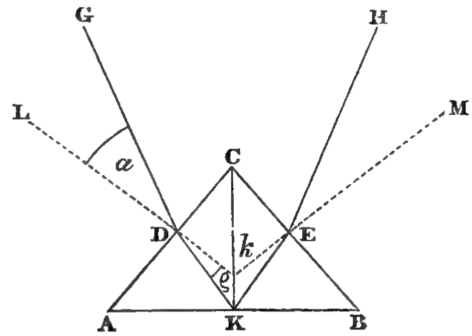
of total reflection of heat from different sources within a prism, would afford much more definite information as to the refrangibility of heat than any other method. After much consideration, an apparatus of the following kind was adopted.

51. It is fundamentally composed of a jointed frame, resembling a box exactly square, ten inches in the side, without top or bottom, and having hinges at every angle, so that it may be formed into a lozenge of any degree of obliquity. This is seen in Plate XIII, Fig. 1, and marked AB. By an arrangement presently to be described, the rays of heat are made to pass parallel to the edge *a c* of one of the sides of the box, and to fall upon the prism P, whence, after undergoing reflection (total or partial) at the posterior surface of the prism, they proceed parallel to the line *a d*, and fall upon the sentient extremity of the pile at *p*. Now, in order that this course may be taken by the reflected rays, it is necessary that, supposing the prism to be an isosceles one, the posterior reflecting surface *a′ b′* Fig. 2, should form equal angles with the incident and reflected rays *c e* and *f d*. It was to effect this that the arrangement of the jointed lozenge was adopted. The prism P (Fig. 1) rests on a column O, moveable round the line of junction of the sides C and D of the lozenge. The column O has connected with it a tail-piece of brass *a* E passing through the diagonal of the frame, and preserved constantly in that position by a slit parallel to its length, through which passes a clamping screw *b*, serving at once to maintain this constancy of direction, to secure the form of the moveable lozenge, and by means of an index pointing to a graduated scale of inches reckoned from *a*, along *a* E, to determine the length of the diagonal *a b* at any moment, and consequently the angles of the lozenge.

52. A little consideration of this mechanical arrangement, will shew how it is adapted to the end in view. The rays from a source of heat S, rendered parallel by the lens of rock-salt L, fall upon the prism P, and, after undergoing two refractions and one reflection, they fall upon the sentient surface of the pile *p*. This will always take place so long as the posterior surface of the prism forms equal angles with the lines *a c*, *a d*, which will be secured by making it truly perpendicular to the tail-piece *a* E, by which it is guided, and which of course always bisects the angle *c a d*. Now, it is evident that, whilst the angle *c a d* remains small, the reflection will continue *partial*, but that as the diagonal *a b* is shortened, a point will be reached when *total* reflection abruptly commences, which ought to be indicated by a *saltus* in the movement of the galvanometer connected with the pile. This critical angle will be soonest attained for rays of greatest refrangibility, and the calculation of the refractive index of the prism is reduced to a simply mathematical problem.

53. Before going farther, we shall proceed to solve this problem, viz.: *A ray of light* GD (fig. next page) *falls upon the surface* AC *of a prism, which has the angles at* A *and* B *equal ; it falls upon the surface* AB *at the critical angle of total reflection ; required the index of refraction* (μ) *of the prism the angle of incidence* (α) *being given.*

What is true of one ray GD, which after refraction meets the posterior surface at K, its middle point, will be true of any other parallel to it; also the incident and emergent rays DG, EH, form equal angles with the surface AB when the angles A and B are equal. By hypothesis DKC = the angle of total internal reflection $= \sin^{-1}\frac{1}{\mu} = \beta$. Let KD$k$, the angle of refraction, $= \varrho$, then $\sin\alpha = \mu\sin\varrho$. Also, considering α positive when G falls between L and C, and the corresponding value of ϱ also +, we have in the triangle KDC



$$180^\circ = \beta + \text{ACK} + (90^\circ + \varrho)$$

And, calling the angle at C, I, ACK $= \frac{1}{2}$I,

$$90^\circ = \beta + \tfrac{1}{2}\text{I} + \varrho.$$

But $$\sin\alpha = \mu\sin\varrho = \mu\sin\left(90^\circ - (\beta + \tfrac{1}{2}\text{I})\right)$$
$$= \mu\cos(\beta + \tfrac{1}{2}\text{I})$$
$$= \mu\left\{\cos\beta\cos\tfrac{1}{2}\text{I} - \sin\beta\sin\tfrac{1}{2}\text{I}\right\}$$
$$= \mu\left\{\sqrt{1-\sin^2\beta}\cos\tfrac{1}{2}\text{I} - \sin\beta\sin\tfrac{1}{2}\text{I}\right\}$$

(Also since $\sin\beta = \frac{1}{\mu}$,) $$= \mu\left\{\sqrt{1-\frac{1}{\mu^2}}\cos\tfrac{1}{2}\text{I} - \frac{1}{\mu}\sin\tfrac{1}{2}\text{I}\right\}$$
$$= \sqrt{\mu^2-1}\cos\tfrac{1}{2}\text{I} - \sin\tfrac{1}{2}\text{I}$$

Whence $$\mu = \sqrt{1 + \left(\frac{\sin\alpha + \sin\frac{1}{2}\text{I}}{\cos\frac{1}{2}\text{I}}\right)^2}$$

54. I had a rock-salt prism constructed, so that the incidence on the first surface might be nearly vertical at the critical angle of total reflection, so as to avoid as much as possible any error arising from imperfections of the surface, or want of absolute equality of the angles at A and B; and likewise, that within the limits of the experiment, the loss of heat by reflection at the two surfaces might be nearly unaltered, as it is believed to be almost constant at incidences tolerably nearly perpendicular.* This prism, constructed for me by Mr John Adie, had two angles of 40° and one of 100°; and so accurately was it made, that (satisfying myself with a careful measurement by the common goniometer, extreme nicety being unimportant) the angles appeared to be true to those quantities within a few minutes of a degree.

55. By a reference to Plate XIII, Fig. 1, it will now be understood that the required arrangement is of this kind. The heat diverging from the source S, is converted into an approximately parallel beam by the lens L. It then passes

* See Melloni on the Reflection of Heat, *Annales de Chimie*, Dec. 1835.

through a diaphragm T, placed on one or other side of the prism (it does not much matter which, as the beam which arrives at the pile is always much wider than the second diaphragm *t*, placed there to admit only the central rays arriving parallel to the line ac). The use of this diaphragm is, that a narrow enough pencil of rays may be employed, to be independent of the variable breadth under which the surface of the prism is presented to the incident beam. The usual dimension of this diaphragm was one inch in breadth, and one and a quarter in height, but in some instances its breadth was reduced to three-eighths of an inch.

56. The pile *p* has its funnel-shaped orifice closed by a screen with a vertical slit, an inch wide, in the direction of its axis. But there is a peculiarity in the arrangement of the pile very essential to the success of these experiments, where the pile itself is moveable, which I must not omit to mention. Its exposure to currents of air would render the observations, when the pile cannot be entirely enclosed by a box or screen, very capricious in its action. I therefore adapted to the end, bearing the conical reflector (II. 6), an adjustable wooden tube *r*, containing a rock-salt lens, which still farther increased its sensibility, and totally protected it from aërial currents.

57. The more important adjustments of the apparatus previous to use, are these: 1. To place the surface $a'b'$ of the prism (Fig. 2) so as to form equal angles with the sides of the lozenge $K\gamma$, $K\delta$, the point K being precisely above the angle of the lozenge frame. To accomplish this, the prism rests upon a brass plate, having an adjusting motion concentric with that of the pillar O (Fig. 1), on which it rests. The adjustment was made by placing a piece of truly parallel mirror-glass in the position of the posterior surface of the prism, suspending two plumb lines in the prolongation of the lines ac, ad, and observing by the eye placed at c whether the reflection of the other was seen in the direction ac, and adjusting the brass plate before mentioned, bearing the mirror, until such was the case, then making it fast by a clamping screw. 2. The next adjustment was to bring the centre of the lens L into the line ac, which was done by placing a small flame of a lamp in the position of the axis of the pile *p*, and regulating the position of the lens until the image of it fell exactly upon the prolongation of the line ac, the prism being so placed that the angles of incidence were almost perpendicular; (reflection at a mirror would have been preferable). 3. The adjustment of the source of heat behind the lens is the next point. When the source is luminous, it is done by causing the axis of the refracted cylinder of light to coincide with the line ac; when not luminous, its breadth being usually considerable, it is found that a small displacement in one direction or another, makes but a small difference in the effect upon the pile.

58. The abruptness of the effect of transition from partial to total reflection is far from being so complete as might be wished; and this is easier accounted for than remedied. It arises mainly from the magnitude of the source of heat, the

consequent want of parallelism of the refracted rays, the scattering of these rays in consequence of the imperfect polish of the surfaces, the unequal intensity of the rays in different parts of the section of the cylinder, and lastly, from the want of homogeneity of the rays of heat from any source, which the method would serve to measure, were the other imperfections removed, just as in the course of the total reflection of light, prismatic colours are successively presented.

59. My first rude attempts shewed all this very clearly. As the diagonal $a\,b$ of the lozenge (Fig. 1) shortened, total reflection obviously succeeded to partial, and the change was not only very great, but near one point very rapid. The point where the most rapid increase took place, is obviously that where the greater proportion of the incident rays underwent total reflection, and might therefore be taken as a mean representation of the quality of the heat. Still the change was too gradual to enable one by mere inspection to determine this point with accuracy, and I speedily resolved to take the sure but laborious method of ascertaining at a number of points intermediate between total and partial reflection the intensities of the reflected heat, and by constructing a curve having measures of the diagonal of the lozenge (a function of the angle of incidence) for abscissæ, and intensities for ordinates, I endeavoured to discover graphically for what value of the former the measure of the latter increased most rapidly, in other words, where the tangent made the greatest angle with the axis, or where was the point of contrary flexure of the curve.

60. Plate XIII, Fig. 3, may represent such a curve. I have found that when the diagonal of the lozenge was 14.5 inches, the reflection was in all cases nearly total, or the galvanometer was little affected by any increase of the angle of incidence. This effect, measured by the vertical line AB, was denoted by 100. When the diagonal was increased to 15.0, the effect was reduced, we shall suppose, to 90, expounded by the line CD, at 15.5 by EF, and so forth. An interpolating curve drawn through the points so fixed, would have its greatest inclination to the axis AX, when, for a given variation of the diagonal, the decrement of the intensity was a maximum, in other words, at the determining angle for the predominating part of the heat used. Such a point of contrary flexure would therefore determine the *mean* index of refraction of the given kind of heat by the aid of the formula above investigated, whilst the form of the curve would lead to some conjecture at least, respecting the distribution of heat of the more or less refrangible kinds in the given ray. Heat of low refrangibility being the last to be totally reflected, would cause the curve to droop fastest near the extremity B, the more refrangible rays would be cut off at the other end I of the curve.

61. I lost no time in verifying the general truth of the principle, and also of the received doctrines respecting heat, by examining the *quality* of the heat which reached the pile at different stages of total reflection. If, as M. Melloni first rendered probable, heat of *low* temperature is *least* refrangible, and *vice versa*;

and farther, if it be admitted that such heat passes most difficultly through such substances as glass, it follows, that after total reflection has proceeded a certain way, so that the more refrangible, and therefore more transmissible, rays have suffered total reflection, whilst the remaining rays constituting the primitive beam continue to be refracted, the heat thus reflected will be more copiously transmitted by glass, than when it came direct from the source. This conjecture was precisely verified.

62. Subsequent experiment still more fully confirmed this result, and by shewing that, during the whole progress from partial to total reflection, the specific quality of the heat changes, gave countenance to the view that the gradation is in a great measure owing to the want of homogeneity of the heat, and that the figure of the curve becomes (as we have said) a real test of the composition of a ray.

63. At the inferior limit of the curve, or when partial reflection takes place, all kinds of heat are equally reflected (in the case of light, the light is white), just as at the superior limit, or after total reflection is complete, the beam has exactly the same relative composition as before. In the intermediate stages the composition is perpetually varying. The first rays totally reflected (and combining with the scattered and partially reflected rays) are the more refrangible, or those more easily transmitted by glass. At a certain point a maximum proportion of these enter into the reflected beam. As the angle of incidence becomes greater, more and more of the less refrangible rays enter into the composition of the reflected heat, which at last possesses the same qualities as at first. This is well illustrated by the following early experiment which I made on the proportion of the reflected rays transmitted by a plate of glass .06 inch thick, at different stages of reflection (7th February 1838).

Diagonal $a\,b$, Fig. 1, in Inches.	Deviations of Galvanometer.		Ratio.	REMARKS.
	Glass.	No Glass.		
14.5	$8\overset{\circ}{.}3$	13.75	60 : 100	Total reflection complete.
15.0	7.85	12.65	62 : 100	
15.25	7.1	10.9	65 : 100	
15.5	5.5	7.85	70 : 100	
15.75	3.4	5.1	67 : 100	
16.0	2.3	3.75	61 : 100	
16.5	1.45	2.3	63 : 100	Partial reflection.

64. The experiments of which I am now to state briefly the results, were made with heat from various sources, and modified by transmission through different media. Considering them of great importance, I have spared no pains in verifying the results, and ascertaining the limits of error. My latest experiments, in which I availed myself of the experience which earlier ones had afforded, are of course most to be depended upon, and to them I shall chiefly refer

(made between the 21st March and the present date); but it is important for the credit of the results to observe, that they not only derive a general confirmation, but exhibit an almost exact numerical coincidence with those obtained formerly, and with less careful adjustments.

65. Avoiding, then (as in these papers I have habitually done), the tedious detail of minute precautions which the experienced operator will soon discover, and which to others would be of little use, it is to be understood that in the following experiments on the law of the Transition from Partial to Total Reflection, the arrangement was that shewn in Plate XIII. Fig. 1, and described (with the adjustments) in arts. 51–57;—that the centre of the pile p was 13 inches from the prism P, and the distance of the source of heat S from P was 12 inches;—that a diaphragm T, whose aperture was 1 inch by $1\frac{1}{4}$, was placed in the path of the ray usually between P and L near P;—that the aperture of the pile was contracted to a breadth of one inch, whose centre was exactly in the line ad;—and that only that part of the prism was employed which was free from flaws capable of producing total reflection.

66. The diagonal of the lozenge frame was varied from 14.5 inches up to 16.5 or 17.0, about eight observations of the intensity of reflected light being made at intervals. The series was then frequently *reversed*, and the mean results of the going and returning series taken to allow for any change which might have occurred in the intensity of the course. In all cases an observation of verification was made and such change allowed for. The dynamical effect on the galvanometer (II. 8) was observed and noted.

67. In reducing the observations the following plan was adopted. The intensity corresponding to the diagonal 14.5 inches being assumed = 100, the other intensities were reduced relatively to it, and projected, as explained in art. 60. By this means different series of observations became at once comparable with each other, and the beauty and regularity of the curves thus formed, and the almost perfect identity of those obtained on different days, and with different adjustments, give a degree of confidence in the results which is extremely satisfactory. When from the nature of the heat the effect was very small (as in the case of alum being interposed, or the source being of low temperature), I have endeavoured to supply the deficiency by multiplying observations, and the uniformity of the curves thus obtained has been the test of my success. Where this test has failed (as in the attempt to work with heat of 212°), I have suppressed the results.

68. I am unwilling to swell this paper by a quotation of individual experiments, of which the number is very great,* but I think it fair to give specimens of the actual work in a few cases.

* It may not be superfluous to state, that during the course of the experiments referred to in this series of papers, I have adopted a uniform and clear system of recording my experiments, which admits of subsequent reference, and, if necessary, of publication. The experiments have been fairly written out,

Dark Hot Brass. March 31. 1838.

Measure of Diagonal *a b*, Plate xiii. Fig. 1, in Inches.	Galvanometer-Needle		Excess.	Ratio to Result with Diagonal = 14.5.
	Stands at	Swings to		
14.5	A 0°.1	A 11°.0	10°.9	100 : 100
15.0	0.0	9.3	9.3	85 : 100
15.25	0.15	8.0	7.85	72 : 100
15.5	0.15	6.2	6.05	55 : 100
—	0.3	6.3	6.0	
15.75	0.25	4.6	4.35	40 : 100
16.0	0.15	3.15	3.0	28 : 100
16.25	0.2	2.25	2.05	19 : 100
16.5	0.15	1.9	1.75	16 : 100
14.5	0.05	10.8	10.75	

Incandescent Platinum. February 10. 1838.

Measure of Diagonal *a b*. Plate xiii. Fig. 1, in Inches.	Galvanometer-Needle		Excess.	Ratio to Result with Diagonal = 14.5.
	Stands at	Swings to		
14.0	B 0°.25	A 15°.25	15°.5	100 : 100
14.5	A 0.05	15.5	15.45	100 : 100
15.0	B 0.15	13.65	13.8	89 : 100
15.25	0.0	11.75	11.75	75 : 100
15.5	0.1	8.8	8.9	58 : 100
15.75	0.1	6.1	6.2	40 : 100
16.0	0.1	4.05	4.15	27 . 100
16.25	0.0	3.3	3.3	21 : 100
16.5	0.15	2.5	2.65	17 : 100
16.75	0.0	2.25	2.25	15 : 100
14.0	0.9	14.4	15.3	
......	1.25	13.75	15.0	
......	1.7	13.75	15.45	

Heat from Locatelli Lamp sifted by a plate of Alum. March 31. 1838.

Diagonal.	Dynamical Effects.		Mean Effect.	Ratio to 14.5.
	Direct Series.	Reversed Series.		
14.5	3°.1	3°.2 3°.25	3°.18	100 : 100
15.0	2.95	2.9	2.92	91 : 100
15.25	2.9	2.65 2.5 2.7	2.69	84 : 100
15.5	2.15	2.1	2.12	66 : 100
15.75	1.75 1°.7	1.75	1.73	54 : 100
16.0	1.2	1.25	1.22	38 : 100
16.25	0.8	0.6 0.6	0.67	21 : 100
16.5	0.75 0.75	0.7	0.73	23 : 100
17.0	0.55		0.55	17 : 100

averages and ratios taken, generally on the same day on which they were made. This methodical plan cannot be too strongly recommended. Much after anxiety is spared, the calculations are lightened, errors avoided in the reduction after minute circumstances have been forgotten, and suggestions are afforded by the result of past experiments for the conduct of new ones.

69. After the observations made as now described have been projected in the form shewn, Plate XIII, Fig. 3, and Plate XII, the diagonal corresponding to the maximum rate of decrease of the intensity was determined, for the purpose of deducing the index of refraction. The following enumerations of the kinds of heat employed, and the results derived from the several projections, will give a just idea of the confidence due to the results. They are distinguished into those made since, and those previous to the 21st March, because some additional precautions have been taken since that time, which do not, however, appear to have produced a sensible change. Of the experiments made in the way above described, only one series is rejected, on account of its discrepancy from others of the same kind, (the discrepancy was so large as to indicate a displacement of the prism, or some fundamental derangement not perceived at the time); and another on account of the irregularity of the points marked out for the curve, although the general form of the curve did not differ from others similarly obtained.

70. *Sources of Heat.*—(1.) The direct rays of the Locatelli Lamp. A slightly concave reflector was employed. (2.) The same lamp, with a reflector having the form of a portion of a sphere concentric with the wick; the heat transmitted through *alum*. (3.) Heat from the same source transmitted by *window-glass* .06 inch thick. (4.) Heat from the same transmitted by opaque *black glass* (through which the disk of the unclouded sun is just visible). (5.) Heat from the same transmitted through *dark coloured mica*, by which direct sunlight is absolutely stopped. This singular substance I long sought for in vain, it is unknown to many practical mineralogists; it transmits green light at small thicknesses, when thicker its colour is hair-brown. By reflected light its colour is between green and black. (6.) Heat from incandescent platinum. (7.) The same sifted by *window-glass* as above. (8.) The same sifted by *opaque mica*. (9.) Heat from dark brass about 700°. This is obtained from a nearly cylindrical cover of smoked brass placed over the flame of a spirit-lamp, so as entirely to conceal it, and which gives remarkably good results, without increasing considerably the angular breadth of the source (which is greatly to be avoided when a lens is used). It is in fact not much greater in size than the helical coil of platinum wire used in (6). (10.) The same, sifted by clear *mica* .0044 inch thick. (11.) Heat from a crucible of mercury about 450°. The crucible was about 2 inches in the side, smoked externally, and heated by a spirit-lamp. The temperature of the mercury which it contained (covered with sand) was noted at each observation by means of an inserted thermometer.

71. The results were the following:

SOURCE OF HEAT.	Diagonal corresponding to Point of contrary flexure of Curve.	
	Before March 21.	Since March 21.
Locatelli; direct, ...	15.47 15.50 15.60*	15.54 15.47
——— with Alum, . . .		15.79 15.73
——— Window-Glass, .	15.64	15.70 15.60
——— Opaque Glass, . .		15.75 15.67
——— ——— Mica, . .		15.60 15.62
Incandescent Platinum, . .	15.51 15.47	15.52
Ditto, with Glass, ...		15.64 15.67
Ditto with opaque Mica, . .		15.62
Brass at 700°, ...	15.44 15.42	15.45 15.47 15.45
Ditto with clear Mica, . .		15.62† 15.55
Mercury at 450°, ...		15.52 15.52 15.45

72. From these numbers we can of course compute the corresponding angles of incidence, and thence the value of the index of refraction by the formula of art. 53. For the purpose of ready comparison I have calculated the following table, which gives the angles of incidence, and consequently the indices of refraction, corresponding to different values of the diagonal computed by the formula of art. 53:

Diagonal ab.	Angle of Incidence $= \alpha$.	Angle of Total Reflection $= \beta$.	Index of Refraction $= \mu$.
Inches.	° ′	° ′	
15.0	−1 25	40 55	1.527
15.1		40 37	1.536
15.2	−0 32	40 20	1.545
15.3		40 3	1.554
15.4	+0 21	39 47	1.563
15.5		39 30	1.572
15.6	+1 16	39 12	1.582
15.7		38 55	1.592
15.8	+2 11	38 37	1.602
15.9		38 20	1.612
16.0	+3 8	38 4	1.622

73. We have the following mean values of ab for the points of contrary flexure, and consequent values of the indices of refraction of the *most abundant* rays in each source:

* This observation was made with a very contracted diaphragm; the readings therefore were very small. It is omitted in the final reductions.

† Omitted in the final reductions on account of the irregularity of the observations.

Source of Heat.	ab.	μ.
Locatelli, direct, ...	15.49	1.571
——— with Alum, . . .	15.76	1.598
——— Window-Glass, .	15.65	1.587
——— opaque Glass, . .	15.71	1.593
——— ——— Mica, . .	15.61	1.583
Incandescent Platinum, . .	15.50	1.572
Ditto with Glass, ...	15.66	1.588
——— opaque Mica, . . .	15.62	1.584
Brass at 700°, ...	15.45	1.568
Ditto with clear Mica, . .	15.55	1.577
Mercury at 450°, ...	15.50	1.572
Mean Luminous Rays, . .	15.8	1.602

74. In the following table I have given the *mean* results of the different series of observations on which the above conclusions are founded, and from these numbers I have projected the curves exhibited in Plate XII, the dots corresponding to the numbers here given, and the mode of projection being that already explained:

Source of Heat.	Values of the Diagonal ab.							
	14.5	15.0	15.25	15.5	15.75	16.0	16.25	16.5
Locatelli; direct, ...	100	93	80	60	41	30	2	13
——— with Alum, . . .	100	91.5	84.5	71.5	51	39	24	18.5
——— Window-Glass, .	100	93	84	66	47	33	23.5	18
——— opaque Glass, . .	100	97.5	89	75.5	55	42.5	26	22.5
——— ——— Mica, . .	100	94	82	67.5	48.5	32.5	23	20
Incandescent Platinum, . .	100	89	75	58	41	30	22	18
Ditto with Glass, ...	100	88	77	62.5	42.5	30.5	23	17.5
——— opaque Mica, . . .	100	92	78	63	46	30	22	18
Brass at 700°, ...	100	84	69	51	35.5	25	20	15
Ditto with clear Mica, . .	100	85	71	52	33	26	13	11
Mercury at 450°, ...	100	92	77	57	42	29	22	13

75. When we compare the preceding results, obtained with a rock-salt prism, with those for light, we find that the *received* index of refraction for that substance would give to heat a higher degree of refrangibility than light, a result contrary to all probability. This, however, is not confirmed by direct experiment. Placing a bright small source of light at S (Plate XIII. Fig. 1), and a screen at p, I find the index of refraction for the most luminous rays to be higher than that of any of the above kinds of heat, being at least 1.602, corresponding to a diagonal $ab=15.8$ inches, as I have given it above. By two series of results derived from a very small oil flame (without wick), I got 15.87 for the diagonal both times; and from the Locatelli-lamp (which on account of the size of the flame forms a better standard of comparison with the experiments on heat) 15.76; so that I consider 15.8 as a fair representation of the case of light.

76. Yet it is quite certain that the index of refraction of the rock-salt used is really much below 1.60. A single experiment with Dr Wollaston's instru-

ment gave me a result between 1.53 and 1.54. Without dwelling more than necessary on this difference (our great object being gained when we have compared heat and light under similar circumstances), I will mention the two causes which I believe to produce it. (1.) It is undeniable that the transition from total to partial reflection takes place much more gradually than is due to the mere heterogeneity of the rays; (this the experiment with light makes very obvious). The angle of incidence throughout the range of experiment (from $ab = 14.5$ to $ab = 16.5$) *within* the prism varies from 42° 22′ to 36° 38′. The intensity at any point is made up of totally and of partially reflected light. Consequently throughout this range of incidence, the partially reflected light must be more intensely reflected as the incidence is greater, and it is easy to see that the effect of this variation in the intensity of the partially reflected rays will have the effect of shifting all the curves towards the right hand in Plate XII. (2.) We have before remarked, that owing to the dimensions of the source of light or heat, the rays do not form a refracted beam of uniform intensity. The central rays are usually brightest. Now, it may be shewn that in consequence of the varying angle of incidence the central rays travel across the front of the pile, and consequently there would be a maximum effect produced at one point from this cause alone.

77. I believe that the former cause is the only one whose effects are sensible, or at least considerable; and having reason to think that its action is similar upon different kinds of heat, and also of light, we shall probably be very near the truth if we substitute for the indices of refraction above found, others .04 or .05 lower. But relative results are in this case by much the most important.

78. The results which we have obtained apply, it must be recollected, only to the *predominant* kind of heat in any source, and that we have as yet got no information respecting the *composition* of a ray and the amount of dispersion.

79. It is very easy to see that were the mathematical conditions of the experiment (art. 55) fulfilled, we should be led to an exact analysis of heat, more perfect far than we have any prospect of obtaining in the case of light, considering the difficulty of applying the photometer to coloured light. Were the curve in Plate XIII. Fig. 3, solely representative of the progress of reflection due to the heterogeneity of the rays, the increment of intensity between any diagonal E and another C, or D *f* would denote the proportion of the entire heat incident, which lies between the limits of refrangibility assigned by the diagonal, and found by the table in art. 72. Thus an entire ray would be decomposed into parcels of known proportions, between given intervals of refrangibility. The case is considerably different. Though the points of contrary flexure agree remarkably well, as we have seen, the curves are in some cases much more flattened than in others, where the source of heat is the same; owing probably to the greater parallelism of the rays at one time than at another, depending on the distance of the source of heat from the lens.

80. We can, therefore, in this way form but an imperfect idea of the comparative homogeneity of the different kinds of heat. Such comparisons can only be made advantageously by comparing the results obtained in immediate succession from one and the same source with interposed screens of different qualities, as in the comparison which we instituted between heat direct from Locatelli's lamp, and that transmitted by glass, (art. 63).

81. The facts respecting refrangibility, which may now be considered as ascertained, serve to render our ideas much more precise in several respects. For instance, (1.) the range of *mean* refractive indices for heat is small, all the modifications which we have considered lying within a range of .04, or between 1.51 and 1.55 nearly, which is little more than the commonly assigned dispersion of light, which, for rock-salt, is between the limits 1.54 and 1.57 nearly. This, however, is for *extreme* rays of light, which can hardly be said of heat; the extremes of dispersion are certainly much wider apart. (2.) The *mean* refractive index of *direct* rays from different sources varies surprisingly little. In fact the differences for *direct* rays of heat from the Locatelli-lamp, incandescent platinum, and from a crucible heated to 450°, seem almost insensible, or within the limits of error of experiment. It is to be recollected, however, that this is compatible with the utmost variety in the *composition* of each. (3.) The effect of interposed screens in modifying the transmitted heat is very remarkable. These, so far as I have tried them, invariably *raise* the index of refraction, (alum, glass, opaque glass, and opaque mica for the Locatelli-lamp; glass and opaque mica for incandescent platinum, and clear mica for dark heat). This is the case even with those substances which suppress light altogether, and which therefore cannot be considered to do more than detach the heat of considerable refrangibility from the light which usually accompanies it, not as stopping the most refrangible rays and admitting the passage of those of lower temperature. Probably no substance acts in this way, though some (as black glass and mica, as the experiments of MELLONI indicate) may probably absorb the heat spectrum at *both* extremities. It is probably to this source that we must attribute the very small fraction of heat transmitted by the black glass I used, being only that constituting the rays of the higher degrees of refrangibility, all those of *low* and *mean*, and also of the *highest*, degrees of refrangibility being probably absorbed. (4.) With respect to the homogeneity of different kinds of heat, I have already stated that we can deduce nothing *certain* from the forms of the curves in Plate XII. They confirm, however, a view which I have long entertained, that heat from non-luminous sources is more homogeneous than any other. I argued this partly on the ground stated in art. 40 of this paper, and still more from the uniformity of results which I have in all classes of experiments obtained from dark heat, which often more than made up for the narrower range of the thermal effect, and which shewed that the discrepancies observed in other cases were due not so much to errors of observation, as to unavoidable changes in the

character of the heat, (art. 14). This result is the more probable from the size of the source of heat necessarily used in the crucible experiments (art. 70), which tends to render the passage from partial to total reflection more gradual, and thus to flatten the curve. To the same cause may also probably be attributed the somewhat greater index of mean refraction obtained for heat from this source than that of dark heat of higher temperature.

82. The following method might perhaps be used with success for obtaining more exact data respecting the refrangibility, and especially the *dispersion*, of heat, than that just described pretends to give. It must insure a beam of parallel rays of heat of sufficient intensity and uniform in every part of its section. A small point of heat placed behind a lens (or two or three lenses to diminish aberration) is the most obvious plan. But the intensity would be inadequate. I would, therefore, propose a platinum-wire, heated by one of Mr DANIELL's constant voltaic batteries, placed behind a refracting semi-cylinder of rock-salt.* The central rays should be alone employed, and the prism for total reflection should be high and narrow as well as the aperture of the pile. It is possible that in this case the transition from partial to total reflection would be so rapid as to make the error arising from the varying intensity of partial reflection (art. 76) inconsiderable. By changing the force of the battery, heat of all temperatures might be employed in succession. The numerical analysis of the heat spectrum would then take place as described in art. 79.

Conclusion.

83. My object in these, as in former researches, has not been to group experiments of mere curiosity indiscriminately selected, but to present a basis for a proper theory of heat. Without some such end in view I should have thought the time and labour spent on these experiments in some degree misapplied. Mere numerical results, though ultimately of the highest consequence to science, should never form the exclusive object of the philosopher. I trust to have shewn that though many of the conclusions in this paper are based upon quantitative results, these have not been the ultimate aim of the inquiry.

84. The mutual bearing of the three sections of this paper, and of all upon what (from analogy to physical optics) we may call *physical thermotics*, is now evident. (1.) In the First Section we have minutely discussed a point apparently perhaps of minor importance, namely, the unequally polarizable nature of the rays of heat. The importance of the doctrine lies in this: that the common theory of undulation recognises no such variation, nor *perhaps* does it exist in the case of light (I know, however, of no decisive experiments on this point), with the excep-

* Such a one I have had executed.

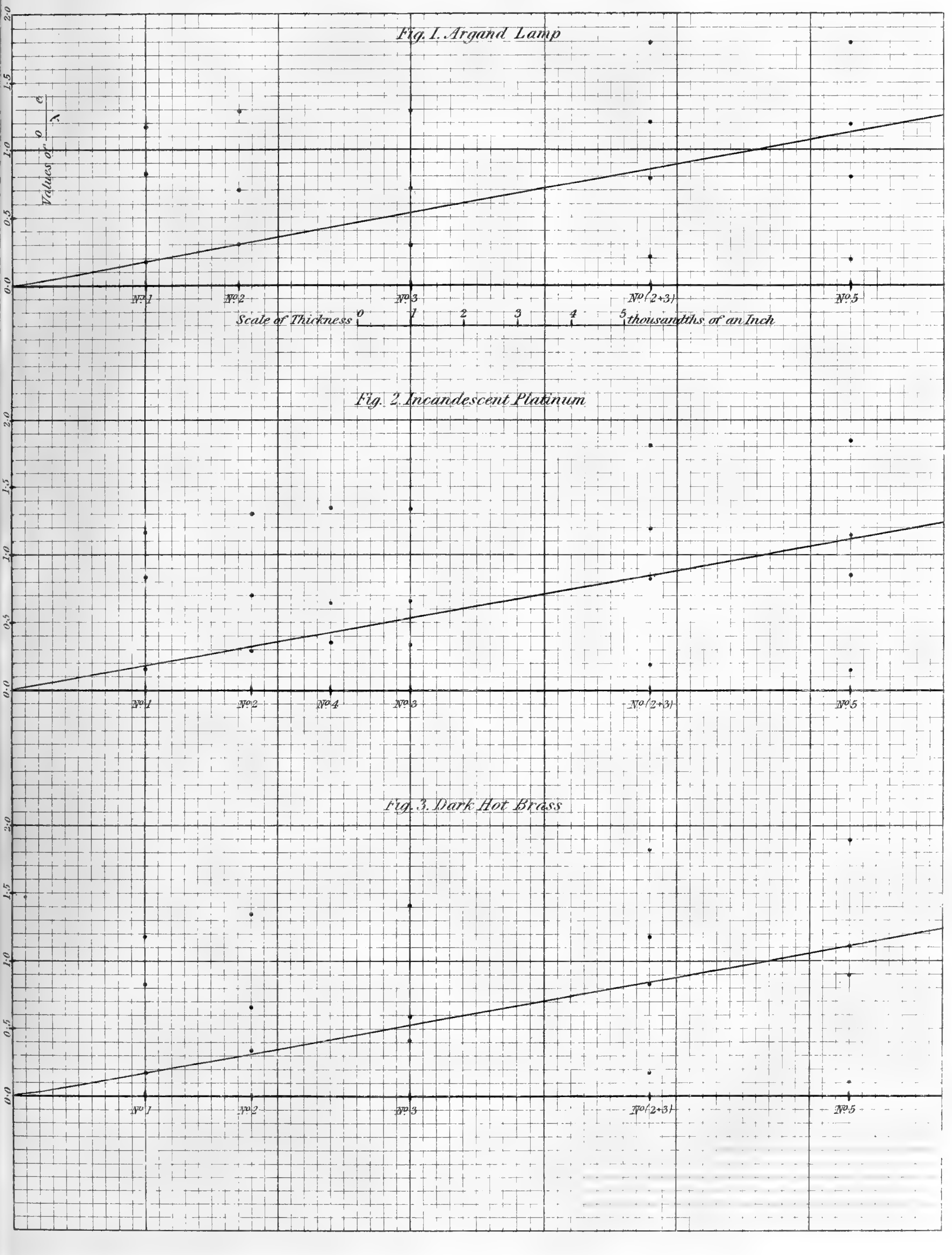
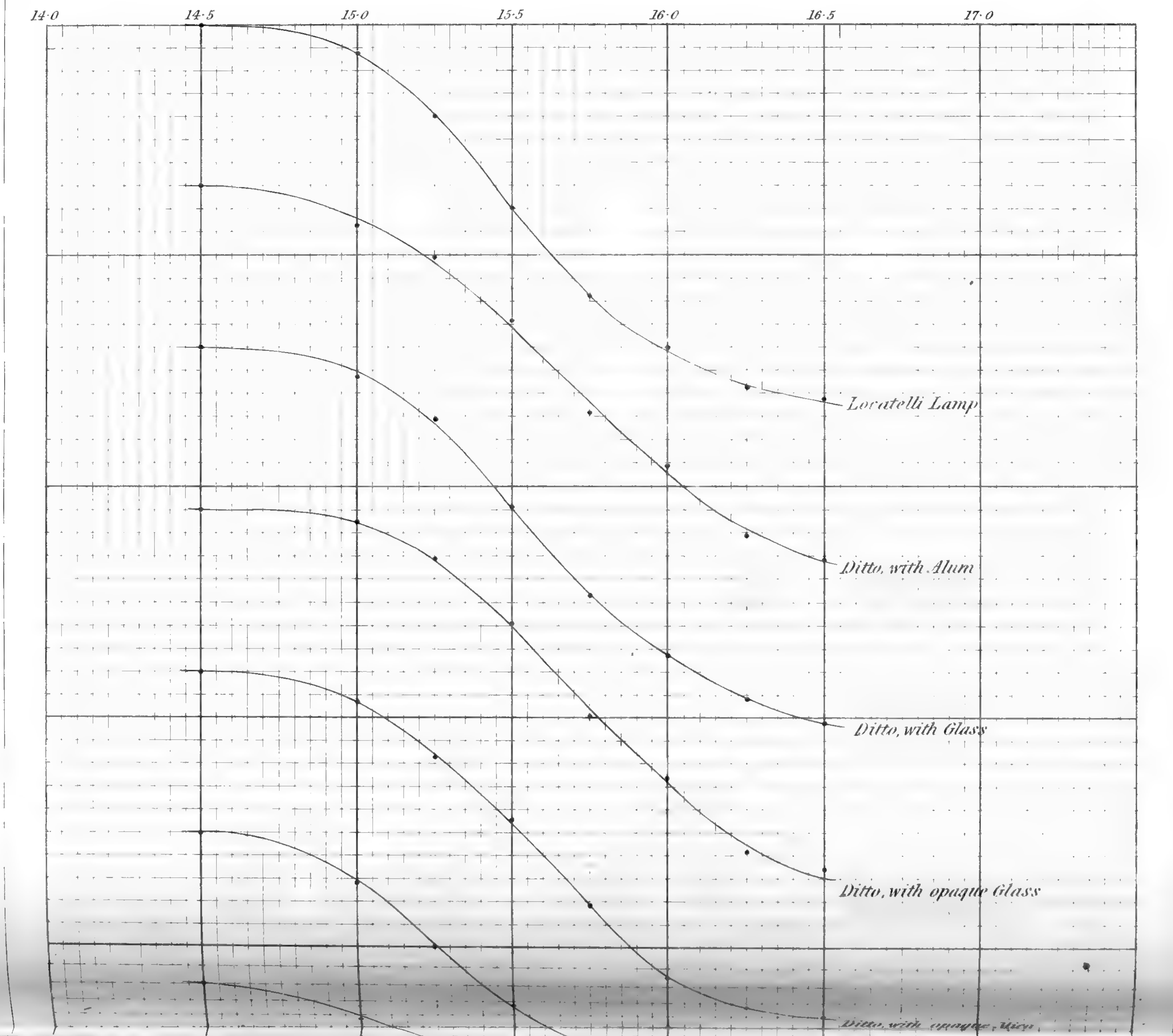
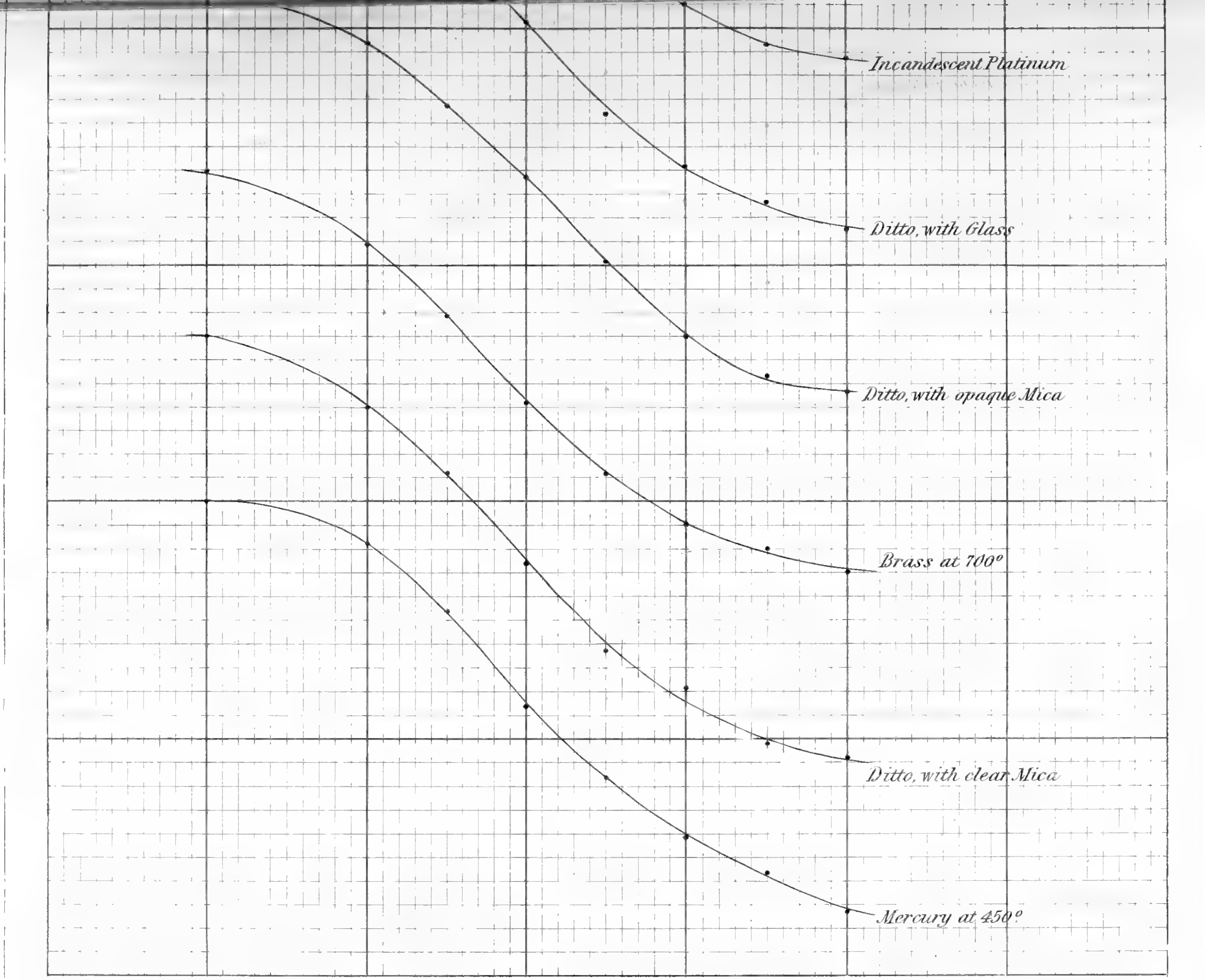


Fig. I. Argand Lamp
Values of $\frac{o-e}{\lambda}$
No 1
No 2
No 3
No (2+3)
No 5
Scale of Thickness 0 1 2 3 4 5 thousandths of an Inch
Fig. 2. Incandescent Platinum
No 1
No 2
No 4
No 3
No (2+3)
No 5
Fig. 3. Dark Hot Brass
No 1
No 2
No 3
No (2+3)
No 5
0.0
0.5
1.0
1.5
2.0

Lizars sc.

REFRANGIBILITY OF HEAT.




Incandescent Platinum
Ditto, with Glass
Ditto, with opaque Mica
Brass at 700°
Ditto, with clear Mica
Mercury at 450°

Izars sc.

Fig. 1.

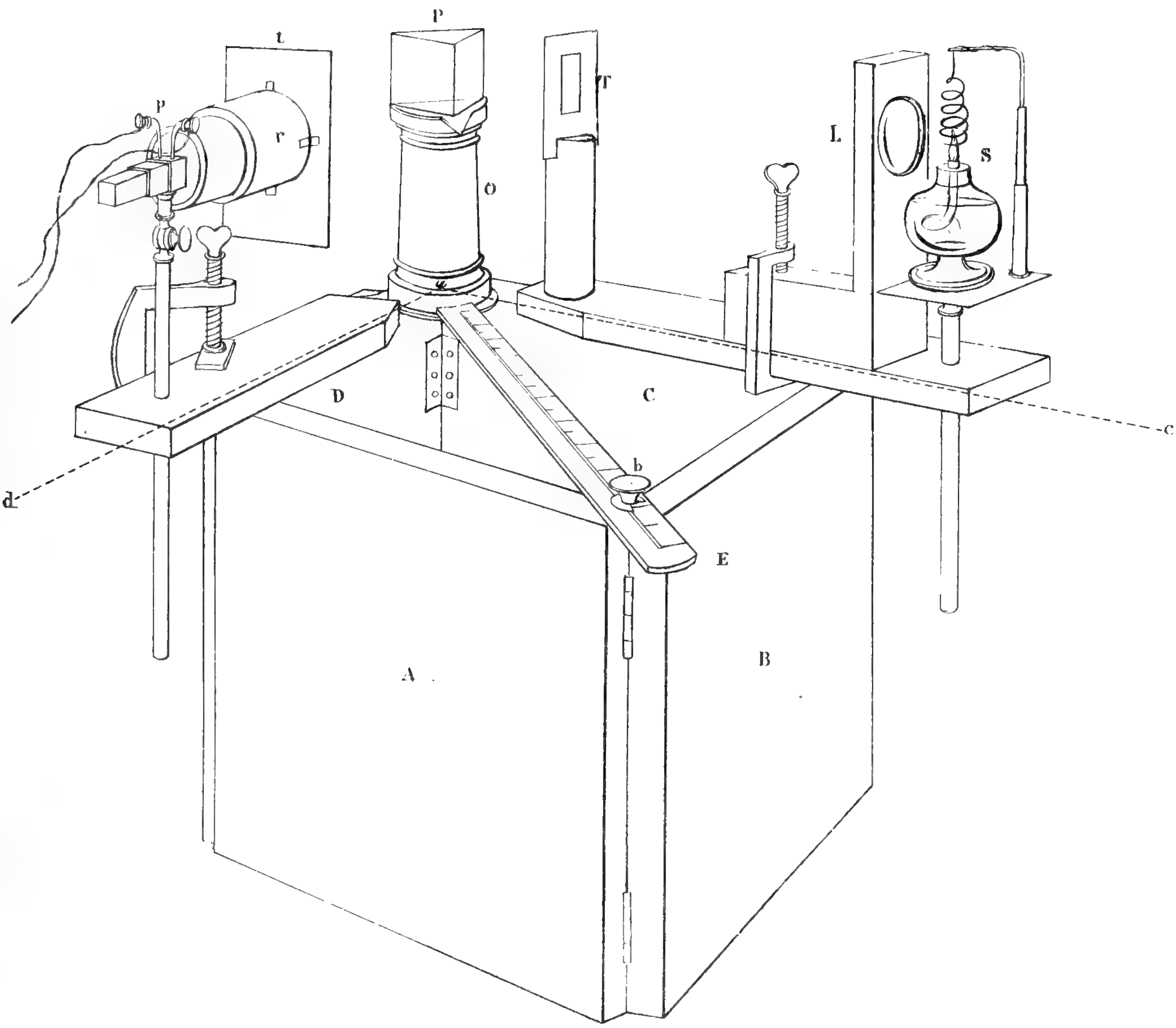


Fig. 2.

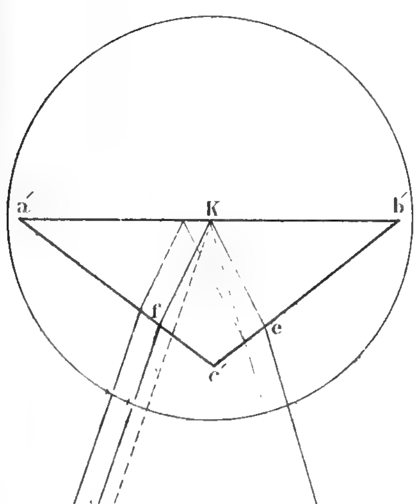
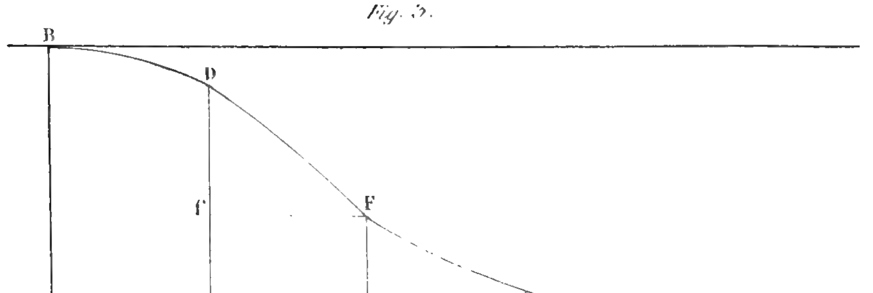


Fig. 3.



tion of the small effect due to the difference of refrangibility. Now, having proved in the Third Section that this difference of mean refrangibility is from most sources very small, which yet differ widely in their polarizability, we infer that that explanation is probably inadequate, and that we must look for a mechanical theory of heat differing in some particulars from that of light.

85. (2.) This latter conclusion is farther confirmed by the results of the Second Section, in which is deduced, from the singularly accordant results of wholly distinct series of experiments with heat from those distinct sources, that the phenomena of depolarization differ surprisingly, *numerically speaking*, from those of light, whilst in their general character they are entirely similar. The results at which we have arrived oblige us to admit, *either* that the length of a wave of heat is several times greater than that of a wave of light, *or* that the velocities of the ordinary and extraordinary ray in doubly refracting crystals are totally different from those of light; *or else* a combination of these hypotheses. Now, of the two first alternatives we are bound at present, I think, to prefer the latter, since we know nothing of the phenomena of double refraction but from this experiment; whilst the subsequent experiments on the refractive index, would, according to the prevalent theory of dispersion, seem to shew that the mean length of a wave of heat cannot differ very materially from one of light. This amounts to admitting that the doubly refractive energy is more feeble for heat than for light; in other words, that a greater thickness of a crystal is required to produce a given effect. The Second and Third Sections also confirm one another in this respect, that the uniformity of the results of depolarization with heat from different sources, and also of the refrangibility, would both be highly improbable did the length of a wave materially differ in those instances.

86. (3.) Of the results of the Third Section, I have already spoken at sufficient length (art. 81). The mean index of refraction for all kinds of heat tried is *less* than for light;—it ranges within narrow limits;—when the heat from different sources is unmodified by transmission through diathermant bodies, these limits are very narrow indeed;—the measure of dispersion is considerable but unascertained, and opens a fair field for experiment;—dispersion is probably least for sources of low temperature.

87. Such are the chief data for speculation afforded by the experimental results contained in this paper:—too imperfect perhaps in themselves to form the basis of a mechanical theory of heat, yet such I hope as may be considered to be fit contributions towards its construction at a future period.

EDINBURGH, 16*th April* 1838.

X. *On the Real Nature of Symbolical Algebra.* *By* D. F. Gregory, *B. A., Trin. Coll. Cambridge.*

(Read 7th May 1838.)

The following attempt to investigate the real nature of Symbolical Algebra, as distinguished from the various branches of analysis which come under its dominion, took its rise from certain general considerations, to which I was led in following out the principle of the separation of symbols of operation from those of quantity. I cannot take it on me to say that these views are entirely new, but at least I am not aware that any one has yet exhibited them in the same form. At the same time, they appear to me to be important, as clearing up in a considerable degree the obscurity which still rests on several parts of the elements of symbolical algebra. Mr Peacock is, I believe, the only writer in this country who has attempted to write a system of algebra founded on a consideration of general principles, for the subject is not one which has much attraction for the generality of mathematicians. Much of what follows will be found to agree with what he has laid down, as well as with what has been written by the Abbé Buee and Mr Warren; but as I think that the view I have taken of the subject is more general than that which they have done, I hope that the following pages will be interesting to those who pay attention to such speculations.

The light, then, in which I would consider symbolical algebra, is, that it is the science which treats of the combination of operations defined not by their nature, that is, by what they are or what they do, but by the laws of combination to which they are subject. And as many different kinds of operations may be included in a class defined in the manner I have mentioned, whatever can be proved of the class generally, is necessarily true of all the operations included under it. This, it may be remarked, does not arise from any analogy existing in the nature of the operations, which may be totally dissimilar, but merely from the fact that they are all subject to the same laws of combination. It is true that these laws have been in many cases suggested (as Mr Peacock has aptly termed it) by the laws of the known operations of number; but the step which is taken from arithmetical to symbolical algebra is, that, leaving out of view the nature of the operations which the symbols we use represent, we suppose the existence of classes of unknown operations subject to the same laws. We are thus able to prove certain relations between the different classes of operations, which, when expressed between the symbols, are called algebraical theorems. And if we can show that any operations in any science are subject to the same laws of combination as these classes, the theorems are true of these as included in the general case: Provided always, that the resulting combinations are all possible in the particular operation under consideration. For it may very well, and does actually happen, that, though each of two operations in a certain branch of science may be possible, the complex operation resulting from their combination is not equally possible. In such a case, the result is inapplicable to that branch of science. Hence we find, that one family of a class of operations may have a more general application than another family of the same class. To make my meaning more precise, I shall proceed to apply the principle I have been endeavouring to explain, by shewing what are the laws appropriate to the different classes of operations we are in the habit of using.

Let us take as usual F and f to represent any operations whatever, the natures of which are unknown, and let us prefix these symbols to any other symbols, on which we wish to indicate that the operation represented by F or f is to be performed.

I. We assume, then, the existence of two classes of operations F and f, connected together by the following laws.

(1.) $\mathrm{F}\,\mathrm{F}(a) = \mathrm{F}(a)$. (2.) $ff(a) = \mathrm{F}(a)$.
(3.) $\mathrm{F}f(a) = f(a)$. (4.) $f\mathrm{F}(a) = f(a)$.

Now, on looking into the operations employed in arithmetic, we find that there are two which are subject to the laws we have just laid down. These are the operations of addition and subtraction; and as to them the peculiar symbols of + and — have been affixed, it is convenient to retain these as the symbols of the general class of operations we have defined, and we shall therefore use them instead of F and f. As it is useful to have peculiar names attached to each class, I would propose to call this the class of *circulating* or *reproductive* operations, as their nature suggests.

Again, on looking into geometry, we find two operations which are subject to the same laws. The one corresponding to + is the turning of a line, or rather transferring of a point, through a circumference; the other corresponding to — is the transference of a point through a semicircumference. Consequently, whatever we are able to prove of the general symbols + and — from the laws to which they are subject, without considering the nature of the operations they indicate, is equally true of the arithmetical operations of addition and subtraction, and of the geometrical operations I have described. We see clearly from this, that there is no real analogy between the nature of the operations + and — in arithmetic and geometry, as is generally supposed to be the case, for the two operations cannot even be said to be opposed to each other in the latter science, as they are generally said to be. The relation which does exist is due not to any identity of their nature, but to the fact of their being combined by the same laws. Other operations might be found which could be classed under the general head we are considering. Mr PEACOCK and the Abbé BUEE consider the transference of property to be one of these; but as there is not much interest attached to it in a mathematical point of view, I shall proceed to the consideration of other operations.

II. Let us suppose the existence of operations subject to the following laws:

$$(1.)\ f_m(a).f_n(a) = f_{m+n}(a). \qquad (2.)\ f_m f_n(a) = f_{mn}(a).$$

Where f_m, f_n are different species of the same genus of operations, which may be conveniently named index-operations, as, if we define the form of f by making $f_1(a) = a$, and suppose m and n to be integer numbers, we have those operations which are represented in arithmetical algebra by a numerical index. For if m and n be integers, and the operation a^m be used to denote that the operation a has been repeated m times, then, as we know,

$$a^m.a^n = a^{m+n}. \qquad (a^m)^n = a^{mn}.$$

We have now to consider whether we can find any other actual operations besides that of repetition which shall be subject to the laws we have laid down. If we suppose that m and n are fractional instead of integer, we easily deduce from our definition that the notation $a^{\frac{p}{q}}$ is equivalent to the arithmetical operation of extracting the q^{th} root of the p^{th} power of a, or generally the finding of an operation, which being repeated q times, will give as a result the operation a^p. Thus we find, as might have been expected, a close analogy existing between the meanings of a^m when m is integer, and when it is fractional. Again, we might ask the meaning of the operation a^{-m}; and we find without difficulty, from the law of combination, that a^{-m} indicates the inverse operation of a^m, whatever the operation a may be. When, instead of supposing m to be a number integer or fractional, we suppose it to indicate any operation whatever, I do not know of any interpretation which can be given to the rotation, excepting in the case when it indicates the operation of differentiation, represented by the symbol d. For we know by TAYLOR'S theorem, that

$$\varepsilon^{h\frac{d}{dx}} f(x) = f(x+h)$$

Or,

$$a^{\frac{d}{dx}} f(x) = f(x + \log a).$$

In the case of negative indices, we have combined two different classes of operations in one manner, but we may likewise do it in another. What meaning, we may ask, is to be attached to such complex operations as $(+)^m$ or $(-)^m$? When m is an integer number, we see at once that the operation $(+)^m$ is the same as $+$, but $(-)^m$ becomes alternately the same as $+$ and as $-$, according as m is odd or even, whether they be the symbols of arithmetical or geometrical operations. So far there is no difficulty. But if it be fractional, what does $(+)^m$ or $(-)^m$ signify? In arithmetic, the first may be sometimes interpreted, as because $(+)^m = +$ when m is integer, $(+)^{\frac{1}{m}}$ also $= +$, and as $(-)^{2m} = +$, also $(+)^{\frac{1}{2m}} = -$: But the other symbol $(-)^m$ has, when m is a fraction with an even denominator, absolutely no meaning in arithmetic, or at least we do not know at present of any arithmetical operation which is subject to the same laws of combination as it is. On the other hand, geometry readily furnishes us with operations which may be represented by $(+)^{\frac{1}{m}}$ and $(-)^{\frac{1}{m}}$, and which are analogous to the operations represented by $+$ and $-$. The one is the turning of a line through an angle equal to $\frac{1}{m}$th of four right angles, the other is the turning of a line through an angle equal to $\frac{1}{m}$th of two right angles. Here we see that the geometrical family of operations admits of a more extended application than the arithmetical, exemplifying a general remark we had previously occasion to make. Whether when the index is any other operation, we can attach any meaning to the expression, has not yet been determined. For instance, we cannot tell what is the interpretation of such expressions as $(+)^{\frac{d}{dx}}$ or $(-)^{\frac{d}{dx}}$, or $(+)^{\log}$.

III. I now proceed to a very general class of operations, subject to the following laws:

$$(1.)\quad f(a) + f(b) = f(a+b).$$
$$(2.)\quad f_{,}f(a) = ff_{,}(a).$$

This class includes several of the most important operations which are considered in mathematics; such as the numerical operation usually represented by a, b, &c., indicating that any other operation to which these symbols are prefixed is taken a times, b times, &c.; or as the operation of differentiation indicated by the letter d, and the operation of taking the difference indicated by Δ. We therefore see what an important part this class of functions plays in analysis, since it can be at once divided into three families which are of such extensive use. This renders it advisable to comprehend these functions under a common name. Accordingly, SERVOIS, in a paper which does not seem to have received the attention it deserves, has called them, in respect of the first law of combination, *distributive* functions, and in respect of the second law, *commutative* functions. As these names express sufficiently the nature of the functions we are considering, I shall use them when I wish to speak of the general class of operations I have defined.

It is not necessary to enter at large here, into the demonstration that the symbols of differentiation and difference are subject to the same laws of combination as those of number. But it may not be amiss to say a few words on the effect of considering them in this light. Many theorems in the differential calculus, and that of finite differences, it was found might be conveniently expressed by separating the symbols of operation from those of quantity, and treating the former like ordinary algebraic symbols. Such is LAGRANGE'S elegant theorem, the first expressed in this manner, that

$$\Delta^n u_x = (\varepsilon^{h \frac{d}{dx}} - 1)^n u_x;$$

or the theorem of LEIBNITZ, with many others. For a long time these were treated as mere analogies, and few seemed willing to trust themselves to a method, the principles of which did not appear to be very sound. Sir JOHN HERSCHEL was the person in this country who made the freest use of the method, chiefly, however, in finite differences. In France, SERVOIS was, I believe, the only mathematician who attempted to explain its principles, though BRISSON and CAUCHY sometimes employed and extended its application: and it was in pursuing this investigation that he was led to separate functions into distributive and commutative, which he perceived to be the properties which were the foundation of the method of the separation of the symbols, as it is called. This view, which, so far as it goes, coincides with that which it is the object of this paper to develope, at once fixes the principles of the method on a firm and secure basis. For, as these various operations are all subject to common laws of combination, whatever is proved to be true by means only of these laws, is necessarily equally true of all the operations. To this I may add, that when two distributive and commutative operations are such that the one does not act on the other, their combinations will be subject to the same laws as when they are taken separately; but when they are not independent, and one acts on another, this will no longer be true. Hence arises the increased difficulty of solving linear differential equations with variable coefficients; but for more detailed remarks on this, as well as for examples of a more extended use of the method of the separation of symbols than has hitherto been made, I refer to the Cambridge Mathematical Journal, Nos. 1, 2, and 3.

As we found geometrical operations which were subject to the laws of circulating operations, so there is a geometrical operation which is subject to the laws of distributive and permutative operations, and therefore may be represented by the same symbols. This is transference to a distance measured in a straight line. Thus if x represent a point, line, or any geometrical figure, $a\,(x)$ will represent the transference of this point or line; and it will be seen at once that

$$a\,(x) + a\,(y) = a\,(x + y);$$

or the operation a is distributive. What, then, will the compound operation $b\big(a\,(x)\big)$ represent? If x represent a point, $a\,(x)$, which is the transference of a point to a rectilinear distance, or the tracing out of a straight line, will stand for the result of the operation; and then $b\left(a(x)\right)$ will be the transferring of a line to a given distance from its original position. In order to effect this, the line must be moved parallel to itself, the effect of which will be the tracing out of a parallelogram. The effect will be the same if we suppose a to act on $b(x)$, since in this, as in the other case, the same parallelogram will be traced out: that is to say,

$$a\left(b(x)\right) = b\left(a(x)\right)$$

or a and b are commutative operations.

The binomial theorem, the most important in symbolical algebra, is a theorem expressing a relation between distributive and commutative operations, index operations, and circulating operations. It takes cognizance of nothing in these operations except the six laws of combination we have laid down, and, as we shall presently shew, it holds only of functions subject to these laws. It is consequently true of all operations which can be shewn to be commutative and distributive, though apparently, from its proof, only true of the operations of number. The difficulties attending the general proof of this theorem are well known, and much thought has been bestowed on the best mode of avoiding them. The principles I have been endeavouring to exhibit appear to me to shew in a very clear light the correctness of EULER's very beautiful demonstration. Starting with the theorem as proved for integer indices, which he uses as a suggestive form, he assumes the existence of a series of the same form when the index is fractional or negative, which may be represented by $f_m(x)$. He then considers what will be the form of the product $f_m(x) \times f_n(x)$. This form must depend only on the laws of combination to which the different operations in the expression are subject. When x is a distributive and commutative function, and m and n integer numbers, we know that $f_m(x) \times f_m(x) = f_{m+n}(x)$. Now integer numbers are one of the families of the general class of distributive and permutative functions; and if we actually multiplied the expressions $f_m(x)$ and $f_n(x)$ together, we should, even in the case of integers, make use only of the distributive and permutative properties. But these properties hold true also of fractional and negative quantities. Therefore, in their case, the form of the product must be the same as when the indices are integer numbers. Hence $f_m(x) \times f_n(x) = f_{m+n}(x)$ whether m and n be integer or fractional, positive or negative, or generally if m and n be distributive and permutative functions.

The remainder of the proof follows very readily after this step, which is the key-stone of the whole, so that I need not dwell on it longer. I will only say, that this mode of considering the subject shews clearly, that not only must the quantities under the vinculum be distributive and commutative functions, but also the index must be of the same class,—a limitation which I do not remember to have seen any where introduced. Therefore the binomial theorem does not apply to such expressions as $(1+a)^{\log}$ or $(1+a)^{\sin}$; and, though it does apply to $(1+a)^{\frac{d}{dx}}$, since both a and $\frac{d}{dx}$ are distributive and commutative operations, it does not apply to $\left(1+f(x)\right)^{\frac{d}{dx}}$, as $f(x)$ and $\frac{d}{dx}$ are not relatively commutative.

Closely connected with the binomial theorem is the exponential theorem, and the same remarks will apply equally to both. So that, in order that the relation

$$\epsilon^x = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \&c.$$

may subsist, it is necessary, and it suffices, that x should be a distributive and commutative function. On this depends the propriety of the abbreviated notation for TAYLOR's theorem

$$f(x+h) = \epsilon^{h\frac{d}{dx}} f(x).$$

Properly speaking, however, the symbol ϵ ought not to be used, as it implies an arithmetical relation, and instead, we ought to employ the more general symbol of $\log^{-1}$. But this depends on the existence of a class of operations on which I may say a few words.

IV. If we define a class of operations by the law

$$f(x) + f(y) = f(xy),$$

we see that, when x and y are numbers, the operation is identical with the arithmetical logarithm. But when x and y are any thing else, the function will have a different meaning. But so long as they are distributive and commutative functions, the general theorems such as

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \&c.$$

being proved solely from laws we have laid down, are true of all symbols subject to those laws. It happens that we are not generally able to assign any known operation to which the series is equivalent when x is any thing but a number, and we therefore say that $\log(1+x)$ is an abbreviated expression for the series $x - \frac{x^2}{2} + \frac{x^3}{3} - \&c.$ But there may be distinct meanings for such expressions as $\log\left(1+\frac{d}{dx}\right)$ or $\log\left(\frac{d}{dx}\right)$, as there are for $\epsilon^{h\frac{d}{dx}}$, that is $\log^{-1}\left(\frac{d}{dx}\right)$. In the case of another operation, Δ, we know that $\log(1+\Delta) = \frac{d}{dx}$.

V. The last class of operations I shall consider is that involving two operations connected by the conditions

(1) $$a\,\mathrm{F}(x+y) = \mathrm{F}(x)f(y) + f(x)\,\mathrm{F}(y)$$

and (2) $$a\,f(x+y) = f(x)f(y) - c\,\mathrm{F}(x)\,\mathrm{F}(y).$$

These are laws suggested by the known relation between certain functions of elliptic sectors; and when a and c both become unity, they are the laws of the combinations of ordinary sines and cosines, which may be considered in geometry as certain functions of angles or circular sectors, but in algebra we only know of them as abbreviated expressions for certain complicated relations between the first three classes of operations we have considered. These relations are,

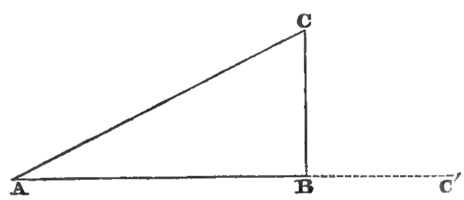
$$\operatorname{Sin} x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5}\ \&c.$$

$$\operatorname{Cos} x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4}\ \&c.$$

The most important theorem proved of this class of functions is that of DEMOIVRE, that

$$(\cos x + (-)^{\frac{1}{2}} \sin x)^n = \cos n x + (-)^{\frac{1}{2}} \sin n x.$$

It is easy to see that, in arithmetical algebra, the expression $\cos x + (-)^{\frac{1}{2}} \sin x$ can receive no interpretation, as it involves the operation $(-)^{\frac{1}{2}}$. In geometry, on the contrary, it has a very distinct meaning. For if a represent a line, and $a \cos x$ represent a line bearing a certain relation in magnitude to a, and $a \sin x$ a line bearing another relation in magnitude to a, then $a \left(\cos x + (-)^{\frac{1}{2}} \sin x\right)$ will imply, that we have to measure a line $a \cos x$, and from the extremity of it we are to measure another line $a \sin x$; but in consequence of the sign of operation $(-)^{\frac{1}{2}}$, this new line is to be measured, not in the same direction as $a \cos x$, but turned through a right angle. As, for instance, if $\mathrm{AB} = a \cos x$, and $\mathrm{BC}' = a \sin x$, we must not measure it in the prolongation of AB, but turn it round to the position BC; and thus, geometrically, we arrive at the point C. Also, from the relation between $\sin x$ and $\cos x$, we know



that the line AC will be equal to a, and thus the expression $a \left(\cos x + (-)^{\frac{1}{2}} \sin x\right)$ is an operation expressing that the line whose length is a, is turned through an angle x. Hence, the operation indicated by $\cos \frac{2\pi}{n} + (-)^{\frac{1}{2}} \sin \frac{2\pi}{n}$ is the same as that indicated by $(+)^{\frac{1}{n}}$, the difference being, that, in the former, we refer to rectangular, in the latter to polar co-ordinates. Mr PEACOCK has made use of the expression $\cos x + (-)^{\frac{1}{2}} \sin x$ to represent direction, while Mr WARREN has employed one which, though disguised under an inconvenient and arbitrary notation, is the same as $(+)^{\frac{1}{n}}$. The connection between these expressions is so intimate, that, being subject to the same laws, they may be used indifferently the one for the other. This has been the case most particularly in the theory of equations. The most general form of the root is usually expressed by $a\left(\cos\theta + (-)^{\frac{1}{2}} \sin\theta\right)$, while the more correct symbolical form would be $(+)^{\frac{p}{q}} a$, since the expression

$$x^n + P_1 x^{n-1} + P_2 x^{n-2} + \&c. + P_n = 0$$

does not involve any sine or cosine, but may be considered as much a function of $+$ as of x, so that the former symbol may be easily supposed to be involved in the root. Hence, instead of the theorem that every equation must have a root, I would say every equation must have a root of the form $(+)^{\frac{p}{q}} a$, p and q being numbers, and a a distributive and commutative function.

XI. *Investigation of a New Series for the Computation of Logarithms; with a New Investigation of a Series for the Rectification of the Circle. By* James Thomson, *LL.D., Professor of Mathematics in the University of Glasgow.*

Read 7th May 1838.

I.

The series $l(1+x) = \mathrm{M}(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \text{\&c.})$, discovered by Mercator, seems to be the origin from which, directly or indirectly, all the series may be derived which are usually employed in the computation of logarithms. A series, which affords remarkable facilities for such computations, and which lately occurred to me, may be investigated in the following manner.

In Mercator's series, change x successively into $\frac{n}{x}$ and $-\frac{n}{x}$; then, by adding $l\,x$ to each of the results, we get

$$l(x+n) = l\,x + \mathrm{M}\left(\frac{n}{x} - \frac{1}{2}\frac{n^2}{x^2} + \frac{1}{3}\frac{n^3}{x^3} - \frac{1}{4}\frac{n^4}{x^4} + \text{\&c.}\right) \ldots\ldots\ldots\ldots\ldots(1)$$

$$l(x-n) = l\,x + \mathrm{M}\left(-\frac{n}{x} - \frac{1}{2}\frac{n^2}{x^2} - \frac{1}{3}\frac{n^3}{x^3} - \frac{1}{4}\frac{n^4}{x^4} - \text{\&c.}\right) \ldots\ldots\ldots\ldots(2)$$

Take half the sum and half the difference of these; then

$$\frac{l(x+n) + l(x-n)}{2} = l\,x - \mathrm{M}\left(\frac{1}{2}\frac{n^2}{x^2} + \frac{1}{4}\frac{n^4}{x^4} + \frac{1}{6}\frac{n^6}{x^6} + \text{\&c.}\right) \ldots\ldots\ldots(3)$$

$$\frac{l(x+n) - l(x-n)}{2} = \mathrm{M}\left(\frac{n}{x} + \frac{1}{3}\frac{n^3}{x^3} + \frac{1}{5}\frac{n^5}{x^5} + \text{\&c.}\right) \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

By multiplying the latter by n, and dividing the product by $2\,x$, we get

$$\frac{n\{l(x+n) - l(x-n)\}}{4\,x} = \mathrm{M}\left(\frac{1}{2}\frac{n^2}{x^2} + \frac{1}{2.3}\frac{n^4}{x^4} + \frac{1}{2.5}\frac{n^6}{x^6} + \text{\&c.}\right) \ldots\ldots(5)$$

Adding this and (3), and by transposition, we obtain

$$l\,x = \frac{l(x+n) + l(x-n)}{2} + \frac{n\{l(x+n) - l(x-n)\}}{4\,x} + \mathrm{M}\left(\frac{1}{3.4}\frac{n^4}{x^4} + \frac{2}{5.6}\frac{n^6}{x^6} + \frac{3}{7.8}\frac{n^8}{x^8} + \text{\&c.}\right) (6)$$

If $n = 1$, this becomes

$$l\,x = \frac{l(x+1) + l(x-1)}{2} + \frac{l(x+1) - l(x-1)}{4\,x} + \mathrm{M}\left(\frac{1}{3.4}\frac{1}{x^4} + \frac{2}{5.6}\frac{1}{x^6} + \frac{3}{7.8}\frac{1}{x^8} + \text{\&c.}\right) \ldots (7)$$

The m^{th}, or general term of this series, is evidently $\mathrm{M}\dfrac{m}{(2\,m+1)(2\,m+2)} \cdot \left(\dfrac{1}{x}\right)^{2m+2}$.

The last two series, besides the simplicity and elegance of their form, are remarkably convergent, when x is large, compared with n or 1. The latter of them gives, with great facility, the logarithm of a whole number from the logarithms of the two numbers immediately preceding and following it, when the number is considerable: and this, as we shall presently see, is a case of continual occurrence in the computation of logarithmic tables.

To exemplify the use of formula (7), suppose that the common logarithm of 2 has been computed by any of the known methods:* then, by doubling and trebling it, the logarithms of 4 and 8 are obtained; while that of 5 is found by subtracting it from 1, the logarithm of 10. From the logarithms of 8 and 10, the logarithm of 9 is obtained by means of series (7), as, by taking $x=9$, that formula gives

$$l9=\frac{l10+l8}{2}+\frac{l10-l8}{36}+\mathrm{M}\left(\frac{1}{3.4}\cdot\frac{1}{9^4}+\frac{2}{5.6}\cdot\frac{1}{9^6}+\&c.\right)$$

In this the convergence is so rapid, that to find the logarithm true for seven decimals, it is not necessary to proceed beyond the first term in the vinculum; and by employing additional terms, any assigned degree of accuracy is easily obtained. By halving the logarithm of 9, we get that of 3; from which, and from the logarithm of 2, that of 6 is found. Then, by series (7),

$$l7=\frac{l8+l6}{2}+\frac{l8-l6}{28}+\mathrm{M}\left(\frac{1}{3.4}\cdot\frac{1}{7^4}+\frac{2}{5.6}\cdot\frac{1}{7^6}+\&c.\right);$$

—a series of rapid convergence.

Now, by adding the logarithm of 2 to the logarithms of 6, 7, 8, 9, and 10, we get those of the even numbers 12, 14, 16, 18, and 20; and the logarithm of 15 is the sum of the logarithms of 3 and 5. We should then find with great ease, by means of (7), the logarithms of the prime numbers 11, 13, 17, and 19. By adding the logarithm of 2 to the logarithms of 11, 12, 13, ... 20, we should have those of the even numbers from 20 up to 40; and those of the primes between the same limits would be computed by means of (7). In a similar manner, we should first obtain the logarithms of the even numbers from 40 up to 80, and then those of the intermediate primes; and thus we might proceed as far as we please, the computations for the primes becoming easier and easier, as the numbers become larger. The logarithm of any whole number, indeed, from 40 upwards, would be obtained by (7), true for seven or more places of decimals, merely by means of the logarithms of the two numbers immediately preceding and follow-

* If the modulus of the common logarithms be supposed to be known, the common logarithm of 2 may be computed with great ease by finding, by MERCATOR's series, the logarithm of $1+0.024$; by adding to the result 3, the logarithm of 1000, and thus finding the logarithm of 1024; and, lastly, by dividing by 10, because 1024 is the tenth power of 2.

ing it, without employing any of the terms in the vinculum, and consequently *without any trouble with the modulus.*

The facility of the process by means of formula (7) will appear from the following example, in which the common logarithm of 61 is computed from those of 60 and 62.

$$\begin{array}{rl} l\,62 = & 1.792391689 \\ l\,60 = & 1.778151250 \\ \hline & 2)3.570542939 \\ \hline \text{Half sum} = & 1.785271469 \\ \hline \text{Difference} = & 0.014240439 \end{array}$$

Now $61 \times 4 = 244$, and dividing the difference by this, we get 0.000058362; the sum of which and of the half sum, found above, is 1.785329831, the logarithm of 61. This is true in all its figures except the last, which ought to be 5.

It may be proper to remark, that when x is large, its logarithm will be obtained very readily by means of formula (3); as, by taking $n = 1$, and transposing, we get

$$l\,x = \frac{l\,(x+1) + l\,(x-1)}{2} + \mathrm{M}\left(\frac{1}{2}\frac{1}{x^2} + \frac{1}{4}\frac{1}{x^4} + \frac{1}{6}\frac{1}{x^6} + \&c.\right);$$

—a formula which will give the logarithms of whole numbers above 2000, true for seven or more decimals, by means of the logarithms of the two numbers immediately preceding and following, without any term of the series.

II.

A series which gives the rectification of the circle with greater ease than any other with which I am acquainted, occurred to me some time ago, and I then believed it to be new. I have lately found, however, that the same series was discovered by EULER, and that it appeared in the eleventh volume (1793) of the *Nova Acta* of the Petersburgh Academy, with two investigations by that distinguished writer. My investigation is altogether different from those given by him, and is very simple—perhaps more so than either of his. It is obtained, also, by means of a method of integration which may be employed with advantage in many other instances: and though, as might be expected, several things in my paper are anticipated in EULER's, yet mine contains others which are not to be found in his. For these reasons, I shall present the paper in almost exactly the same state in which it was before I saw the article by EULER.

If we put $\tan^{-1} x$ to denote the circular arc, whose tangent is x, we have, by the formula for the differential of the arc in terms of its tangent to the radius 1,

$$d\tan^{-1} x = \frac{dx}{1+x^2}, \text{ and therefore } \tan^{-1} x = \int \frac{dx}{1+x^2}.$$

The integral of the second member of this, in the form that will suit our purpose, will be obtained in perhaps the easiest manner by means of the formula,

$$d\left(\frac{u}{v}\right)=\frac{v\,du-u\,dv}{v^2}=\frac{du}{v}-\frac{u}{v}\cdot\frac{dv}{v};$$

which, by integration and transposition, gives

$$\int\frac{du}{v}=\frac{u}{v}+\int\frac{u}{v}\cdot\frac{dv}{v} \quad\text{.......................}\quad (8)$$

By taking in this $u=x$, and $v=1+x^2$, the expression found above becomes

$$\tan^{-1}x=\frac{x}{1+x^2}+\int\frac{x}{1+x^2}\cdot\frac{2x\,dx}{1+x^2},\ \text{or}\ \tan^{-1}x=\frac{x}{1+x^2}+\int\frac{2x^2\,dx}{(1+x^2)^2}.$$

The integral of the last term of this is obtained in a similar manner, from formula (8), by taking $du=2x^2\,dx$, and $v=(1+x^2)^2$, and is found to be

$$\frac{2}{3}\cdot\frac{x^3}{(1+x^2)^2}+\frac{2.4}{3}\int\frac{x^4\,dx}{(1+x^2)^3}.$$

It is plain that this process may be continued without limit; and, the law of continuation being manifest, we obtain

$$\tan^{-1}x=\frac{x}{1+x^2}+\frac{2}{3}\frac{x^3}{(1+x^2)^2}+\frac{2.4}{3.5}\frac{x^5}{(1+x^2)^3}+\frac{2.4.6}{3.5.7}\cdot\frac{x^7}{(1+x^2)^4}+\text{\&c.}\ldots(9)$$

This is the series proposed to be investigated; and, for giving an arc in the first quadrant, it requires the addition of no constant quantity.

When x is a fraction $\frac{p}{q}$, the foregoing series may be exhibited, after some modifications, in the convenient form,

$$\tan^{-1}\frac{p}{q}=\frac{pq}{p^2+q^2}\left\{1+\frac{2}{3}\cdot\frac{p^2}{p^2+q^2}+\frac{2.4}{3.5}\left(\frac{p^2}{p^2+q^2}\right)^2+\text{\&c.}\right\}\ldots\ldots\ldots(10)$$

By putting A, B, C, &c. to denote the successive terms of the last series, and k to denote the fraction $\frac{p^2}{p^2+q^2}$, we get the following expression, which answers best for the purposes of computation:—

$$\tan^{-1}\frac{p}{q}=\frac{pq}{p^2+q^2}+\frac{2}{3}k\text{A}+\frac{4}{5}k\text{B}+\frac{6}{7}k\text{C}+\text{\&c.}\ \ldots\ldots\ldots\ldots\ldots\ldots(11)$$

We have thus obtained the means of computing a circular arc in terms of its tangent. The well known series,

$$\tan^{-1}x=x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\text{\&c.}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(12)$$

given, first by James Gregory, and afterwards by Leibnitz, serves the same purpose, but is far inferior in practice. Like (12), the series above investigated, converges the more rapidly, the smaller the tangent is in comparison of the radius.

Yet, even in the very unfavourable case in which $x = 1$, and the arc $= 45°$, we should have, by series (9),

$$\frac{1}{4}\pi = \frac{1}{2} + \frac{2}{3}\left(\frac{1}{2}\right)^2 + \frac{2.4}{3.5}\left(\frac{1}{2}\right)^3 + \&c.;$$

less than twenty terms of which would give the circumference true for six places of decimals; while many thousand terms of the series,

$$\frac{1}{4}\pi = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \&c.$$

derived from (12), would be required to effect the same object.

In the actual computation, however, of the circumference to a great degree of accuracy, the series found above is applied with most advantage in connexion with the curious and elegant principle first employed by MACHIN, and afterwards extended by EULER,—that of finding arcs whose tangents are rational, and are small known fractions, and the sum or difference of which arcs, or of their multiples, is a known part of the circumference. Such arcs are innumerable; and, by taking them sufficiently small, any degree of convergence whatever may be obtained. Rapidity of convergence, however, is far from being the sole important consideration. The convergence may be very great, and yet the fraction k may be of such a form as to render the computation laborious and difficult. No arc, indeed, answers well, unless $p^2 + q^2$ be of the form $\frac{10^m}{2^n}$, m and n being whole positive numbers; and even of arcs having this property, many are, in other respects, inconvenient. Of a great number of tangents which I have tried, those which seem to answer best are $\frac{1}{3}$, $\frac{2}{11}$, $\frac{1}{7}$, and $\frac{3}{79}$; which give respectively for the values of k, 0.1, 0.032, 0.02, and 0.00144: and, since it is easy to shew that $3\tan^{-1}\frac{1}{3} - \tan^{-1}\frac{2}{11} = \frac{1}{4}\pi$, we get, by quadrupling,

$$\pi = 12\tan^{-1}\frac{1}{3} - 4\tan^{-1}\frac{2}{11} \quad \text{.................(13)}$$

In a similar manner, it would appear that

$$\pi = 8\tan^{-1}\frac{1}{3} + 4\tan^{-1}\frac{1}{7} \quad \text{.................(14)}$$

$$\pi = 10\tan^{-1}\frac{1}{3} - 2\tan^{-1}\frac{3}{79} \quad \text{.................(15)}$$

$$\pi = 8\tan^{-1}\frac{2}{11} + 12\tan^{-1}\frac{1}{7} \quad \text{.................(16)}$$

$$\pi = 20\tan^{-1}\frac{2}{11} - 12\tan^{-1}\frac{3}{79} \quad \text{.................(17)}$$

$$\pi = 20\tan^{-1}\frac{1}{7} + 8\tan^{-1}\frac{3}{79} \quad \text{.................(18)}$$

By means of any of these six formulæ, in connexion with series (11), the value of π may be computed with great despatch and facility. As an example, let us take formula (18), and putting successively $p=1$, $q=7$, and $p=3$, $q=79$, in formula (11), we get

$$\pi = \left\{ \begin{array}{l} \quad 2.8 + \tfrac{2}{3}\,\mathrm{A} \times 0.02 + \tfrac{4}{5}\,\mathrm{B} \times 0.02 + \tfrac{6}{7}\,\mathrm{C} \times 0.02 + \&c. \\ + 0.30336 + \tfrac{2}{3}\,\mathrm{A}' \times 0.00144 + \tfrac{4}{5}\,\mathrm{B}' \times 0.00144 + \&c. \end{array} \right\}$$

Hence, by carrying the decimals out to twelve places, the computation will stand thus:

First Series.

A =	2.800000000000
B =	37333333333
C =	597333333
D =	10240000
E =	182044
F =	3310
G =	61
$20 \tan^{-1}\frac{1}{7}$ =	2.837941092081

Second Series.

A′ =	0.303360000000
B′ =	291225600
C′ =	335492
D′ =	414
$8 \tan^{-1}\frac{3}{79}$ =	0.303651561506
$20 \tan^{-1}\frac{1}{7}$ =	2.837941092081
π =	3.141592653587

This value of π is true in all its figures except the last. The computation of the terms is effected with great ease. Thus in the first series, B is found by doubling A, subtracting from the result one-third of itself, and rejecting the last two figures; C by doubling B, taking from the result one-fifth of itself, and rejecting two figures; and so on: while, in the second series, B′ is found by multiplying A′ by 144 (which is easily done on account of the repetition of the figure 4), by taking from the result one-third of itself, and rejecting five figures: and, in both the computations, various arithmetical contractions will suggest themselves as the work proceeds.

Numberless other expressions for π might be obtained, and of any degree of convergence whatever. Those given above, however, are preferable perhaps to any others, on account of the simple form of k, and the consequent facility with which it is managed in the computation. The following may be mentioned in addition to those already given:

$$\pi = 28 \tan^{-1}\frac{3}{79} + 20 \tan^{-1}\frac{29}{278} \quad \ldots\ldots\ldots\ldots (19)$$

$$\pi = 48 \tan^{-1}\frac{3}{79} + 20 \tan^{-1}\frac{1457}{22049} \quad \ldots\ldots\ldots\ldots (20)$$

$$\pi = 68 \tan^{-1}\frac{3}{79} + 20 \tan^{-1}\frac{24478}{873121} \quad \ldots\ldots\ldots\ldots (21)$$

$$\pi = 22 \tan^{-1}\frac{1}{7} + 2 \tan^{-1}\frac{685601}{69049993} \quad \ldots\ldots\ldots\ldots (22)$$

$$\pi = 88 \tan^{-1} \frac{24478}{873121} + 68 \tan^{-1} \frac{685601}{69049993} \quad \text{..................(23)}$$

$$\pi = 88 \tan^{-1} \frac{3}{79} \qquad - 20 \tan^{-1} \frac{685601}{69049993} \quad \text{..................(24)}$$

These all converge rapidly, and more especially (20), (21), (23), and (24). Such, indeed, is the convergence in the last mentioned formula, that, in one of the series arising from it, each term is less than a seven-hundredth part, and in the other less than a ten-thousandth part, of the term preceding it! The value of k, however, for the tangent $\frac{685601}{69049993}$, is inconvenient in practice, being the unmanageable decimal 0.00009857636367356395552: and though this inconvenience might be obviated in a considerable degree by tabulating the multiples of k by 2, 3, &c., up to 9, and thus facilitating its multiplication by the preceding terms of the series in employing formula (11), there is little doubt but the computation would be found to be easier by means of some of the other formulæ already given.

XII. *Of the Third Pair of Nerves, being the first of a series of papers in explanation of the difference in the origins of the Nerves of the Encephalon, as compared with those which arise from the Spinal Marrow.* By Sir Charles Bell, *K. H., F. R. SS. L. & Ed., M. D. H. Gott., &c.*

(Read 2d April 1838.)

It is not a little remarkable, that in an age which assumes to itself the character of devotion to science, in anatomy, a science which embraces the best interests of humanity, this question should remain unanswered; *What is the meaning of the nerves of the spinal marrow being in regular order and perfectly symmetrical: whilst the ten nerves arising from the brain present no similarity one to another, and agree neither in origin, size, nor distribution?*

It is plain that we must be in the dark, not only with respect to the knowledge of the nervous system, but of the animal frame generally, whilst such a question is open and courts inquiry, and yet remains without an effort being made towards its solution. We must, I fear, attribute this neglect in part only to the difficulty of the inquiry, and much to the indifference to all that does not tend directly to profit; on which account it has the better demand on the attention of a learned and philosophical Society.

So far back as the year 1811, I ventured to announce this principle, *that in the nervous system a filament possessed the same endowment, performed the same function through its whole course, whether that filament be in a nerve, or traced from the nerve into the spinal marrow, or from the spinal marrow into the brain.*

The truth of this is apparent as soon as expressed, and inquiries directed on this principle have given it countenance and importance. It enabled me to shew that what was called a common nerve, being such as was supposed to possess all the vital properties, consisted of two nerves joined in the same sheath—one presiding over motion, and the other the seat or organ of sensation. Following out the principle, I found that the roots of the so-called "common" nerves differing in function, arose from distinct columns of the spinal marrow, and that these columns corresponded with the roots of the nerves to which they gave origin. That is to say, that the anterior of these columns presided over motion, and the other over sensation: That they were distinct, but not separated in their course, and preserved their parallelism and resemblance even till lost in the cerebrum. For at the point where the anterior columns join, and decussate in the medulla oblongata, a similar junction and decussation takes place between the posterior columns.

In tracing these columns upwards, we come on ground necessary to our present inquiry, viz.—the *Nodus Cerebri* or *Pons Varolii.* This most conspicuous part of the base of the brain is an intricate mass of fibres, whose commissures and columns interweave, and to what purpose? Can it be doubted that it is for general union?—in order that organs seated apart may be united through the connection of their nerves?

Below the *nodus,* the course of the columns is regular; above it, the course of the corresponding tracts is simple. Here, then, must be seated the mystery, since, but for this intricacy in the *nodus,* order and simplicity would be displayed throughout the whole nervous system. We are directed in this inquiry by another circumstance. When we consider the nerves of the Encephalon according to the enumeration of WILLIS (which hitherto has been the acknowledged system), and count them and describe their course, we find *six sent to the eye!* The 2d, 3d, 4th, 5th, 6th, and 7th, wholly or in part pass into the orbit, into a space not larger than a wallnut shell. It is obvious, that if we discover why these nerves crowd into the orbit, the reason of the variety in the nerves of the base of the brain must also be disclosed to us.

This consideration points to the organ which has most engaged philosophers of every age and country—the human eye.

There are many reasons for considering vision as the compound operation of the sense seated in the retina, and the sensibility to the muscular movement of the eyeball. But without entering upon this demonstration, it is sufficient to our present purpose that we observe the surprising power of muscular adjustment of the eye in the direction of its axis to the sensation in the retina, or in other words to the object contemplated—as when the attention is directed to the minutest speck,—or the property by which the eye follows objects in motion, the flight of a bird or the track of a bombshell.

Since, then, the relation between the motions of the eyeball and the sense enjoyed by the proper nerve of vision is intimate beyond all comparison, and, I had almost said, comprehension, our first inquiry may take this shape—Ought the motions of the eye, so necessarily conjoined with the proper sense of vision, to be trammelled by the complex relations of the general frame?—those motions which, from familiarity, appear the simplest possible, but which are in fact the most complex. There is not a muscle in the body, nor a system of muscles, in which combination to a very great extent is not necessary to action.

The spinal marrow is a system through which the whole body, and especially the four quarters, are combined in action. There is not a limb stretched out without a conforming motion and balancing of the whole body. Walking, running, leaping, swimming, exhibit instances of this combination of the limbs, and of the trunk in consequence. The whole active machinery of the frame is in most intimate union.

If we consider the office of the eyes— the necessity for their free and uncontrolled motions—their sole dependence on the sensation in the retina—we shall be ready to acknowledge that they must be relieved from the train of concatenated actions which occurs in every movement of the frame besides. Thus unembarrassed, the *recti* muscles of the eye are suited to that sympathy, and that extraordinary minuteness of accordance with the state of sensation in the retina, which are necessary to vision.

Resuming the consideration of the columns of motion and sensation:—they may be traced up into the cerebrum; the great *crus cerebri* is formed of these combined columns, as each diverges, and is lost in the hemisphere. The column of motion is still anterior, so that the anterior part of the crus belongs to muscular action, and the posterior part to sensation.

Now, considering that the essential difference in these columns is this—that in the anterior, the course of impulse is outward from the *sensorium commune*, and inwards, or towards the sensorium in the posterior, the origins of all the nerves must conform, or the system is overthrown.

We look with increasing interest on the roots of nerves, as conclusive on this subject.

The *first* nerve, the olfactory, being traced backwards, divides into three roots, and disperses in the inferior part of the anterior lobe of the cerebrum, without the intervention of the columns, and without interference with them.

The *second* or *optic* nerve, though in direct contact with the column of motion, takes no origin from it; but in a long and circuitous course, under the name of *Tractus opticus*, turns round the *crus cerebri*, to fall into the rear or back part of the column of sensation; and so, is combined with those nerves, through which the impulse is towards the sensorium, or inwards.

Even on proceeding so far, it is fair to infer that a nerve of sense gives off no branch—that it can communicate no endowment—that it is unequal to confer either motion or sensation, or any property but that which is its limited office.

Further, if we take the pen, and trace the nerves of sense—the olfactory, optic, auditory, and gustatory—we shall find them all avoiding the anterior column, and falling into the back part of the sensitive column; so that already in part we perceive the cause of irregularity in the base of the brain, in the necessity of the nerves of sense avoiding the anterior column, to gain the posterior or sensitive column.

We come next to the *Third Nerve*. This nerve is distinguished from all others; its origin is peculiar, and its distribution limited. By universal consent, it has got the name of *motor oculi*, being distributed to the voluntary muscles of the eye, and to none others; so that it directs the axis of the eye in vision, both controlling the muscles, and having the further property of conveying to the mind the impression of the condition of the muscles. I entertain this idea because it is a double nerve.

Its origin.—Our best authors describe this nerve as arising from the *crus cerebri*, and so it does, above all the intricacies of the nervous system. It does not enter into the mixture of originating filaments in the *pons* or *nodus*. It does not communicate with the decussation in the *medulla oblongata*. It is in direct communication with the brain. But its precise origin deserves more particular inquiry.

As I have elsewhere shewn, that the *crus cerebri* consists of two columns, one of motion, the other of sensation, and that the *corpus nigrum* divides these columns. If a section be made of the *crus*, just anterior to the origin of the third nerve, we shall find that we cut through the *corpus nigrum*. And now if we take the curette, and gently divide the two columns, and so separate them in the direction towards the root of this nerve, we shall divide or split it, shewing that part of it arises from the anterior column, and part of it from the posterior column. If we carefully dissect and lay out the third nerve, we have a very interesting view, as illustrative of its function, and of the nervous system in general. The roots, as they arise, and for some way in their course (see Plate XIV., Figs. 1 and 2), are in round distinct cords, running parallel to each other. They then join together, and form a dense body, in which the filaments are separated, rejoin, and are matted together, after which their progress is as a common nerve. Their distinct origin from the divisions of the *crus*—the two distinct fasciculi of parallel fibres—the course of these for some way without exchange of filaments, and then afterwards running into intimate union—are circumstances of much interest, as shewing the distinction of the *crus cerebri*, the distinct nature of the roots of the third nerve, and that it is a double nerve, dedicated to the finer motions of the eye, peculiar in its structure, and yet in conformity with the system which I have followed.*

A question is naturally suggested here, Is the third nerve a sensitive nerve, as well as a motor; and if so, how comes it that there is no regular ganglion on the root which it receives from the sensitive column?

This would incline me to believe, that the ganglionic root is an organization on the spinal nerves and fifth pair, suited to that sensibility which the body universally and the surface especially enjoys, which gives pain, and becomes a guard upon the frame.

* The objection which will be naturally suggested, is, that the *abducens* nerve arises behind the *pons*. We shall afterwards shew why it does so. And, let it not be forgotten, that the relations of this nerve are the cause of frequent disturbance to the condition of the eye, a consequence, certainly, of its greater complexity.

At the same time, it will not be overlooked, that the texture of the nerve at the union of the fasciculated roots very much resembles the texture of the spinal ganglion (Fig. 2, D). The difference may be reasonably attributed to the distinction in office, *i. e.* that it has no reference to the sensibility of the surface, but only to the condition of the muscle.

The very peculiar and unique position of the roots of this third nerve, whilst it places the function of volition directly in communication with the sensorium, and unembarrassed by communication with other nerves, has also this superior advantage, that it is in direct relation to the sensitive column. This connection, as I have just said, has no reference to common sensation, for the nerve is strictly limited to the muscles, but only to that property of estimating the condition of muscular activity.

We pass on to the consideration of the Fourth Nerve. To comprehend its relations, we must take a wide range, and a different course.

XIII. *Of the Origin and Compound Functions of the Facial Nerve, or Portio dura of the Seventh Nerve;—being the Second Paper in explanation of the difference between the Nerves of the Encephalon, as contrasted with the regular Series of Spinal Nerves.* By Sir Charles Bell, *K. H., F. R. SS. L. & Ed., M.D. H. Gott., &c.*

(Read 9th April 1838.)

In following out the principle formerly laid down—that the study of the organization and functions of the part to which the nerve is distributed, will explain the peculiarities of its origin and connections—I have in this paper entered on a subject of great extent and difficulty.

As the *Facial Nerve* is one of a distinct class, it will be necessary to shew in what that class is peculiar—that it essentially belongs to the act of breathing—that the act of respiration being, in its ordinary condition, independent of the will, there are nerves appropriated to that function. At the same time, it must be shewn, as the apparatus of breathing is made subservient to other purposes in the economy, by what relations it is brought under the influence of the will.

The excited act of breathing extends to the features of the face; and the face, so influenced, is especially the seat of expression. The same parts are the instruments of speech. Hence, it appears, that the nerve which animates the features must have a compound root. It will be my object to shew that the facial nerve has an origin corresponding with its complex operations.

It has been affirmed that I have given up the term Respiratory Nerves. This betrays an ignorance of the whole subject. But as it may have arisen from my imperfect description, I beg the Society to permit me to illustrate this subject, as the necessary foundation of what I have to offer on the nerve of the face. In an inquiry of this kind, the observation of natural phenomena is more agreeable, and more conclusive, than experiments on living animals. With this object, let us notice the actions of the frame at a time when sense and volition are withdrawn.

It is sometimes the severe duty of the physician to watch the act of dying, and to mark its successive stages. A man is not dying, whilst yet the respiration is unembarrassed. But whether he die from violence and loss of blood, or by gradual exhaustion and lingering disease, the act may be said to commence with an excited state of the respiratory organs.

When the vision is clouded, and the eyes want speculation or direction, and the hand is insensible to the pressure of affection, then the chest rises high at each inspiration, and the muscles of inspiration are prominent in action.

When, through increasing insensibility, the limbs lie relaxed and powerless, the muscles of the shoulders, neck, throat, and nostrils, are visibly excited, and at each inspiration (although we cannot say there is effort, that being an influence of the mind), yet each fibre of the class of respiratory muscles is like a cord in violent tension.

When the decay of life reaches the respiratory system, it first affects the lesser muscles which expand the air-tubes, the muscles of the *glottis* and *velum palati* lose their tone, and these parts becoming relaxed, vibrate in the inspiration of the breath, and cause stertor.

At length the regularity of the respiration is disturbed—there is an interval between the inspiration—the interval is prolonged and irregular, and the action returns with sudden violence, every muscle starts convulsively into action, but with no voluntary effort or struggle. The longer the interval of rest, the more sudden and startling is the return of action, and when we deem all at rest, once more the breath is drawn. At last the action ceases in the chest, whilst yet the throat and cheeks are pulled with a regular succession of actions, and the last fibre which answers to the presence of life, is the *Risorius Sanctorini* and muscles of the nostril. Two or three times the *Risorius* is drawn with spasmodic twitchings, and then all is still. It is the ultima moriens.

We can hardly miss noticing the resemblance here to natural sleep, the absence of all sense and voluntary motion, and the continuance of the respiration by a property of action which knows neither lassitude nor debility.

In a Society which does not reject the cultivation of literature with science, I may be permitted to quote the beautiful description of HALLER: " Nocte redeunte sensim torpor percipitur in musculis longis, ineptitudo ad cogitationes severiores, amor quietis in animo et corpore. Tunc peculiariter vires corpus erectum tenentes laborant, et oculi nolentes clauduntur, et maxilla inferior pendet, et oscitationis necessitas ingruit, et caput antrorsum nutat, et objectorum externorum actiones minus nos adficiunt, et denique turbantur ideæ," &c.

But, whilst the body, as far as it is subject to the mind, or subject to change through the will, or through passion, is thus at rest, a class of muscles of great extent, seated remote from each other, are combined in simultaneous action. No weariness or exhaustion reaches them. They are most perfect, most regular in action, whilst all besides are at rest.

Thus, we may contemplate the body under two conditions: *First*, Where all is animated, sensitive, and expressive: *Secondly*, Where the body has the semblance of death, and the active powers are at rest. It is natural to seek in the anatomy, and especially in the nervous system, for some correspondence in the structure: nor shall we have far to seek.

Having distinguished that symmetrical system of nerves, the functions of which are sensation and volition, we find nerves which at first appear superfluous. But these superadded nerves, although they produce, when entangled with the others, the appearance of extreme intricacy, are, when taken separately, regular also. For we trace all to a centre, and that centre giving them an origin and a source of power different from the other nerves.

When we see to what order of parts these nerves are distributed, it is impossible to refuse to them the term of Respiratory Nerves. The distinction of animal functions from vital and natural, has always been noticed, and the distinction explained on the supposition of distinct nerves ministering to each; which suggestion was resigned, because anatomists would not agree to any such distinction of nerves. Nevertheless the reasoning was just, and the objection of the anatomists unfounded.

Comparative anatomy affords us the most pleasing view of this respiratory system. From the lowest link of the chain of beings to the highest, there is a progressive series of nerves, increasing in complexity. It is natural to inquire, On what does this complexity depend? In the lower link of the chain of animals, we see the essential operation of decarbonization performed by the contact of air with the fluids circulating over all the frame. We see the same object attained by a sac or cavity, which admits the air, and which sac alternately opens and closes again in other creatures; such cavities communicate with the atmosphere through prolonged and intricate tubes. Witnessing all this, we also perceive the necessity of new nerves of connection. When we further see a new power or faculty bestowed by means of the air which plays through these tubes—voice issuing by their vibration; when we observe that the air drawn through the tubes is diverted into another channel, and made subservient to smelling; when, still ascending to the highest link of the scale, we find the faculty of speech bestowed through the same means,—it would be strange, indeed, if anatomy did not in the same ascending scale disclose an increasing number of nerves.

Again, in the mouth and in the throat are two passages. How shall the one only admit air, and the other food? How shall breathing, deglutition, and speech, coughing, vomiting, be performed, each action differing from another, in the arrangement of some fifty muscles of these tubes? How are these actions ordered, but by a minute and seemingly intricate supply of nerves?

In all animals, man included, the same symmetrical system of nerves, unvarying in any essential circumstance, is devoted to sensibility and locomotion. But the other system, that which is superadded, varies in a remarkable manner; comparatively simple in the animals which merely breathe, complex when the organs of breathing become instruments under the will, they are at once essential to life, and in their higher office minister to the qualities of mind. It would be a strange anomaly, if, with these new faculties, sympathies, and relations, there were not also an increasing complication of nerves. This intricacy, this fine dependence of the functions, render experiments delusive and unsatisfactory. For we may divide a nerve, one which appears to our conception essential, and no consequent results! We cut a nerve going to the tongue or the throat, and the animal breathes, barks, and swallows. It would be dangerous therefore to conclude that the nerve were superfluous. It is only by an enlarged view of the anatomy that we shall be brought to just conclusions.

From this system I have to select one nerve, and shew how through it, two distinct offices—vital respiratory actions, and voluntary actions—are combined in the face.

The base of the brain being carefully taken out, without tearing the roots of the nerves, and the whole being for a twelvemonth preserved in spirits, we may commence the dissection. The *medulla oblongata* and *pons varolii* being cleared of their membranes, and the places of the *sixth* and ninth nerves noted, we clear and arrange the filaments of the eighth pair, and the *portio dura* of the seventh.

We see the *facialis* or *portio dura* of the seventh nerve coming out from the depth between the convexity of the *pons* or *nodus cerebri*, the *corpus olivare*, and the root of the auditory nerve. This nerve we have now to trace inwards, and in the substance of the *pons* or *nodus*.

We shall not find this nerve arising in separate filaments, but in a flat layer of nervous matter, which fan-like spreads into the nodus.

To understand the full consequence of this form of the root, we must make a section of the *nodus* or *pons*, to shew the manner in which the motor tract expands within it. Previous to this let the sixth nerve, and *portio dura* of the seventh, be thrown forwards, and the glosso-pharyngeal and nervus vagus laid aside. If we now dissect close round the corpus olivare, the motor column will be found bending round that body; and now, by following the root of the *portio dura* inwards, its origin from the column of voluntary motion will be apparent. One portion diverging towards the sixth nerve, the other towards the glosso-pharyngeal nerve. See Fig. 3, (6, 7, and 8,) also Fig. 4, in which the relations of the nerves are made more distinct. By proceeding differently, we obtain a better view of the common origin of the eighth pair and *portio dura*. Cut across the *processus ad cerebellum*, and open up the fourth ventricle. Trace the roots of the eighth pair inwards. You find the column from which they arise in the form of a *tractus* ascending to the *corpora quadrigemina*, the *valvula cerebri* forming the commissure of the two respiratory tracts. From this tract the *portio dura*, now viewed from behind, will be seen to take an origin.

We may now have a view of the relation of the respiratory nerves to the sensitive column of the *medulla oblongata*, either by tracing up the sensitive column from the spinal marrow, or by tracing down the sensitive root of the fifth nerve.

We shall now find that the eighth pair, that is to say, the *nervus vagus* and *glosso-pharyngeus*, is situated so as to draw roots from the sensitive column.

By such a mode of dissection, it will be found that the facialis or *portio dura* of the seventh nerve has direct connection with the motor and respiratory columns, and hardly less directly is related to the fourth and sixth nerves.

The facial nerve, thus arising, allies itself with the auditory nerve, and passes into the temporal bone. In its passage through that bone, it exchanges fibres with the branches of the fifth nerve, and after some intricacies, escapes by the *stylo-mastoid foramen*, to expand upon the cheek, and finally to reach every part on the side of the head, with the exception of the muscles of the jaws. Although its connection on the side of the neck countenances the view I am about to give of this nerve, yet we must draw our inferences chiefly from the origin and functions of the nerve.

Of the Function of the Facial Nerve, or Portio Dura.

In the facial nerve we have an organ of most complex operation. It combines the passages with the great internal organ of respiration. It animates the lips and cheeks in combination with the organs, so as to give both speech and expression. It is the source of all the sympathetic actions which illuminate the features in unison with the condition of the mind. It has some remarkable effects on the eyes, which subject we shall reserve to be taken apart from the present inquiry.

That the facial nerve is the respiratory nerve, I early shewed, by dividing it in brutes; when, although sensibility remained, all action in the face was cut off, excepting the motion of eating. Many occurrences in the practice of my profession have exhibited the same results from the same cause in man.

Though one of the most celebrated philosophers of our day, Dr YOUNG, asked rather querulously, "What had the face to do with respiration?" yet must it be obvious (unless, indeed, the mind be exclusively engaged in observing the chemical phenomena of the economy), that the tubes which give passage to the air, being soft and pliant, and subject to the pressure of the atmosphere, must be dilated, and their sides held apart by muscular action. How also are they to admit of breathing, and more especially, how is the expansion of the tubes to be adapted to the excited condition of breathing? I have already alluded to the stertor consequent on the relaxation of the tubes in apoplexy. And when this nerve is deprived of power, we find the relaxed lips playing in the act of breathing like the flapping of a sail.

It will not therefore be again asked, why a branch of that system of nerves which animates the organs of respiration extends to the lips and nostrils, as other branches tend to the *velum palati*, the throat and larynx.

Considering the nerve in this, perhaps its most important function, that is operating upon the tubes or passages for the breath, during sleep and insensibility, we have next to contemplate it, as combining the effort of the will in unison with that of respiration. It is in this combined exercise that we have to be most grateful for the effect,—the vibrations of the tubes modulated into articulate language,—the performance at once of that function most necessary to existence, and that faculty of speech essential to the developement of the powers of the mind as the *instrument* of thought.

Finally, the facial nerve is the source of expression. If the properties of this nerve through any accident be lost, accidentally cut across, pressed on by a tumour, or engaged in inflammation, the corresponding side of the face remains motionless and blank. The cheeks and lips are blown out like a window-blind. They have neither tension nor action. *Expression*, whether in laughter or in tears, and all the intermediate conditions, continue to influence the other side of the face, but with frightful distortions, pulling upon the side which has lost power.

There are instances recorded, now that the cause is understood, of entire loss of expression on *both* sides of the face. A young woman, in whom the roots of the nerve on both sides were involved in disease, exhibited the most distressing consequences,—for whether she laughed or cried, the features were immoveable. She laughed under a mask, a sad thing to witness, a light heart behind a face in the repose of death.

There is an animation coincident with speech, and a reflection of the mind in the human countenance, *at all times*. We have the full sense of this only from the effects of this nerve being cut, for then the features are completely fallen, more divested of expression than a mask or a bust, for it is not the fixed state of a *statue* which has *meaning*, but something worse than death.

If this condition of total inaction continue, the plumpness of the face is lost, and the skin becomes like a piece of parchment stretched over the bones. It is a remarkable thing to see, in one sense, the life and sensibility of the parts remaining, whilst there is a ruin of all which is a reflection of the mind. The muscles of the jaws, however, remain as powerful and active as before, having their energies excited through the nerves of the other system.

We now perceive the correspondence between the roots of the facial nerve and the offices it has to perform in the face. We recognise its double roots, its relation to two distinct columns in the performance of two distinct functions. We perceive its lively subjection to the will, because of its relation to the motor column, whilst its origin in common with the eighth pair of nerves explains to us how it is that the nostrils and lips move simultaneously with the other parts engaged in the act of respiration; in other words, how the vital actions through the influence of this nerve are continued during the repose or annihilation of sensation and volition.

It is not possible to account for all the finer operations of the features by the investigations of anatomy. Yet we see that this nerve arises in most peculiar circumstances, that its roots are connected with the *nodus cerebri,* a name well chosen, since in it, without exaggeration, fibres are crossing in every possible direction. We may display these fibres, and we may suppose that each filament has its influence; but it is better to stop short of conjecture, and to rest on the demonstrated fact that this nerve is special, in one sense a double nerve, not, however, like the double nerves of the spine, where action and sensibility are conjoined, but double in as far as two modes of action are effected through it; one independent of mind, the other answering to its slightest emotions.

There is indeed nothing more remarkable than those distinct offices, and that variety in the motion of the features, arranged and controlled through a single nerve not larger than a thread, combining the features in the general act of respiration, giving utterance in speech, and indicating every degree and variety of emotion.

I had at one time been deceived into the belief that laughter, and all the changes of the face indicative of what is pleasurable or ludicrous, were but the result of the degrees of relaxation of the muscles, as it were a defect of action. Such an opinion is untenable when we perceive the consequences of the loss of this nerve, for its defect of influence, so far from giving place to a smile, reduces the features at once to the most painful and melancholy relaxation.

Laughter, and all the changes of the countenance indicative of pleasurable emotion, are neither the effect of relaxation nor of spasm incidentally produced. It is a balanced condition of the features, in which certain muscles are in activity, whilst others are thrown out of action. In the painful emotions, still influencing the features through the same nerve, another classification of muscular actions takes place; some muscles in tension, others incontrollably relaxed, both conditions, and all the intermediate states, are designed as the outward signs of passion; and from which are afforded the highest and the most unceasing gratification, a language which is the charm of life, and the bond amongst men.

But I am somewhat trespassing, and deviating from the proper object of the paper, which was to shew in what the facial nerve or *portio dura* is distinguished from the symmetrical nerves—that it is in a different sense a compound or double nerve, and that its roots correspond so far with its various functions.

In my next paper, I shall endeavour to shew the necessity of combination between the Facial Nerve and those which enter into the orbit.

EXPLANATION OF PLATE XIV.

Fig. 1, A, B, Section of the *Crus cerebri.*

A, Motor column.

B, Sensitive column.

C, The Third Nerve, arising from both columns.

D, E, *Tractus opticus,* passing round the motor column.

Fig. 2, A, Section of right *Crus cerebri.*

B, Distinct fasciculi of the Third Nerve, arising from the muscular column.

C, Similar fasciculi of the nerve arising from the sensitive column.

D, The union of the fasciculi in a dense ganglionic texture.

Fig. 3, Represents the origins of the nerves from the *Pons Varolii* and *Medulla Oblongata.*

5, The Fifth Nerve in its two portions.

6, The Sixth pair ; the nerve of the right side unravelled.

P. D. 7, The Portio dura of the Seventh Nerve.

P. M. 7, The Portio mollis.

8, The Glosso-pharyngeal, Nervus vagus, and Spinal Accessory, forming the Eighth pair.

9, The Lingualis.

The foramina, both large and numerous, mark the provision for the entrance of blood-vessels, from which we may deduce the vital importance of the nerves.*

Fig. 4, An enlarged view of the roots of the Sixth, of the Portio dura or Facialis, and of the Glosso-pharyngeal nerve.

A, The Pyramidal body.

B, Corpus olivare.

* Consult the interesting paper by Sir Astley Cooper on the obstruction of the Vertebral Artery.

Trans. Roy. Soc. of Edin. Vol. XIV p. 236.

Fig. 3.

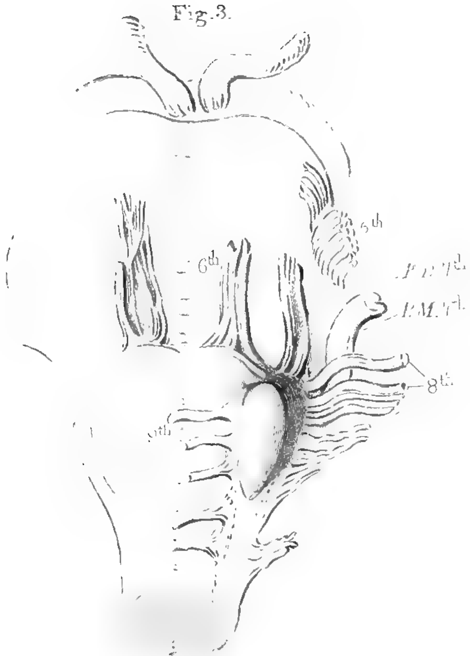


Fig 2

Fig 4.

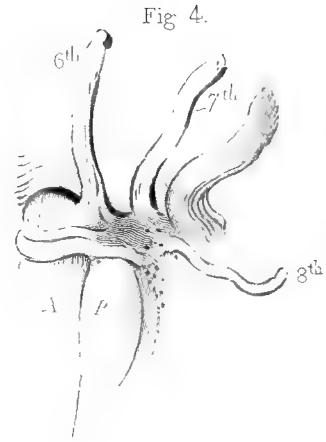
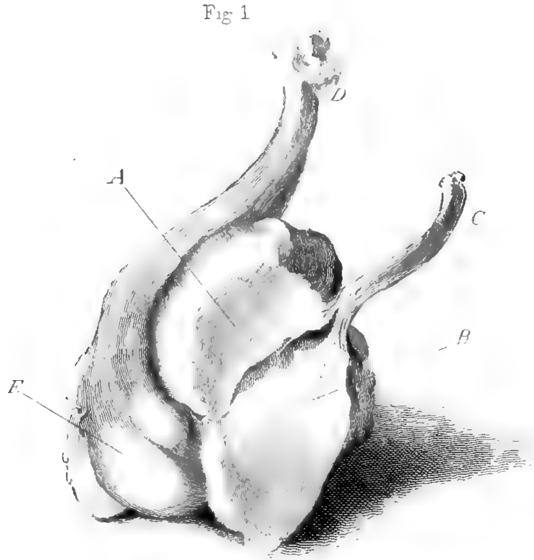


Fig 1



Sir C. Bell delt.

XIV. *Of the Fourth and Sixth Nerves of the Brain ;—being the concluding paper on the distinctions of the Nerves of the Encephalon and Spinal Marrow.* By Sir CHARLES BELL, *K. H., F, R. SS. L. & Ed., M. D. H. Gott., &c.*

Read 7th May 1838.

INTERESTING as theoptical properties of the eye have been to philosophers in every age, there are conditions of this organ which are no less curious, and which have not had their share of attention.

In the year 1823, I introduced the subject to the Royal Society of London, nearly in the terms I am now using, but there is much more in the subject than I then conceived, although I see no reason to change the mode of contemplating it.

The eight muscles of the eye, and the five nerves, exclusive of the optic nerve, which pass to them, imply the complex nature of the apparatus exterior to the globe, and I fear it is too plain that the subject has not been satisfactorily treated.

It is chiefly with respect to the protecting motions of the eye that the difficulty occurs, for I hope the dependence of the proper organ of vision on the voluntary muscles of the eye, has been proved and acknowledged.

Permit me to draw the attention of the Society to what appears a very simple piece of anatomy, the circular muscle which closes the eyelids, *orbicularis palpebrarum.*

It will be necessary to divide the muscle into three parts—and that not in an arbitrary manner, but according to the action of each division.

Around the margin of the orbit, there is a portion of the muscle red, fleshy, and strong—within it, and lying on the eyelids, there are fibres, which though they converge to the same point in the angle of the eye, are distinguishable from the outer circle by their delicacy and paleness. Lastly, the *ciliaris* is a thin slip of muscle which lies along the margin of the lower eyelid.

These divisions of the muscle authorized by ALBINUS and others, have different actions and different relations. The larger and external circle is brought into action when the eye is irritated or excited. The lesser paler fibres gently close the eyelids as in sleep, or in winking when the eyelids fall together in their usual rapid closing, to moisten the surface of the cornea,—an action which is quite peculiar, and so quick as not to interfere with the permanence of the impression on the retina.

The ciliaris draws the lower eyelid horizontally, and belongs properly to the *sacculus lacrymalis,* squeezing and emptying the sac.

Let us observe the necessity of certain relations between the larger and fleshy circle of the orbicularis when the eye is excited, as when something offensive is thrown into it. It is here necessary to notice the resemblance in action of the human eye to that of quadrupeds which possess the *haw.*

In the *caruncula lacrymalis* and *membrana semilunaris,* we have an apparatus less perfect, certainly, than that of the horse—but for the same purpose, to gather together and thrust out what is irritating the eye.

To the exercise of these parts, however, other muscles must consent, so that the cornea shall be turned in towards the inner *canthus* of the eye at the moment that the eyeball is forcibly squeezed by the action of the orbicularis. The difficulty which the oculist encounters, is to prevent the cornea turning into the lesser angle of the eyelids; for this direction of the eye towards the nose, is taken the instant that the knife or needle touches the eye. This action of the muscles which is a provision for the protection of the eye, is often the source of mishap in operation.

The position of the eyeball in which it is drawn towards the *os planum,* and the axis turned inwards, could not be the effect of the external orbicular muscle alone—nor could it be performed by the combined action of the recti. It is obvious that a relaxation must take place in the *rectus externus* or *abducens* muscle. How is this relaxation to be effected in correspondence with the action of the larger or outer portion of the *orbicularis?* There is no direct connection between these muscles: they do not touch: no nervous filaments pass between them. We must therefore turn to the origins of the nerves which supply these muscles. We find them related at their origins though proceeding by different courses to their destination.

Here let us consider the nature of that relation which exists between two classes of muscles engaged in any action. I mean not those only which are excited together to contraction, but those also which are relaxed, and which relaxation is as necessary to the effort or movement, as the contraction of their opponents.

In every action there are two conditions of the muscles, and philosophically considered, both might be called states of action, for that which is called relaxation, is not like the throwing loose of a rope which gives no resistance, but a condition of yielding, apportioned in the finest degree to the state of contraction of the other class.

This relation established between the two opposite classes of muscles is not always as in the limbs where they lie in juxta-position,—but often, more especially where the muscular action is related to the internal functions, muscles may be far apart, which yet through nervous connection are in intimate correspondence, though in opposite conditions, the one contracted, the other relaxed.

And now as to the motions of the eye, we cannot be surprised if the muscle exterior to the eyelids, and the muscle lying deep in the socket should be related in the condition of relaxation and contraction, and that the relation should be established through the connection of their respective nerves at their roots—this is the matter to be enquired into.

Let us now attend to the relation of the lesser portion of the orbicular muscle—that paler set of fibres which lies on the eyelids.

When we recollect that the margins of the eyelids in closing, touch only at their outer edge, and that when closed, a gutter is left between them and the cornea, the winking of the eye, far from clearing the surface of the cornea, would suffuse it, by leaving the tear upon it, were it not that in the act of winking the eyeball is turned up. It is by this revolving of the ball, that the cornea is moistened and wiped clear.

How is this revolving motion of the eyeball accomplished? No action of the eyelids could produce this effect. It can only be done by the action of superior *rectus* or inferior oblique. It is not performed by the superior *rectus*, which I determined, by cutting that muscle in a monkey, when the upward rolling of the cornea continued accompanying the effort of the eyelids to close.

The two oblique muscles being opponents, the relaxation of the upper one will give power to the lower one, so that we have, as before, to seek for a relation between the paler fibres of the *orbicularis* and the superior oblique. The superior oblique stands in the same relation to the internal and paler circle of the *orbicularis* that the *abducens* or *rectus externus* does to the stronger exterior circle of the same muscle.

Thus brought to acknowledge the relation between the muscle seated on the eyelid and external, to a muscle seated deep in the orbit, and seeing no direct connection between them, we must turn once more to notice the relations which establish the connection at the roots or origins of their respective nerves.

By such study of the muscular apparatus of the eye, and their actions for the protection of the organ, we are following out the principle, that the structure and action of a part will direct us to the peculiarity of its nerves. That the eye stands distinguished from the other organs of sense by the multiplicity of its guards, is owing to its extreme delicacy of structure, and its necessary exposure.

In my second paper, I had occasion to notice that the *facial nerve*, or *portio dura* of the seventh, coming circuitously through the *foramen stylo-mastoideum* to supply the face, is the nerve also of the *orbicularis palpebrarum*, or, in other words, of the exterior muscles of the eyelids. It was also noticed, that the *facialis* at its origin is in direct relation with the sixth or *abducens* nerve. Now, this sixth nerve takes a direct course forwards into the bottom of the eye-socket, crosses all the other nerves, and, though crowding with them into the orbit, forms no relation with them, but is altogether given to the abductor muscle.*

Again, in following the root of the *facialis*, and especially that portion of it which arises from the respiratory tract, we find a relation established between it and the root of the fourth nerve or *trochlearis;* and this delicate nerve passes forward like the other, enters the *foramen lacerum*, and gains the bottom of the orbit, touches no other nerve, but is wholly given to the superior oblique muscle.

If an objection should be made to the circuitous course of these nerves, as the source of the relations between the muscles, how else are the sympathies to be accounted for? and what interpretation are we to put upon the fact, that wherever there are *recti* and *obliqui* muscles of the eye,—wherever the eye possesses protecting motions, we find the same arrangement of the third, fourth, and sixth nerves, from man downwards.†

Surely it is time to read off the anatomy of the nervous system,—to seek for the relations of parts through their nerves,—to ask ourselves why there are such deviations in their course,—and why these deviations are constant, not in individuals only, but in the different classes of animals? If, with this view, we ask ourselves why the facial nerve takes a course different from the fifth, the functions of the part to which it is distributed do sufficiently inform us, that the features must be joined in sympathy or unison with the act of respiration. If we inquire why its branches reach to the eyelids, to the very part through which the branches of the fifth pass, we have only to notice the necessity for a guarding action of the eyelids in all excited conditions of the respiratory system. I need not here repeat my former observations.

In the same manner we may interpret the course of the spinal accessory nerve, or the reverted course of the recurrent of the *par vagum*.

In looking generally to the remarkable deviations in the course of the nerves, we shall find that they are confined to those which join distant parts in the act of breathing, or modify the act of breathing, as in speaking, swallowing, &c. No such irregularities are found in the system of nerves which minister to voluntary motion and sensation. They are distributed with a perfect symmetry or regularity.

And in the orbit, if we take away the respiratory of the face (the facialis), and the nerves resulting from that connection, viz. the fourth and sixth, the intricacy of the orbitary nerves is removed; there would remain one for vision, another for common sensibility, and a third for motion.

* Where there is a retractor muscle, this abducens nerve supplies it, which strengthens the supposition of a relation between the retraction of the eye and its simultaneous direction inwards.

† Taking the facts of the anatomy into account, and the actions of these muscles, the subject becomes of great interest, as connected with the expression in the eye.

In conclusion, and in explanation of the marked difference in the origin of the nerves of the spine and of the encephalon, we shall find that it is principally owing to the columns of motion and sensation.

1. All the nerves of Sense,—the olfactory, optic, auditory, not excepting the gustatory, though running into the base of the brain, must deviate from the anterior column, which is that of motion; and, accordingly, they twist round that column to enter directly into the sensorium (as the olfactory does), or to associate with the column of sensation, like the optic tractus, and roots of the auditory nerve.

2. The next source of irregularity of form and apparent intricacy in the base of the brain, is a consequence of the various sensibilities and active powers of the eye and its appendages, which require no fewer than six nerves to this one organ! Hence the ophthalmic of the fifth nerve, in addition to the proper optic nerve,—hence the motor independently of the fifth,—hence the sixth and the fourth, to associate the deep and superficial muscles of the eye and eyelids.

3. Another source of intricacy is the intrusion, as it were, of the fifth nerve among those of the encephalon. The fifth, a nerve of double root, a sensitive and motor nerve, a nerve with a distinct ganglion on its posterior root, I early announced to be a spinal nerve, that is, the anterior or (in the human body) the superior of the spinal nerves. This nerve not only giving motion to the jaws, but being the nerve of sensibility, mingles its branches with every other nerve, goes every where in the head and face, and enters into every organ, and is therefore a source of exceeding great intricacy, until the just principle be obtained.*

4. Lastly, a main source of intricacy in the nerves of the base of the brain, and of the whole nervous system, is the existence of a distinct source of motion independent of volition, and yet necessarily conjoined to it,—a source of power which shall govern the act of respiration, and yet permit a union of the apparatus of breathing with those of speech; for example, the necessity of joining the motion of the features with the pneumatic office of the lungs, being the cause of that deviation which we perceive in the facial nerve from those of simple volition.

In short the double origin and double function of the nerves of the spine is the reason of their uniformity and simplicity of distribution. The nerves of the brain differ from those of the spine, and from each other, inasmuch as each has a peculiar endowment, and necessarily a distinct origin. They therefore vary in course and distribution, and hence their apparent intricacy.

* The discovery of this fact has been caught at by men, as incapable of the induction which led to it, as of following it up in its consequences.

XV. *Inquiry whether Sea-Water has its Maximum Density a few Degrees above its Freezing Point, as Pure Water has.* *By* THOMAS CHARLES HOPE, *M. D., V. P. R. S. E., F. R. S., Professor of Chemistry in the University of Edinburgh.*

Read 2d April 1838.

Sir CHARLES BLAGDEN concludes a memoir in the Philosophical Transactions of London, vol. lxxviii., for the year 1788, "On the Effect of various substances in lowering the Point of Congelation in Water," with the account of an experiment to determine the effect of salt upon the expansion of water by cold.

From that experiment, he imagined that he had reason to conclude, as far as one experiment goes, that the combination of a salt with water has no other effect upon its quality of expanding by cold, than to depress the point at which that quality begins to be sensible, just as much as it depresses the point of congelation.

The scientific world appear to have admitted this general deduction of BLAGDEN, and, trusting to the result of a single experiment, extended it so far as to lay it down as a general law, that, as pure water has its maximum of density at $7\frac{1}{2}°$ above its freezing point, so every saline solution has a maximum of density at a temperature above its point of congelation, and that this temperature will be exactly $7\frac{1}{2}°$ distant from the freezing point of the solution.

The water of the ocean holding in solution saline matter of different kinds, was believed to obey the same law to which the solutions artificially made were supposed to be subject; and as the congealing point of sea-water was alleged to be, at an average, about 29°, it was consequently imagined that it had its maximum density at nearly $36\frac{1}{2}°$.

As sea-water collected in different latitudes, at different depths and at different distances from shore, contains different quantities of saline matter, its point of congelation must be liable to variation, and with it the temperature of its supposed maximum of density.

Every one conversant with the writings of Count RUMFORD, will remember how much use he makes of the strange anomaly in the constitution of water to account for several interesting aqueous phenomena of Nature; and several writers on oceanic occurrences have availed themselves of the belief that the same anomaly exists in sea-water, and employed it to explain several hydrographic facts of curiosity and interest, and, in particular, some remarkable currents in the ocean.

When it is considered that a general law has been deduced from, and much reasoning on natural phenomena founded upon, the solitary experiment, it is a matter of surprise that no one thought it necessary to investigate the matter more fully than Sir Charles Blagden had done, either at the period of the publication of his paper in the year 1788, or during the succeeding thirty-one years.

Dr Marcet, in his memoir on the specific gravity and temperature of sea-waters, in different parts of the ocean, &c., Philosophical Transactions for 1819, records some experiments relative to this matter. He operated in two modes—first, by examining the specific gravity at different temperatures, by means of the weighing bottle; and, second, by observing the apparent changes of volume in a thermometric-like vessel. From these experiments, he inferred, that sea-water does not expand when cooled from 40° to its proper freezing point, as fresh-water does. In spite of these experiments of Marcet, the opinion derived from Blagden continued to prevail.

The Annalen der Physick und Chemie of Poggendorf, for the year 1828, contains a dissertation by Mr G. A. Erman junior, entitled "Observations on the Expansion of Salt-water by a temperature between + 8° and — 3° R. = 50° and $25\frac{1}{4}$° Fahr. From his experiments he came to the same conclusion which Marcet had adopted.

Having taught the doctrine of Blagden for half a century, I felt unwilling to relinquish it upon the authority of the experiments of Marcet or Erman, to the latter of which my attention had been called last summer by the allusion made to them by Mr Lyell, in the last edition of his Geology, and therefore determined to investigate the fact by experiments upon artificial solutions, and the natural one presented in sea-water.

I shall preface the detail of my own experiments by a more particular account of those of Blagden and Erman than the short allusion already made to them.

I beg leave to quote the words of Sir Charles Blagden, Philosophical Transactions, vol. lxxviii. for year 1788, p. 311. "I shall conclude this paper with the account of an experiment to determine the effect of salt upon the expansion of water by cold. Pure water begins to shew this expansion about the temperature of 40°, that is 8° above its freezing point. I put a solution of common salt, in the proportions of 4.8 parts of water to one of the salt, and consequently whose freezing point was $8\frac{2}{3}$°, into an apparatus I had used for other experiments of the same kind, and found that the solution continued to contract till it was cooled to 17°, but had sensibly expanded by the time it was cooled to 15°. Suppose the expansion to have begun at $16\frac{1}{2}$°, it would be just 8° above its new freezing point. Hence we have reason to conclude, as far as one experiment goes, that the combination of a salt with water has no other effect upon its quality of expanding by cold, than to depress the point at which that quality begins to be sensible, just as much as it depresses the point of congelation."

Sir CHARLES does not describe the apparatus which he employed; he only says that he introduced the solution of salt into an apparatus he had used for other experiments of the same kind. I have no doubt the apparatus was a vessel of glass, having a comparatively slender neck, in which small variations in bulk of the contained fluid could be easily discerned, analogous to that originally employed by Dr CRONE in 1688 to demonstrate the expansion of water as it approaches the freezing point.

It is familiarly known, that when water at temperature 50°, included in such an apparatus, is exposed to cold, it descends in the stem till its temperature reaches the 41°, and continues nearly stationary while the temperature falls to $39\frac{1}{2}$°; but when the temperature of the fluid declines below that point, the water begins to rise in the stem, and continues to do so as the temperature sinks to the 32°.

From the detail of Sir CHARLES above quoted, it appears that, having introduced a solution of common salt, in the proportion of 4.8 parts of water to 1 of the salt, and whose freezing point was consequently $8\frac{2}{3}$° into the apparatus, he found the solution continued to contract till it was cooled to 17°, but had sensibly expanded by the time it was cooled to 15°. As no mention is made of the continuance of the experiment, we are led to conclude that it terminated at this point. But from it he considered himself warranted, as far as one experiment goes, to draw the conclusion, that the combination of a salt with water has no other effect upon its quality of expanding by cold, than to depress the point at which that quality begins to be sensible, just as much as it depresses the point of congelation.

Relying upon this experiment, subsequent writers considered the general law established, and some of them made application of it to sea-water, and reasoned upon the effects of it in accounting for some oceanic phenomena.

It cannot be denied that a solitary experiment, if judiciously contrived and carefully conducted, may afford sufficient proof of a particular fact; but no single experiment can warrant a general law, embracing circumstances different from those of the individual trial.

Two circumstances regarding the experiment now quoted cannot fail to excite much surprise. The first is, that Sir CHARLES stopped in the very threshold, and did not continue it to its legitimate termination. It is astonishing that so able and so patient an experimenter did not, to render the experiment conclusive, or indeed worthy of any confidence, continue the application of cold till the inclosed fluid was reduced in its temperature to its freezing point, with the view of observing whether it continued to expand till it arrived at that temperature, as is the case with pure water; and farther prolong the experiment, by applying heat to the apparatus, to see whether the liquid contracted as water does when its temperature is elevated to $39°\frac{1}{2}$.

The second circumstance creating surprise is, that the scientific world were satisfied with so imperfect an experiment, and based a general law upon so narrow a foundation.

In investigating the question I prosecuted the inquiry in two very different methods. In the first series of experiments I employed an apparatus similar to what I conceive BLAGDEN'S to have been; it was a very large thermometer glass.

Experiment, No 1. The apparatus containing a saturated solution of chloride of sodium, *i. e.* common salt, was immersed for some time in a mixture of snow and water, and then plunged into a mixture of salt and snow, whose temperature was —2. The liquor descended in the tube regularly, till it became stationary upon reaching the temperature of the mixture. In this progress there was not the smallest indication of the existence of any peculiarity in the salt-brine in regard to the usual effect of cold in causing contraction. I next transferred the apparatus from the freezing mixture into melting snow, and after some time into water at 40°. The solution immediately began to rise in the tube, and continued to do so regularly, without the smallest interruption or retrocession, and obeying the usual law of expansion by heat.

Experiment, No. 2. was performed with the same apparatus and fluid as the preceding, and in all respects in the same manner, with the exception of suspending the apparatus in the air, which at the moment had a temperature of 45° or thereby, when in the commencement of the second stage of the experiment, it was withdrawn from the frigorific mixture. The object of doing so, was to prevent the *saltus immersionis* which accompanies the removal of any similarly constructed apparatus from a cold medium into one considerably warmer, by securing the slow and gradual ingress of heat from the atmosphere.

It is indispensably necessary to attend to the *saltus immersionis,* when very small alterations of volume are to be estimated by slight risings or fallings of the fluid in the tube, else there will be a chance that we deceive ourselves. For example, when the apparatus having the saline fluid cooled to zero is plunged into water of 40°, the liquor instantly falls somewhat in the stem, and so assumes the appearance of its being contracted by heat. This, however, proceeds from the sudden expansion of the glass, and consequent enlargement of the capacity of the vessel.

The result of this experiment in both its stages agreed with the preceding, and both led to the conclusion, that a saturated solution of common salt obeys the general law of expansion by heat, and contraction by cold, in the temperatures near its congelation, as it does at all higher temperatures, and that it does not possess its greatest density at a temperature of 7° or 8° above its freezing point, as it ought to have, were the principle of Sir CHARLES BLAGDEN sound.

Experiment, No. 3. A solution of salt was made in the proportion of 1 part of salt and 4.8 of water, which were the proportions employed by Sir CHARLES BLAGDEN in his solitary experiment, and introduced into a similar large thermometer glass. The apparatus was immersed first in a mixture of snow and water, and then in a frigorific mixture at temperature 8°. The fluid descended in the tube uniformly and regularly till it reached the temperature of the frigorific mixture. It did not exhibit any interruption in its progress, or any retrocession.

I then removed the apparatus from the mixture, and suspended it in the air, the temperature of which was nearly 45°, the fluid ascended slowly but steadily. After a while I plunged it into water, and it continued to ascend without any halt or retrocession.

Experiment, No. 4. In this case I made use of a more powerful frigorific mixture, which brought the liquor more speedily to its stationary point, but it descended as in the preceding trial, without any interruption in its progress, or any retrogression. I then withdrew the apparatus from the freezing mixture, and suspended it in the air. The fluid immediately began to ascend, and continued to do so steadily. It was then immersed in water, which caused an acceleration of the ascent.

From experiments Nos. 3. and 4, it appears that a solution of salt of the strength employed by BLAGDEN, contracted by cold, and expanded by heat, conformably to the common law, and exhibited no indication whatsoever of the peculiar anomaly which exists in pure water.

Experiment, No. 5. As the particular object of my research was to discover whether sea-water observes the same relation in regard to the effects of heat and cold as pure water does, I next employed an apparatus containing sea-water taken from the Frith of Forth, a couple of miles to the westward of Leith. Its specific gravity at temperature of 60° is 1.024. It contains 3.6 per cent. of saline matter. Its congealing point is 29°.

According to the principle deduced from BLAGDEN'S experiment, did it possess the same peculiarity which fresh-water has, it ought to begin to expand at temperature $36\frac{1}{2}°$, and the expansion should go on increasing as its temperature falls to the 29th degree, its freezing point; and again, when heat is applied to it at that temperature, it onght to contract till it reach temperature $36\frac{1}{2}°$, and then begin to expand. The result of the experiment, however, was very different.

The sea-water at the commencement was somewhat above 40°, and when the apparatus was immersed in a frigorific mixture of temperature 20°, the fluid progressively descended in the stem till it became stationary, without halt or retrocession in its course. The instrument was then withdrawn from the cold mixture and supended in the air; the fluid immediately began to ascend, and continued to do so steadily, as it gradually increased in its temperature.

Experiment, No. 6. The same experiment was repeated, and afforded a similar result. These experiments lead to the conclusion that sea-water contracts as it is cooled to its freezing point, and expands from that point when it is heated.

I conceive that perfect confidence may be reposed in these conclusions, though drawn from trials made in the thermometer-like apparatus, the changes of whose capacity by heat and cold necessarily affect the apparent movement of the included fluid, whether in ascent or descent. When during refrigeration the diminishing capacity of the instrument proceeding from the contraction of the glass necessarily causes the fluid to ascend in the stem, and the fluid notwithstanding continues to descend, the descent unequivocally proves that the fluid is undergoing contraction. The case, however, is very different when the fluid during cooling ceases to descend, and then begins to rise in the stem, and continues to do so as it is further cooled. Here the indication is equivocal, as the occurrence may arise either from the diminution of the capacity of the instrument, or the expansion of the fluid, and it is no easy matter to decide from which of the causes it proceeds.

It was in consequence of this circumstance that I discarded this description of apparatus formerly when investigating, in 1803 and 1804, the singular anomalous fact of water at low temperatures expanding by cold, and contracting by heat.

Though convinced by these experiments that Sir CHARLES BLAGDEN had fallen into an error, I did not rest satisfied with them, but instituted a second series, conducted upon the same plan which I introduced in 1803. But before proceeding to describe these experiments, I shall call attention to the observations of Mr ERMAN.

He did not make any experiments upon sea-water, but upon a solution of common salt, of the specific gravity of 1.027, which he considered as equivalent to the water of the ocean, and upon two weaker solutions, and conducted the inquiry in four different methods.

He first tried the question by taking the specific gravity of the representative of sea-water at different temperatures by the weighing-bottle, and then by NICHOLSON's hydrometer; but he put little reliance upon the results, though favourable to the idea of the fluid having no maximum density at some degrees above its freezing point as water has, for the reason that the phenomena and apparent results may depend in either of these modes, as much upon a change of the dimensions and capacity of the instruments employed, as on a change of density of the fluid, at different temperatures.

He next made trial of what he calls the HOPE method, being that which I had employed in establishing the existence of a maximum density in water, but he also put no great trust in his experiments conducted in this manner, obviously, I conceive, from want of care in executing them. Mr ERMAN then adopted a method which might at first sight appear to have no connection whatever with the object of inquiry. This consists in observing the progress of cooling of the fluid when approaching its point of congelation.

He observed that water as it cools from + 6° of R. (45°.5 FAHR.) to + 2° (36°.5 FAHR.), exhibits a great retardation at that temperature, at which it attains its maximum of density. Thus, a small quantity of water contained in a cylindrical vessel one and a half inch high and one inch in diameter, having a thermometer suspended in it within one line of the bottom, exposed to a very cold atmosphere, required to cool from the 6° of REAUM. (45°.5 FAHR.) to the 2° R. (36°.5 FAHR.), at an average, for each half degree, fifty-seven seconds, but required one hundred and ninety-eight between the 4° and $3\frac{1}{2}$° R. Now, the moment of this retardation corresponds with that of the maximum density.

This retardation was still more remarkable in a second experiment, in which the small cylinder containing the water was immersed in a very powerful frigorific mixture of snow and muriate of lime, by which the whole period of refrigeration was much shortened, and during which the thermometer fell from 7° to 1° R. The average number of seconds required for each degree was 25.5 seconds, but the cooling from the 4° to the 3° required 208.2.

Mr ERMAN, perceiving that the period of retardation coincided with that of the maximum density, concluded that he could easily ascertain whether the solutions of salt obeyed the same law as pure water, by observing whether they exhibited any retardation during cooling.

He found upon trial, that a solution of salt of specific gravity 1.027, the representative of sea-water, fell in its temperature from + 6° to — 4° R. in a regular progression, exhibiting no indication of a period of retardation. He next submitted a solution having a specific gravity of 1.020, and obtained a similar result. Lastly, he tried a solution of the specific gravity of 1.010, and found that it did display a retardation while cooling between the 6° and 1°.5 R.

From these experiments, he concluded that such a quantity of sea-salt added to water as increased the specific gravity to 1.010, had no other effect on the peculiarity in the constitution of this fluid, than to lower the temperature of its maximum density to 1°.5 R. That solutions of specific gravity 1.020 and 1.027 have no maximum density at a temperature above their points of congelation.

Mr ERMAN does not mention the circumstances which led him to investigate the rates of cooling, nor does he attempt to explain the cause of the temporary retardation which occurs when the temperature is in the immediate vicinity of that of the maximum density; which might, at first sight, suggest the startling idea that water at that temperature becomes reluctant to part with its heat.

I must here take the liberty of remarking, that the experiments in my memoir in 1804, not only indicate the fact of the retardation, but also afford an explanation of it. The detail of experiment 2d in that memoir, shews, that a thermometer situated near the bottom of a cylindrical vessel eight and a half inches deep and four and a half in diameter, filled with water of temperature 49°, and immersed in cold water, fell from this degree to the 40th in thirty-eight minutes, which, on an average, is four minutes and one-third for each degree, but required twenty minutes to descend to the 39th degree, or one degree more, after which it fell three degrees, at the average rate of nine minutes for each degree.

The first experiment recorded in the memoir, likewise indicates that a period of retardation takes place in the acquisition of temperature by a thermometer, the bulb of which reaches very nearly the bottom of a vessel containing water, at temperature $39\frac{1}{2}°$, which is the period when the anomaly in the relation of water to heat terminates, and the water becomes obedient to the usual law.

In the experiment alluded to, the same jar and quantity of water were employed as in the preceding experiment.

The temperature of the water was 32°, and it was exposed to an atmosphere of temperature 60°—62°. The thermometer, situated within half an inch of the bottom, rose from 32° to 38°, that is, gained 6° in fifty minutes; it then remained for twenty minutes with scarcely any perceptible rise. It afterwards ascended for 6° uninterruptedly, at the average rate of 1° in thirteen minutes. We cannot wonder that the occurrence of a *remora* or retardation in the rate of cooling or heating of water, at a temperature coincident with that at which the strange alteration of water in relation to the effects of heat and cold upon its volume takes place, should create surprise. At first sight, it seems to indicate that water, at the moment when it deserts the general law of expansion by heat and contraction by cold, does, at the same instant, acquire a degree of reluctance either to receive or part with heat.

This, indeed, would be a very extraordinary circumstance, but there is no occasion to have recourse to such a supposition. The *remora*, which appeared so surprising to Mr Erman, admits of an easy explanation. It obviously proceeds from the change in the direction of the currents in the water, accompanying the change of law.

When a vessel of water at 50° is exposed to a cold medium, the cooling water descends to the bottom in consequence of its contraction, and a thermometer placed near the bottom of the fluid will exhibit an uninterrupted fall of temperature. As soon, however, as the water attains the $39\frac{1}{2}°$ it begins to expand, and the currents in the water as it farther cools now ascend to the top, while the warmer portions descend, and thus the *remora* in the cooling of the fluid at the bottom is produced.

Had Mr ERMAN employed a vessel of larger capacity, and such a quantity of water as must have afforded time for precise observation, and had placed the ball of the thermometer near the top instead of the bottom of the vessel, he would not have seen any indications of *remora*.

The retardation, therefore, in the cooling of water, which excited the surprise of Mr ERMAN, depends upon the change in the direction of the currents consequent upon the anomalous law and the position of the thermometer.

I remain satisfied, that there is no method of ascertaining irregular changes in the density of aqueous fluids so fit as the one I employed in 1804, as it is entirely independent of every change in the density or capacity of the instruments themselves. It consists in ascertaining, by means of thermometers placed near the top and bottom of a column of fluid, the direction which the currents of heated or cooled fluid assume; every one admitting it to be an incontrovertible truth, that the contracted denser fluid will fall to the bottom, and the expanded rarer fluid will rise to the surface.

By pursuing this method, I conceived, in 1804, that I put beyond all doubt or cavil the reality of the existence of the singular anomaly in the constitution of water; I therefore determined to have recourse to the same method in the present inquiry: I did not, however, deem it necessary to perform more than the two most striking of the series of experiments. I used an apparatus similar to what I had formerly employed. It consists of a glass jar twenty-one inches high and four in diameter, to which there is adjusted in the middle of its height a perforated basin of tinned iron, two inches in depth and ten in diameter, from the upper edge of which there issues a small spout. In this jar two thermometers are suspended, one with its ball near the bottom and the other near the top In my memoir of 1804, it is recorded that, when this vessel was filled with water at temperature 32°, and the encircling pan was kept full of water at a temperature between 70° and 80°, the thermometer at the bottom gained nearly 7° of temperature in three-quarters of an hour, while the thermometer at the top was very little affected, plainly indicating that ice-cold water, by being heated, contracts, and becoming more dense falls to the bottom of the apparatus.

Experiment, No. 7.—I filled the jar with the sea-water of the Frith of Forth cooled to 29°, which is its congealing point, and furnishing the pan with a constant succession of water at 84°, the upper thermometer immediately began to rise, and continued to do so uninterruptedly, and gained 19° before the thermometer at the bottom had gained more than 1°. This experiment proves, in a satisfactory manner, that this sea-water cooled to its point of congelation expands upon the application of heat.

Experiment, No. 8.—The preceding experiment was repeated and afforded the same result. Again, as recorded in my memoir so often referred to, when the jar was filled with water of $39\frac{1}{2}°$, and the basin was charged with a mixture of salt and snow, the thermometer at the top was cooled in the course of two hours $6\frac{1}{2}°$, while that at the bottom did not fall half a degree; thus indicating, that water at $39\frac{1}{2}°$ expands as it cools below that point.

Experiment, No. 9.—I filled the jar with the sea-water of temperature 40°, and the basin with a frigorific mixture. The lower thermometer immediately began to show the approach of a cold descending current, and in the course of one hour and ten minutes fell to 30.° The temperature of the upper thermometer was in no measure reduced, the temperature of ambient air being 44°.

From this experiment, it appears that sea-water during cooling contracts, and becomes more dense as it approaches its congealing point.

The concurring testimony of this second series of experiments, and of the preceding, shews that the same anomaly which exists in pure water does not exist in sea-water. Hence the conclusions and general law deduced from the solitary experiment of Sir Charles Blagden must be corrected, and all the reasonings founded upon them respecting peculiar currents in the ocean and other oceanic phenomena, must fall to the ground.

As the subject of inquiry in the preceding memoir relates to the law observed by sea-water, in the changes of volume occasioned by heat and cold for some degrees above its freezing point, I have not thought it necessary to call the attention of the Society to the same circumstances at temperatures below the freezing point. Every body knows, that though the temperature of water while congealing is invariably 32°, yet it permits its temperature to fall many degrees without freezing, provided it be kept free from every kind of disturbance and be cooled very gradually; that this is also the case with saline solutions, sulphuric, and nitric acid, &c. and that the temperature of each fluid, when it begins to congeal, instantly rises to the proper freezing point of that fluid. Marcet in his memoir above quoted concludes, upon the authority of four concurrent experiments, "that if a vessel filled with sea-water of the specific gravity of about 1.027, and of any temperature above the freezing point, be gradually and slowly cooled, the water contracts in bulk; and that this contraction continues to proceed, though in a diminishing ratio, till the temperature has reached 22°. At this point the water appears to expand a little, and continues to do so till its temperature is reduced to between 19° and 18°, at which point the fluid" (congeals and) "suddenly expands to a very considerable degree, shooting up with great rapidity, and forcing itself out at the open end of the tube."

M. Despretz of Paris, in his second memoir on the Maximum of Density of Liquids, as we learn from an extract contained in the Comptes Rendus for March

1836, agreeswith Dr MARCET, that sea-water of 1.027 specific gravity has a maximum of density some degrees below its freezing point, but places it at a temperature of 25°4.

M. DESPRETZ tried the effect of the addition of a great variety of saline bodies in different proportions upon the maximum density of water, and concludes that all of them exert an influence upon this peculiarity of water, but that none of them destroy it altogether, though many of them reduce the temperature of the maximum considerably below the real freezing point of such fluids. As both MARCET and DESPRETZ arrived at their conclusions by means of experiments conducted in thermometric vessels, it may be a question how far confidence can be reposed in them; it being a matter of the greatest difficulty, as already stated, to determine whether the whole, or how much, of the appearance ought to be imputed to the changes in the capacity of the vessel from the contraction of the glass. At all events, if sea-water has a maximum at some degrees below its freezing point, such a circumstance cannot be supposed to have an influence of moment, if any whatever, in the general economy of nature, or in giving birth to any oceanic phenomena, as the action of the winds and the agitation of the sea must for ever preclude it from coming into play, by preventing this fluid from falling in its temperature below its freezing point.

XVI. *On the Mid-Lothian and East-Lothian Coal-Fields.* *By* David Milne, *Esq.*

Read 19th February, 5th March, and 7th May, 1838.

I am not aware of any account having been published of the Coal-fields in East and Mid Lothian, or of any attempt to institute a geological survey of the country in which they are situated. Sinclair, the author of a well known work intituled "*Satan's Invisible World*," published also in 1672 a treatise on Hydrostatics, in which he takes notice of the Prestongrange coals, and of the whinstone-dike that intersects them. Williams, in his "*Mineral Kingdom*" (published) in 1810, gives some information regarding the direction of the Gilmerton and Loanhead coal-seams. But the information contained in both these works, even respecting the coal-strata,—which alone they professed to treat of, is extremely vague, and generally very erroneous. Dr Hibbert was the first geologist who with a scientific eye entered on the district, in order to describe with fulness and accuracy any of its rocks. His discovery of the Saurian remains in the limestone-quarries of Burdiehouse, led him to a minute inspection of the strata in which they were imbedded, and to a consideration of the relative position of these particular strata in the Mid-Lothian coal-field. The paper which he read to the Royal Society on this subject has been published in their Transactions, and it contains a good deal of valuable information in regard to the character and position of the rocks which are in the immediate neighbourhood of Burdiehouse.

It would have been most desirable, that this eminent geologist and intelligent observer, instead of confining his views to a small portion of the field, had extended his survey over the whole of it. It might have suggested new views to him, even with reference to the favourite subject to which he devoted his attention. He would also, I am sure, have drawn up an account of much practical value, and general interest. *Coal proprietors* and *lessees* would have been benefited, by having had explained to them the connection of the edge seams and flat seams, and of having had unravelled the mystery and confusion arising from the innumerable slips, troubles, hitches, and dykes, which intersect and reticulate their coal-fields. I need hardly add, that such an extended survey, by that distinguished geologist, would have been no less important to science; for if I know any thing at all of the district I am now about to describe, sure I am that he would have found and shewn, that remarkable and celebrated as this part of the island is on account of its *unstratified* rocks, and their effects on the adjoining strata, it is no less remarkable on account of its *stratified* rocks—the prodigious extent of country over which they individually spread, and the extraordinary changes they have undergone since their original deposition.

Considering, then, the interest and importance of the subject, I cannot but regret it should have fallen to be handled by me. Though Dr Hibbert is not now amongst us, there remain many other members, whose professional pursuits, aided by scientific attainments, admirably fit them for investigating and explaining this extensive subject. I allude not merely to the *geologists* of the Society, but to those gentlemen who devote themselves to the business of mineral surveyors and mining engineers, and who possess opportunities of acquiring information which no other persons can reach.

I shall be extremely happy if the memoir I am now to read, should have the effect of inducing any of these gentlemen to lay before the Society, in a proper shape, the valuable materials which must have accumulated in their hands. Although the Society should derive no other benefit from it, than that of urging some of its more efficient members to this undertaking, I would venture to hope, that I had contributed in no small degree to promote the cause of geology.

I shall divide what I have to say, under two heads; the first containing an account of the *Rocks* of the district, stratified and unstratified,—the second containing an account of the *Superficial Deposits* which cover these rocks. These heads I shall subdivide into two parts; confining the first to a mere relation of facts, and attempting in the second to explain or account for these facts. I need hardly observe, with regard to both branches, and indeed the whole subject, that the memoir I have drawn up, so far from exhausting the field of inquiry, gives only an *outline* of some of its more important features; so that it yet contains enough, to excite and gratify the ambition for discovery of the keenest geologist.

The district which I propose to describe, is from fifteen to seventeen miles square, extending from the Garleton Hills to Arthur's Seat in an east and west direction, and from the Frith of Forth to the Lammermuir Hills in a north and south direction. The extent and boundaries of this district will best be understood by a reference to the accompanying Map, Plate XV. Through this district, there runs in a nearly north-east and south-west direction a ridge of high ground, which divides the district into two parts. The ridge I allude to, commences at Tranent (where it rises from the sea), and stretches towards the south-west by Elphinstone, Carberry, and the Roman Camp, as far as Arniston. This ridge, in respect of elevation above the sea, varies from 850 to 1000 feet. On the north-west side of this ridge, lies what has been called the valley of the Esks. It is a valley of about six or seven miles in width, the trough or lowest part of which no where rises above the sea to a greater height than 150 feet. On the other side of the ridge, the district (it can hardly be called a *valley*) is much flatter. Its flanks rise perhaps as high above the sea as the valley of the Esk; but its trough is much higher, being 300 or 400 feet above the sea. The only river which waters it, is the Tyne.

In my account of the rocks which occur in the district, I shall describe first those that are *stratified*, and next those that are *unstratified*.

I. STRATIFIED ROCKS.

I. The stratified rocks consist of the following strata (and I mention them in the order of their relative quantities): 1. Sandstone; 2. Shale; 3. Limestone; 4. Coal; 5. Argillaceous strata. I am enabled also to give some notion of the *absolute quantities* in depth of these several strata, for the greatest number have been bored through, and the thickness of each particular stratum nearly ascertained. All the *coal-seams* in the district above 6 inches in thickness have been precisely ascertained; and similar information has been procured with regard to more than one-half of the other strata which lie between the coals. The whole strata, of all kinds, form together a series of layers about 1000 or 1050 fathoms in thickness; and of this deposit there are between 500 and 600 fathoms in which the relative quantities of all the several different kinds of rocks may be pretty accurately stated. In the following calculation, therefore, I at present throw out of view the coals, in that half, the component parts of which are unknown. In the other half, which is entirely known, the following is the aggregate thickness of the several strata:

1. Sandstone, ...	286 fathoms.
2. Shale, ...	188 ...
3. Lime, ...	27 ...
4. Coal, ...	21 ...
5. Clay, ...	12 ...
	534

The aggregate thickness of coal-strata in the whole deposit, above 6 inches in thickness, is about 34 fathoms. This I have ascertained from measurements of each particular seam. If it be assumed, that the other strata maintain the same relative proportions to each other throughout the half which is unknown, as they do in the other half, then the aggregate thicknesses of these several deposits, in the entire basin, would be as follows:

1. Sandstone about	550 fathoms.
2. Shale, ...	360 ...
3. Lime, ...	51 ...
4. Coal, ...	30 ...
5. Clay, ...	22 ...
	1013

Having thus stated the aggregate *thickness* of each class of stratified rock, I may mention the *number* of *strata* composing respectively these different classes, as well as the average thickness of *each particular stratum.*

There are between fifty and sixty *coal-seams* exceeding a foot in thickness. None of them are more than 13 feet thick. The average thickness is about 3½ feet.

I cannot, for the reasons already noticed, state the actual number of the other strata; but in that part of the coal basin best known, which as I have said includes about one-half of it, the following are the respective numbers:

Sandstone strata above	4	feet thick	47
Shale	3	...	52
Lime	2½	...	9
Clay	3	...	8

The maximum thickness which the *sandstone* strata reach appears to be about 200 feet; that of the *shale* 130 feet; that of the *limestone* 40 feet; and that of the *clay* 28 feet.

II. I come next to notice the *superficial extent* of the strata. I here allude not merely to the occurrence of the same *kind* of rock over a given tract of country, but to the extension of *individual* strata. One of the strata of coal worked in the district, is known by the name of the "*Great Seam.*" It may be seen in a quarry at the east end of Portobello. This seam was formerly worked at the following places, viz. *Brunstain, Niddry, Edmonstone, Woolmet, Drum,* and *Gilmerton.* It is (on the same line of bearing) still worked at *Loanhead* and at *Dryden,* places that are eight or ten miles from Joppa, in a SW. direction. But the same seam is also worked along the ridge mentioned at the outset of this memoir, viz. the ridge which runs from the Roman Camp to Tranent, and it is worked on both sides of that ridge. On the NW., or Esk side of the ridge, it is now worked at the following places; *Arniston, Bryants, Prestongrange,* and at the pit recently opened by the Duke of BUCCLEUCH at *Cowden.* Formerly it was worked at several intermediate points, as *Carberry* and *Walliford.* On the SE. or Tyne side of the ridge, this "great seam" is or has been worked at *Vogrie, Edgefield, Oxenford, Elphinstone, Birsley, Tranent,* and *Prestonpans.* Along this ridge, therefore, it runs on both sides for twelve or thirteen miles. But the same seam is known even beyond this line, and on the extreme eastern limits of the coal-field, viz. at the *Bents* (E. of Cockenzie), at *St Germains, Blindwalls,* and *Cinderhall.*

I may add, for the sake of giving a more distinct idea of this matter, though it is anticipating a little what I will afterwards more fully explain, that this coal stratum, on the west side of the basin (as at Gilmerton, Niddry, &c.) dips down rapidly to the SE.,—becomes flat in the middle of the Esk valley,—and rises up to the Roman Camp and Carberry Ridge;—it then mantles over that ridge, sinks again into the valley of the Tyne, and thereafter rises and crops out entirely towards the east. We thus see, that this one stratum or deposit of coal stretches through or covers a space whose horizontal surface measures in one direction (viz. from NW. to SE.) ten or twelve miles. As to the distance to which this particular coal-seam extends towards the north, no information can of course be obtained, for it is there covered by the sea;—towards the south, it is supposed to reach even to the base of the Lammermuir range near *Carlips* and *La Mancha*—a distance from Musselburgh of about eighteen miles.*

It may probably be asked, how is it known that it is one and the same seam which is wrought at all these different places? There is no difficulty in proving this to any one who has attended to the subject. In the *first* place, the thickness of the seam at most of these points is the same. In the *second* place, the roof and the pavement of the seam are almost every where the same, and not merely the roof and pavement, but also the strata adjoining the roof and pavement. In the *third* place, by laying down on a map, the outcrop of the seam at any known point (as, for instance, Joppa), and drawing a line in the direction in which it is there seen to run, that line will run through or very nearly through all the places above mentioned, situated on the west side of the basin; and where this line is not exactly coincident with the actual outcrop, the deviation can be accounted for by special causes. In like manner, by observing the direction in which the seam *dips*, we see that it sinks towards that very part of the basin, from which a similar coal-seam rises on the opposite side.

I may mention a second example, to prove the important fact now adverted to. There is another coal in the series, which is equally remarkable, as the "*Great Seam*"—not for thickness, but for certain no less valuable properties. It is called the "*North Greens*" Coal. It is from this stratum that the largest supplies of *Parrot* coal (so much used for the manufacture of gas) are obtained. It occupies a very low place in the basin, being, generally speaking, 250 or 300 fathoms beneath the great seam. It also can be recognised at a multitude of different and distant places, though not so many as the *Great Seam;* for, being in the lowest part of the basin, it is not every where so easily reached. But the fact of its being in this position, affords one test for its identification, which does not exist in the other case. It lies immediately above a thick bed of limestone; and as there are very few limestone strata in the whole deposit, it is on this account the more readily identified. I might add as another test, the quality of the coal, which is shared by few others, and by none to the same

* The Great Seam has not yet been *identified* at the places last mentioned; but, judging from the direction of the strata, and the fact that several other coal-strata, that are not far distant from the Great Seam in other parts, have been recognised at these places, I have no doubt it also exists there.

extent. This *North Greens* coal is worked or known at eight different points on the west side of the basin, in a line of ten miles,—viz. between Joppa and Dryden; and I have traced it to Carlips and La Mancha. It has been recognised also round the whole of the south and east margin of the coal-field; and it makes its appearance on both sides of the Roman Camp ridge.

These examples I have taken from coal-seams, because they have been more diligently traced, and are more accurately known than any other kind of strata. But it is not merely coal-seams, which individually extend over large tracts of country. I mentioned that one of the many proofs of the identity of particular coal-seams, is afforded by their roofs and pavements being the same. Over the "Great Seam" there is every where a mass of red and yellow coarse sandstone, between forty and fifty feet thick, which forms a continuous bed over at least the greater part of the district. It may be seen in *Joppa* quarry—on the shore at *Prestonpans*—in *Tranent* quarry—and at various places on the north side of the *Roman Camp* ridge. Beneath the Great Seam, there is another mass of sandstone, but of an entirely different appearance. It is generally white, fine-grained, and divided with thin seams of shale, or, (to use the terms of the quarrymen), "parted with black faikes." This sandstone is quarried in several places for the slabs or pavement stones which can be raised from it. At the Sink quarry on the Marquis of Lothian's property, there is a quarry where this white sandstone, with the Great Seam, and the incumbent red sandstone, can all be seen very perfectly.

I might in like manner particularize various limestone strata, which may be traced throughout the entire district. The limestone formerly very extensively worked at *Moredun* and *Gilmerton* (lying below the North Greens coal) can be traced to *Loanhead*, and indeed all the way to *Whitfield*, near Carlips. It makes its appearance also on the Roman Camp ridge—dipping on the north side towards the north—on the west side towards the west,—and on the south side towards the south; and it rises on the opposite side of the Tyne valley, at *Middleton*, *Crichton Dean*, and other places.

But what is true of these individual strata of coal, sandstone, and limestone, is true generally speaking of all the rocks in the district. There are very few of them which may not be traced for a great many miles in all directions.

III. The next point of inquiry, regards the respective positions of these different strata.

In the *first* place, in regard to position in the basin, they seem to be all indiscriminately interspersed, except the *limestone*. The whole of the calcareous strata are situated in the *lower half of the basin;* and the thickest beds are in the lowest part of this half. There is no general rule of this kind characterizing the strata of sandstone, shale, coal, and clay.

Then, as to the position of these several strata in regard to one another, several important remarks occur.

(1.) There is almost invariably a stratum of fire-clay below a seam of coal. In no instance have I seen or heard of coal resting immediately on sandstone or limestone.

(2.) On the other hand, there are frequent instances of a seam of coal being immediately overlaid by sandstone or by shale.

(3.) Sometimes there is a seam of clay or shale in the middle of a coal-seam. Occasionally, but very rarely, is there sandstone or limestone in such a position.

IV. I have alluded to the average, as well as to the maximum thicknesses of these strata. But it must not be supposed, that every individual stratum maintains, in all parts of the district, a uniform thickness. On the contrary, their thickness varies,—some of them to a very large extent.

It might lead to very important inferences, could a correct table be constructed so as to shew at one view the thickness of the same individual strata at a great many different points. Such a table would shew not only the maxima and minima of variation, and the rate of variation with reference to space, but, what is much more important, it would shew whether the strata *thickened*, or whether they *thinned* away in a particular direction.

I have attempted to construct such a table, applicable to the coal and limestone strata of the East-Lothian and Mid-Lothian coal-field. The great practical utility of such a table to miners, coal-proprietors and coal-lessees, this is not the place to point out. I refer to it now, for the purpose of shewing the important geological truths which may be derived from such classifications of facts.

(1.) On examining the table, for the purpose of ascertaining where the coal strata attain a maximum thickness, it will be found, that there is a certain part of the district in which a maximum thickness is attained. This part of the district embraces Niddry, Drum, and Gilmerton, on its west side; Prestongrange, Wallyford, and Elphinstone, on its east side; Cowden and Bryants on its south side. The coals which run through the district now described, are thicker there than they are in any other parts of the district.

(2.) It will farther be seen from the table, that the part of the district where the coal-seams become thinner, is chiefly to the E. and S. of the places above mentioned; and that along the E., SE., S., and SW. margin of the coal-field, all the coal strata gradually thin away, so as not only to become unworkable, but to disappear entirely.

Towards the north of this part of the district, there are not the same means of estimating the diminution of thickness, because the space is so small between it and the sea. That there is a diminution in the thickness of the seams in that direction, there can be no doubt.

(3.) The thickness of the limestone strata has been marked on the table at a number of different points. It will be observed, that in the three uppermost limestones (those, viz. which occur about the middle of the series) a very remarkable regularity of thickness prevails. The case is quite different with the limestone beds in the lower part of the series. The limestone immediately below the North Greens coal, has a very different measure of thickness in some parts of the district from what it has in others. It will be seen that, in the north and along the west part of the coal-field, it is only from 4 to 6 feet thick,—too thin to be workable. On the south and east parts of the coal-field, it attains a thickness of 20 and 30 feet. It is important to notice, that, in this respect, the limestone at the bottom of the series presents features directly opposite to what is presented by the coal-strata.*

I have attempted to form a similar table of other strata, comprehending the Sandstones, Shales, and Fire-clays. These are, for the reasons already hinted at, less accurately known than coals and limestones; so that any inferences from the table *now* referred to are less to be depended on. But I may notice the results which it indicates.

Some of the shales are so hard, arising chiefly from an admixture of calcareous matter with which they are impregnated, as to have obtained the popular name of *Bastard Limestone.* These are, generally speaking, thickest to the south, and are situated in the lowest part of the series.

(4.) The shales (properly so called) as well as the sandstones are thickest towards the north. In the *New Mills level* (south of Dalkeith) there are six beds of *sandstone*, each of which has been recognised on the shore between *Musselburgh* and *Portobello.* At the former place, these six beds present an aggregate thickness of 284 feet; at the latter 475 feet. So far in regard to their tendency to thin or thicken towards the south and north parts. In regard to the east and west parts, it would rather seem that the sandstone rocks become thinner towards *Wallyford* and *Tranent.*

(5.) In the same section of the New Mills level, already referred to, there happen to be six beds of *shale*, which have been also recognised at or near Magdalen Pans, on the sea-shore. At the former place, they exhibit an aggregate thickness of 181 feet—at the latter of 285 feet—being nearly the same rate of increase as the sandstones.† But, on the other hand, this result is in some degree compensated by the fact, that the thick beds of bastard limestones which lie almost entirely on the south side of the district, and which are not reckoned

* The table referred to in the above remarks having been considered too bulky to be published in these Transactions, extracts from it have been put into the Appendix A.

† The table from which these results were obtained, has not been published, with this paper, for the reasons applicable to the other table. The data on which it was constructed, were derived, chiefly from sections given by FAREY in his valuable report on the Duke of BUCCLEUCH's coal-field.

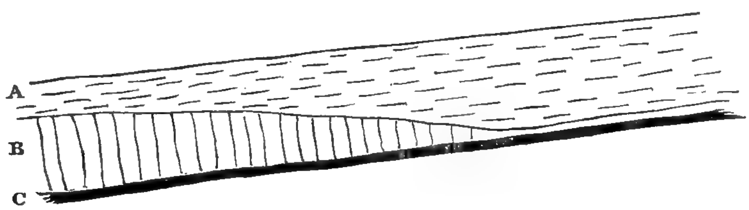
in this computation, consist chiefly of argillaceous matter;—so that in a north and south direction, the argillaceous rocks are more equally distributed over the district than the arenaceous rocks.

In an east and west direction, the shales and clays, like the sandstones, are much thicker on the *west* side of the basin, than on the *east.*

(6.) There is one other inference which an examination of the above tables warrants us in drawing, and it is one of some importance, in reference to the history and formation of these several classes of rocks. That class in which the greatest and most sudden variations of thickness occur, is the *arenaceous* class; the class in which these variations are the least, are the *carbonaceous;* whilst the *argillaceous* and the *calcareous* hold in this respect, a middle place.

In illustration and proof of this remark, that it is in the arenaceous or sandstone strata, that the greatest variations of thickness in the same bed occur, a few cases may be here given.

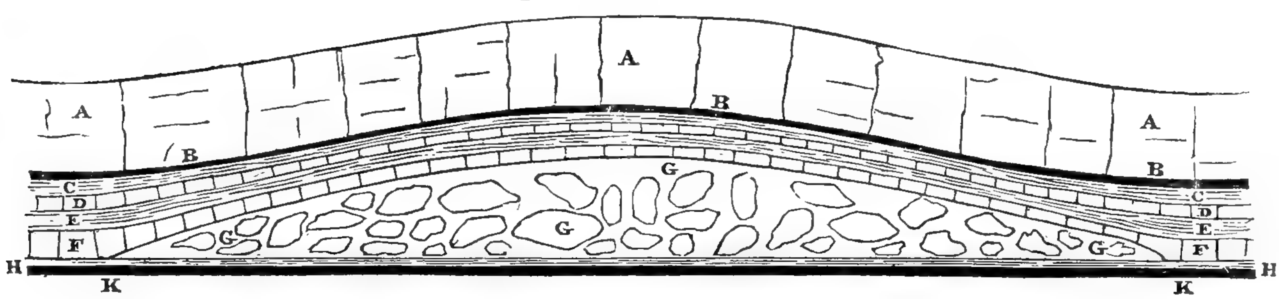
In the workings at Preston (one-half mile east of Prestonpans) there is the following section; A is a stratum of fire-clay generally 4½ feet thick, B is a stra-



tum of sandstone generally 4½ feet thick, C is a seam of coal. At one place the sandstone thins away to nothing, and the fire-clay thickens, so as to fill up the hollow in the sandstone, and to come in contact with the coal. It is there between 8 and 9 feet thick.

The next example I shall mention of a sudden thickening of sandstone is still more remarkable. At New Craighall there are two seams of coal, which go by the name of the *splint-coal* and the *coal-rough.* These are generally apart from each other about 8 fathoms, the interval between them consisting of the following strata:

B, Splint Coal generally 5 feet thick.
C, Fire-clay 1½ fathoms thick.
D, White Sandstone 4 to 6 feet.
E, Blue-coloured shale 1½ fathoms.
F, White slaty sandstone 2½ fathoms.
H, Fire-clay 1 fathom thick.
K, Coal-rough.



In the New Craighall workings, a mass of sandstone (marked in the preceding figure by letter G) occurs between F and H. It there goes by the name of a "saddleback," and has been mined in several places. It has very much the shape shewn in the above figure, which is drawn to a scale of 10 fathoms to an inch.* This mass of sandstone is full of oval-shaped concretions, some of which exceed a stone in weight. They are extremely hard, and give fire with steel. The colour of this sandstone and of these concretions is reddish. The greatest depth or thickness of this sandstone-bed is about 10 fathoms; the width of it at its base about 120 fathoms. It runs in a direction nearly S. by E., and has been traced for about two and a half miles, viz. from the fork of the Dalkeith and Fisherrow Railway, through the west part of the village of Old Craighall to the wall of Dalkeith park. The above figure shews a section of the saddleback, at right angles to the course now indicated.

It will be remarked, that the coal-rough and its roof are not affected by this saddleback. On the other hand, all the superincumbent measures including the roof of the splint-coal (which is a red-sandstone about 8 fathoms thick) mantle over it. I learn from Messrs Wilson and Telfer, the overseers of Sir John Hope at New Craighall, that this splint-coal is only 7 or 8 inches thick at the top of the saddleback, being therefore reduced to about one-seventh of its ordinary thickness; and their opinion is, that the strata immediately subjacent to the splint-coal are also reduced in thickness.

The peculiarity of the foregoing section is, that the sandstone mass shewn in it, does not extend laterally beyond a certain distance.

It also deserves to be particularly noticed, that these two coal-seams, as they mantle over the sandstone, thin away to a very considerable extent. This is more particularly the case with the coal-rough. It is generally 4 feet thick, but it gradually thins away to a mere shell 9 or 10 inches thick, when it reaches the upper part of the protuberance.

Let it not, however, be imagined, that the coal-strata are entirely exempt from those sudden variations in thickness, which more frequently characterize the other strata. One of the seams belonging to the upper series of coals, is known by the name of the Diamond Seam. It was worked formerly at various places in the parish of Newton. It is known to exist in the trough of the basin, and on the east side of the Esk valley; but it does not exist to the north or north-west of Millerhill; at or near that place, the coal thins away and entirely disappears,—the roof and pavement meeting.

This coal-seam, even where it exists and was worked, exhibited very anomalous variations in its thickness and component parts. At Sheriffhall, it was described to me as occurring in "balls" or nodular masses;—so that when in one place the seam was 2 or 3 feet thick, at another place, not a yard distant,

* This was the case in the figure shewn to the Society. The above wood-cut is on a reduced scale.

the coal was entirely absent. The following sections, taken at three places where the seam was worked, shew its extreme irregularity in regard to thickness and composition:

Diamond Coal-Seam.

At Sheriffhall Pit, No. 21.	Ft.	In.	At Sheriffhall Pit, No. 16.	Ft.	In.	At Bell's Law Pit.	Ft.	In.
Coal, ...	1	3	Coal, ...	0	11	Coal, ...	0	8½
Fire-clay,[1] . .	1	6	Fire-clay,[3] . .	22	6	Sandstone, . .	34	0
Coal,[2]	2	7	Coal, ...	5	6	Shale, ...	46	0
Fire-clay, &c. .	3	2	Fire-clay,[4] . .	6	5	Coal, ...	3	8½
Coal, ...	1	0	Coal, ...	1	2	Shale, ...	3	8
						Coal, ...	1	1
	9	6		36	6		89	2

[1] This band was, in the workings, sometimes only 6 inches thick.

[2] This band of coal, in the workings, sometimes reached a thickness of 4 feet.

[3] This band was occasionally only 2 feet thick.

[4] In this bed, a band of sandstone sometimes made its appearance, having a thickness of 5½ feet.

But one observation occurs here applicable to the statements made above, regarding the varying thicknesses of all the strata, coal included. These variations have been inferred from examinations only at particular, and after all, not very numerous points. Now, it cannot be supposed, that the exact spots have been hit on where the maximum and minimum thicknesses respectively occur. Over a district about thirteen miles square, and the strata in which, if spread out horizontally, would extend greatly beyond that space, it is obvious that 100 or even 1000 bore-holes must give but a partial and imperfect notion of their varying thicknesses, and of every thing else connected with them. It is probable, therefore, that we ought to consider the variation in the thickness of the several strata, to be at least double of what is shewn by the tables above referred to.

It is not in *thickness* only, but in *composition*, that the same individual strata change at different places. The best examples of this are afforded by the coal-seams, they having every where been more minutely examined than the others.

The *Great Seam* of coal contains generally a band of parrot;—but at some places, the parrot disappears from it. At *Drum* this parrot band is 2 feet thick, —at *Niddry* it is 2 feet 6 inches thick; towards its northern stretch it thins off, for at Brunstain and Joppa it is only from 1 foot to 1½ foot. It thins off also towards the south, and disappears altogether; at *Gilmerton* and *Loanhead* it does not exist, its place being occupied by a bed of shale. On the opposite side of the basin, the *Great Seam* contains also a similar band of parrot;—it occurs in it in that part of the basin which is opposite to Niddry, viz. about two miles east of Dalkeith, in the Duke of Buccleuch's grounds at *Cowden*, where it is 1 foot 2 inches thick. On this side of the basin, in like manner as on the other it thins off to the north and the south. At *Preston* the place of the parrot is occupied by a seam of shale; at *Bryants* there is parrot in the seam, which is a few inches thick; at the Sinks quarry (about one mile to the west) this parrot band is only 2 inches thick; but the whole seam is also diminished in thickness to 5 feet. Near Stobs (which is two or three miles west of Newbattle) the seam of parrot is only about $1\frac{1}{2}$ inch thick.

The *North Greens* coal affords a parallel example. At many places on both sides of the basin it has parrot in it. On the *west* side, at *Gilmerton*, the parrot band is 10 inches thick; at *Portobello*, it is 4 inches; at *Loanhead*, the parrot band is only a few inches thick; at *Glencorse*, it is reduced to 4 inches, and the quality is very inferior. On the *east* side of the basin there is parrot also in the seam. Towards the north on this side of the basin (just as on the other side) the parrot on the North Greens thins away and disappears. At *Bryants* the seam is altogether $2\frac{1}{2}$ feet, consisting on the top of splint, at the bottom of rough coal, and in the middle from 10 to 14 inches of parrot. At *Kippilaw* (about one and a half miles to the east), the whole seam is only 3 feet thick, and the parrot band is reduced to 6 or 8 inches. At *Edgehead* (which is about two and a half miles to SE.), the North Greens seam is 1 foot 8 inches thick, and contains in the middle of it 5 or 6 inches of parrot. At *Wetholm*, which is about one-half mile still farther east on the line of crop, the parrot band is only 2 inches thick. A little farther east (viz. at *Fuffet*, where a colliery existed twenty or thirty years ago) the North Greens contained no parrot whatever, consisting chiefly of splint. I believe that the parrot band thins off also on the west stretch. At *Arniston*, the whole seam is only 26 inches thick, of which 7 inches consists of parrot. At *Middleton* and other points farther SW., there is hardly a vestige of parrot, properly so called, in the seam.

I may add, that the Great Seam and North Greens are not the only coal-seams which contain parrot bands. For instance, the seams called the "South Parrot" affords parrot at *Gilmerton* and *Loanhead*. The coal called the Laverocks or Mavis, affords parrot at Loanhead. The coal called the Splinty or Stony coal, affords parrot at *Gilmerton*, *Wetholm*, *Blackdub*, and *Fuffet*. Various other examples may be seen from the tabular chart of coal-seams.

The same changeableness which characterizes the parrot, marks also the splint, rough, and other kinds of coal. At some places they thicken, at others they thin away and disappear. But let it be observed with regard to one and all of these different kinds of coal, that though they generally form distinct and separate bands, they, at the line of contact, run into each other; that is, they appear united, as if by some process, which has taken place subsequent to their original formation.

Before quitting this particular subject, I may mention that the bands of clay and shale which frequently occur in coal seams, though they in general preserve a pretty uniform thickness, do occasionally vary in thickness in an extraordinary manner. At New Craighall, in the Beefie coal worked there, a band or stratum of clay occurs 1 foot thick. At Sheriffhall (about two miles distant) this band of clay thickens to 18 feet, thereby, of course, dividing the Beefie coal into two separate seams.

In the example now mentioned, it seems probable that the variation in thickness is characteristic of the original formation of the stratum, and arose from a larger quantity of sediment being deposited in one part of the district than in another. The example next to be given seems to indicate a different cause for the variation in thickness,—and one which operated subsequently to the deposition of the stratum.

At Bryants, the " Coal Patie " is generally 3 feet thick, and is overlaid first by a bed of shale $2\frac{1}{2}$ feet thick, and then by a stratum of coarse sandstone. At one spot, however, the shale disappears, and about a foot of the coal from its upper surface also disappears. In this way an expanded hollow or trough is produced, which is filled up by a protuberance of the sandstone stratum.

A similar example occurs in the workings about a quarter of a mile to the SE. of East Houses. There is or was formerly a small " rough coal " worked there, about 3 feet thick, covered, as in the former case, first by shale, and next by a soft yellowish sandstone. At one place the shale disappears, and the coal is diminished in thickness to $2\frac{1}{3}$ feet, the trough being occupied by the sandstone.

V. The next subject to which I would advert is the form or outline which a section of the whole deposit presents in superficial extent, as well as in depth.

I have, with the view of making my explanation more intelligible, drawn some figures, representing along particular lines (crossing the entire coal-field), the form of its surface, and the supposed undulations of the several strata.

Before pointing out the details of these sections, let me mention how they have been formed, so that the Society may judge what degree of weight, or whether any weight, is to be attached to them.

I traced out, in the first instance, on a map of the district, the croppings of the principal coal and limestone strata, and laid down the course of all the dykes and slips intersecting these strata, of which I could get information. In this way I got the distances between the coal-seams at the surface, and knowing the angle at which they dipped, it was easy to represent the strata in vertical sections, with their different inclinations and undulations. I have also shewn, wherever these lines of section are intersected by slips or dykes, on which side the strata are downcast or upcast (as it is termed);—and I have endeavoured to give a correct outline of the surface of the country along these lines of section, with reference to its elevation above the sea. In constructing these figures, I have made the vertical section on the same scale as the horizontal section,—a circumstance too little attended to in geological diagrams; for otherwise, a distorted and erroneous representation must be given of the whole phenomena. I have drawn these figures on a large scale, so that the real depth of the basin might be easily seen, in reference to its true superficial extent.*

It will be seen from these figures what is the general shape of the basin formed by the coal strata of the district. The elevation in the middle of the figures, and which separates the basin into separate fields, is the ridge which I have already spoken of as running from the Roman Camp to Prestonpans. On the west end of these lines of section the strata dip rapidly to the east, and at some places are almost perpendicular; but on the opposite side they are much less steep. At several points along the line of ridge, the individual strata mantle over, rising up on one side and sloping down on the other. At other points the strata have been apparently broken across, so that they crop up on both sides of the ridge, having a gap or chasm between them, which in some places is three-fourths of a mile in width. The fact now stated may be very distinctly seen at *Fuffet*, a place about three miles SE. of Dalkeith. The two ridges of limestone (the lowest of the series) are there distant from each other only about 200 yards.

Along the west side of the basin the dip of the strata is by no means regular. At *Joppa* it is about 50°. It gets steeper towards the SW., and at *Niddry* and *Edmonstone* they are exactly vertical. The verticality of the strata may be well seen in a quarry near Edmonstone (West Gate), which for many years supplied the yellow-sand brought to Edinburgh to be sold for domestic purposes. The sandstone-rock which was there worked lies between the Corby Craig coal, and the Glass coal, the perpendicular walls of the excavated strata running in a SW. direction. At *Gilmerton* the strata dip at an angle of about 60°; and at *Loanhead* at an angle of 52°;—farther west, the dip gradually diminishes.

On the east side of the Esk valley, the strata generally are less steep than on the west side,—rising up with an angle varying from 20° to 30°.

In the Tyne valley, the strata form a basin much flatter than the basin of the Esk. On the west side, the dip is not more than 8° or 10°; and on the east side it does not exceed 5° or 6°.

It will be seen from the annexed sections, that the coal-seams of the district form as it were three series,—which may be described as the upper, the middle, and the lower series. The interval between these series consist chiefly of shale and sandstone, forming an aggregate thickness in the former case of 40 fathoms, and in the latter of 150 fathoms. But let it not be imagined, that there are no coal-seams at all in these intervals. The tabular chart shews that there are some. But they are so thin, and so far distant from each other, that the

* The sections referred to are shewn on a smaller scale in Plate XV. at the end of this volume.

parts of the deposit now referred to, present a complete contrast to the other parts.

VI. Having described the general arrangement and position of the strata, and said something of the varying character of particular strata, I should wish next to offer some general remarks on the *internal structure* of the strata. On this subject, however, I regret to say, that my information is exceedingly scanty. But if for no other purpose, than to shew how much remains to practical men on this branch, I will state the little that I have gathered in my rambles.

(1.) I begin with *coal*, because the qualities of that mineral have naturally been more closely and accurately examined than those of any other.

There are three or four different kinds of coal in the district. The most easily distinguishable are the splint, the cubical or cherry or cheery* coal, and the parrot. Each of these can be pointed out at once by colliers, as each of them possesses characters, which to their eyes are obvious and decisive. Each kind of coal has a different internal organization. That difference is shewn in various ways. I may mention one way in which it is very easily and very decisively shewn. If a large fragment is smashed with the hammer, or dashed violently to the ground, it will be found to have been intersected by a number of fissures, which give a peculiar shape and form to the morsels broken off. These fissures cause each kind of coal to break up in one way, rather than in another.

In the *Splint*-coal there are three sets of fissures, (1.) one set parallel with the surface of the coal-seam; (2.) another set perpendicular to the surface; and (3.) a third set, also perpendicular to the surface, and intersecting the second set at a constant angle. The blocks are not in the splint coal exactly cubical,—they form thin tabular masses, owing to the predominance in them of the longitudinal fissures. These thin tabular masses are not rectangular in shape. The vertical fissures intersect each other at an angle, which is between 80° and 83°. Farther, the figure of the fragment, on its surface, is rhomboidal,—that is to say, one side is always longer than the side intersecting it. The reason appears to be, that the two sets of vertical fissures which form these sides, are not equally continuous or open. So that there is always a tendency in the coal to split in one way more than another. Those fissures which run farthest and are the widest, of course form the longest side of the rhomboid. These are called slines, backs, or length-way joints, in the language of the colliers. The other set are called cutters or end-joints; and both sets are made good use of, in working the coal, as they allow the wedges to be inserted, and enable the collier to bring down masses of coal of any size he pleases. It is in consequence of these three different sets of fissures, that the splint coal breaks so readily into oblong tabular masses.

* So called, probably, from its blazing better than the other kinds of coal.

The rough or cherry coal has the same three sets of fissures; but the strength and length of each set is in that kind of coal, not precisely the same as in the splint coal. The set first mentioned, viz. those which are parallel to the surface of the coal, are not so numerous or so wide or so extensive, as in the splint, so that it has not the same tendency to a slaty shape; moreover its cutters and its slines are more nearly equal in these particulars; and the consequence is, that the rough or cherry coal when broken, assumes a shape or form more or less cubical. Its vertical fissures intersect at an angle of about 85°.

The same three sets of fissures are discoverable in the parrot-coal. But they are greatly less numerous. The consequence is, that blocks free from cracks and flaws, can be got of a much larger size in this kind of coal than in the other kinds. These different fissures, in respect of continuity and width, are, in the parrot-coal, very nearly equal;—so that the blocks taken from it are, in shape, not very different from cubes. The acute angle formed by the sides is about 87°.

These fissures have of course been produced at a period subsequent to that of the formation, or at least the deposition of the beds they traverse. I have a specimen of splint coal filled with the spines, teeth, and scales of fish. A fissure intersects these organic remains, and has separated the relative parts by a very visible interval. The fissure in the specimen alluded to, is about 1-10th of an inch in width, and is filled with pearl spar. It is obvious, that not only the coal-seam has been fissured, but that there has been a movement of the coal on one side or both sides of the fissure.

Before quitting the subject of slines and cutters, I may allude to a subject which opens up a very interesting inquiry. I have said that these slines and cutters intersect each other at a constant angle. I rather think also that they lie invariably in a direction, which is independent of the dip of the coal. I have observed at a great number of places, the directions of the *backs* or lengthway joints, and also of the *cutters* or end-joints. The former appear to me to run every where in this district in a direction between N. and W. by compass. The cutters, of course, therefore run between E. and N. I will afterwards revert to this subject.*

I have to add with regard to these fissures in coal, that many of them are filled with thin films or veins of a white coloured spar. I gave to my friend Mr A. CONNELL a quantity of this substance to be analyzed, and he reports it to be carbonate of lime, containing also some admixture of magnesia and iron. Sometimes this substance is found beautifully crystallized, coating drusy cavities, in the heart of the coal-seams. Sulphuret of iron, too, in a crystallized state, is abundantly met with filling both cavities and fissures in the coal.

* It is impossible to form any correct opinion on such a point as this, except upon a very extensive range of facts. I have commenced a table shewing the direction of the backs and cutters in different parts of the district, which will be found in the Appendix B.

(2.) It is unnecessary to say much in regard to the internal structure or composition of the *limestone* strata. They have all the dull and argillaceous appearance common to the limestones of the coal-measures, without the least symptoms of crystalline structure.

There are the same sets of fissures in limestone as in coal. But they are by no means so numerous. It is impossible to find a piece of cubical coal at the pit mouth free from fissures;—whereas large blocks of limestone are quarried without fissures. The principal fissures in limestone are distinguished from those in the coal by being more open, and by frequently having their sides covered with cup-shaped cavities. These are hollows in the rock, and are generally from 1 inch to 2 inches in diameter, and about half an inch to 1 inch in depth. The sides of a fissure in all other rocks are generally smooth. These fissures, when near the surface, are generally filled with clay and various debris; at greater depths, they contain veins of calcareous spar.

With regard to the texture and composition of the limestone, all who have read Dr HIBBERT's paper on the Burdiehouse limestone must be aware, that a great difference exists, in these respects, in the limestone strata of the district. The *Burdiehouse limestone* is slaty in its structure, whilst the *Gilmerton*, *Coldcoat*, and superior strata, are solid and massive. After the former has been in the kiln, it can be easily separated or split, even with the hand, into thin plates. This peculiarity of structure appears to be owing to the abundance of vegetable and carbonaceous matter in the Burdiehouse limestone, and which is generally situated between its lamina. The nature and names of the vegetables found in that deposit have been so fully stated by Dr HIBBERT, that it would be presumptuous for me to touch upon that subject: and for the same reason, I refrain from alluding to the discovery in it of the fishes' teeth, scales, and spines, which Dr HIBBERT has so fully and ably noticed. In the superior limestone strata, none of these fossils have been discovered: the only fossils known in them are marine shells and zoophytes. There are no vegetable impressions in these upper strata.

(3.) In regard to the *argillaceous* strata of the district, the only interest attaching to them in respect of composition or internal structure, arises from the fire-clay and ironstone they contain.

That they vary in thickness and in composition, like the other strata previously mentioned, is undoubted; but I am unable to state the amount or the direction, or the exact nature of that variation.

In truth, none of these strata have been worked so extensively as the coal and lime strata, so that there are not the same means of obtaining information regarding them. No attempt whatever has been made in the district to work ironstone, though in some places it has been ascertained to be in considerable quantity, and of good quality. This valuable ore occurs in two states, viz. in a continuous stratum (termed "black band"), and also in small nodules or lenticular balls. At Dryden, there is a band in the lower series of coals (between the glass and the stoney coal), 14 inches thick, which yields about 33 per cent. of iron. It is understood that there is a still thicker band there, near the south parrot coal. Whether or not this band runs, like the coals, throughout a great extent of country, has not yet been ascertained. I have seen, in a bed of shale, thin seams of black band, continuous for a few hundred yards, and then entirely cease.

The clay-ironstone of the district appears to occur most frequently in the form of balls or irregular lumps. It is a remarkable circumstance that every one of these is found to contain more or less organic matter. At *Wardie* they contain the scales, teeth, coprolites, and other remains of fish.* At *Cowpits* and *Pinkie Burn*, they contain large quantities of bivalve shells resembling a *Unio.* There are apparently two species among the shells in my possession,—one of them much more elongated than the other. The last has the round and tumid shape of the species noticed in Dr Hibbert's paper as having been found at Burdiehouse; but all the specimens I have seen from Cowpits are of a larger size than the Burdiehouse shell. The nodules containing these bivalves, are imbedded in a stratum of shale about 2 feet thick, which forms the roof of a coal called the "three feet coal." The stratum immediately above the shale is whitish sandstone, about 30 feet in thickness; and above the sandstone is a bed of shale, which forms the pavement of the Bar's coal. This *muscle-band* (as it is termed) was seen not only at the two places just mentioned, but also at *Midfield* engine, so that it covered a district in one direction of about three miles. There are strong reasons for thinking that the same muscle-band runs south to Smeaton, and that it also appears on the west side of the basin, at *Joppa* (on the shore), at *Easter Duddingston* old engine-pit, at *Wanton Walls* deep level, and near *Somerside*;† so that it now covers between twenty and thirty square miles of horizontal surface.

VII. The next subject connected with the stratified rocks of the district, is the slips or fissures which traverse them. The fissures now referred to are very different from those previously described,—which concerned merely the internal structure of individual strata. Those now to be described, traverse all the strata of a particular district, for many hundred yards or even for miles, and reach to a depth which has never been ascertained.

These fissures (known to the colliers under the various names of hitches, faults, dykes, troubles, slips) show that the strata they intersect have been broken across; and that they have thereafter, on one or both sides of the fracture, changed in their position, so that the fractured ends of the strata are no longer

* For an analysis of this ironstone, see Appendix C.

† All these places, except the first, are stated on the authority of Farey.

opposite to each other, but on one side have sunk down, or on the other side risen up,—in some cases to the extent of several hundred feet.

I have been at considerable pains to obtain information regarding the slips which occur in the district. The subject is one of extreme difficulty, arising from the circumstance, that these slips can very seldom be discovered except in carrying on mining operations. They are seen, therefore, only by the working collier; no geologist need attempt to find them and trace them himself. He must depend entirely for his knowledge of the subject, on information to be elicited from others; and those others unfortunately belong in general to a class of persons, who are neither very intelligent observers, nor very able to explain with accuracy what they have observed.

Notwithstanding these obstacles and disadvantages, I have been able to collect information regarding about 120 different slips. I have drawn green coloured lines on the accompanying map, indicating the *places* through which the most important of these slips run, and the *direction* of their course. The particular spot where they have been proved is in general marked by a × drawn upon these lines. It is unnecessary for me here to enter into any particular account of the effects of each particular slip. There are about 110 of them marked on the map, and numbered with reference to a table annexed to this memoir, which shews exactly the circumstances and effects of each.* I may merely mention, that the greatest slip known in the district is what is called the Sheriffhall slip. It runs N.W. by W., and has produced a dislocation of the strata to the extent of 400 or 500 feet; that is to say, the coals which are worked on the south side of the slip near the surface, are on the north side of the slip 400 or 500 feet down below the surface.

I shall now notice what appears to me the most important circumstances characterizing the slips of this district.

(1.) They are all of unfathomable depth. There is no instance of any slip which comes to the surface, having been found to end or disappear at any depth to which coal operations have reached.

There is one example known of a slip which does not come to the surface. It is in the Sheriffhall colliery. There are there three seams of coal, viz. the *Beefie* coal, the *Diamond* coal, and the *Jewel* coal, the last mentioned being the lowest. This slip cuts through the two last mentioned seams, but it does not reach so far up as the *Beefie coal*. This slip runs due N. and S. It is not marked on the map or on the table of slips.

(2.) The next point deserving of notice on this subject, is the *direction* of the slips as they appear on the surface. This information is afforded by the table I have compiled; and it may be obtained also by inspecting the map.

On an examination of the table, it will be seen that, out of 109 slips the

* For this table, see Appendix D.

directions of which have been exactly ascertained, there are 94 which lie between the north and west points of the compass, and of the remainder 7 run in a direction due west.

(3.) The slips are here as in other places more or less vertical. I have heard of none which deviates more than 15° from a vertical plane. This, however, is a branch of the subject on which my information is yet imperfect.

But though I cannot state the precise angle which the slips make with a vertical line, I have ascertained on which side they deviate from the vertical line, or in other words, towards what point of the horizon they dip. There is a separate column in the table for this information. The point now mentioned is easily ascertained by means of a remarkable law, which prevails here, as every where else, viz. that if a dislocation of the strata has taken place, and the fractured ends of the strata are no longer opposite to one another,—then it is invariably found, that the side of the slip on which the fractured strata are lowest is the uppermost side of the slip;—and knowing which is the uppermost side of the slip, and at the same time its horizontal direction,—we learn at once the quarter towards which it dips. Now, in almost every instance where a slip is met with in the working of either coal or lime, it is an object of practical inquiry whether, and how much, the metals are thrown down or cast up by the slip;—and there are few places where this point has not been most accurately ascertained. There are 78 slips in the district, whose dip I have ascertained by the rule just mentioned. Of these, 35 dip to the south, and 43 to the north.

If each of these 78 slips produced an equal derangement of the strata, the result on the whole would of course be to throw the strata down more towards the north than towards the south, and in the proportion of the respective number of slips. The strata of the district are, in point of fact, thrown down more to the north than to the south;—but they are thrown down to a much larger extent than the above proportion; for it appears from the table, that, whilst the strata are thrown down by the slips dipping to the south 385 fathoms, they are, by the remaining slips, thrown down to the north 754 fathoms.

(4.) These remarks are made with reference to the slips of the district generally. But the subject is one of such importance, both in a practical and in a scientific view, as to deserve a more particular analysis.

a. It will be seen from the map, and by comparing the directions of the slips with the croppings of the coal-seams intersected by them, that the slips are in general nearly at right angles to the crop of the coals; or, in other words, that the rent or fracture of the strata is in the line of the dip and rise. There are *no examples of slips*, (at least deserving any notice), which run *parallel* with the *cropping* of the strata.

b. Farther, it will be seen from the map, that where the dip or the direction of the strata continues uniform through a given district, *the slips are all parallel to each other*. This, indeed, is a corollary which flows directly from the rule just stated. On the other hand, where the dip and direction of the strata present considerable deflections, the slips cease to be parallel. For example, at the Roman Camp, where the same individual strata wind round the hill, having almost a quaquaversal dip, the slips present a similar deflection, and converge towards the top of the hill.

(5.) I may add, in reference to the slips, that they vary materially in *width*. In some of them the sides of the slip are in contact with each other; and in these cases, I am informed, scratches or ruts are occasionally visible on the sides. The late Mr Grieve mentioned to me one remarkable instance of this he had seen in the great ninety fathom slip of Sheriffhall. The scratches, he said, were neither vertical nor horizontal, but formed an angle with the horizon of 30° or 50°, and dipped towards the south-east.

In other cases, however, the sides of the slip are a few inches or a few feet apart. The table notices several where they are nine feet apart. The chasm is generally filled with the debris of the adjoining strata.

(6.) Sometimes the strata are found to be cast up as well as cast down on the same side of a slip. This arises from the individual strata having been tilted up at their opposite ends after being fractured, so that they, as it were, intersect each other at some intermediate point. An example of this occurs at Stobsgreen.

Another singular effect of a slip occurs at Prestongrange, where there is a cast-down of a few fathoms at the west end of a slip, on the north side. This cast-down lessens, however, towards the east, so that the coal-seam at last comes up to the level of the same coal on the other side. But a little farther east, the cast-down again increases. The explanation is, that, on the north side, the strata dip from the common point of contact towards the east and west, more rapidly on the north side of the slip than on the south.

(7.) Another circumstance deserving notice is, that the coal and other strata intersected by slips are much shattered. This prevails sometimes to a distance of many yards from the slip. But it has been often observed, that this condition of things exists only on one side of a slip, not on both sides.

(8.) Very frequently, the strata near the side of a slip, rise suddenly up towards it:—and it is important to remark, that this is always the case (so far as I can learn) on that side of the slip where the fractured edges of the intersected strata are lowest. The great slip of Sheriffhall affords an example of this.

(9.) It would appear from the table, that the amount of dislocation produced by slips increases towards the centre of the basin or dip of the strata, much more frequently than towards the edge of the basin, or out-crop of the strata. The proportion shewn by the table is as two to one.

Besides these points, there are many others of great importance,—such as, whether the strata are always most shattered on the up-cast or the down-cast side of a slip,—whether every slip has one which joins it at right angles, or any other angle. On these, however, I will not enter. The data I have collected in regard to them, are too meagre to warrant any conclusion.

VIII. Before concluding what I have to say regarding the stratified rocks of the district, let me observe, that, in working the coal, *Hydrogen gas and Carbonic acid gas* are met with.

I am not aware, however, of hydrogen gas having been met with at more than one place, viz. Prestongrange. The workings there are in the immediate neighbourhood of a whinstone dyke, to be afterwards described. About a century and a half ago, there was coal worked at Wardie; and SINCLAIR mentions, that one of the reasons why the working of it was abandoned, was the danger arising from what he calls " wild-fire," and which, from his description of its effects, could have been nothing else than hydrogen gas. His words are—" The place where this (*i. e.* the wild-fire) was most known, was in a coal be-west Leith, in a piece of land called Wardy, which, for want of level, *and the violence of that fire*, the owners were forced to abandon." p. 294. I think that SINCLAIR'S account must be erroneous, for Captain BOSWALL informs me, that, during his recent operations in working and boring for coal, no hydrogen gas was met with.

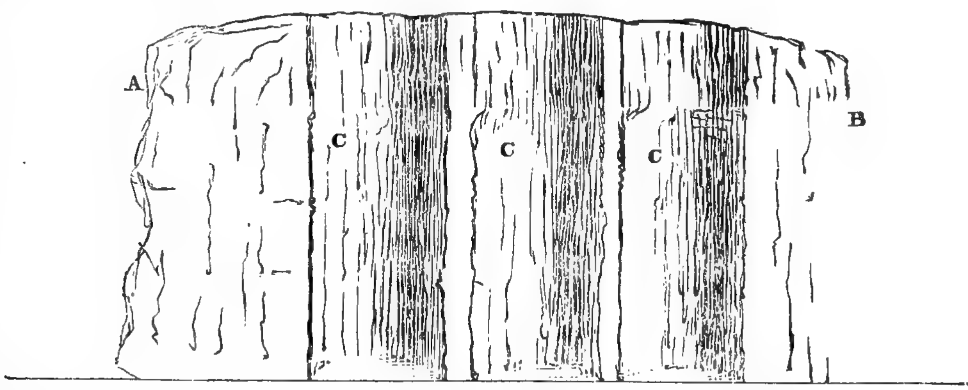
The carbonic acid gas occurs in most pits throughout the district. It occurs in the greatest quantity in the Roman Camp workings, which are close upon the great body of limestone; it is also very abundant in the workings of the North Green at Gilmerton, and in the workings at Tranent. It has been supposed that this gas is disengaged by the partial disintegration of the subjacent limestone, and that it rises up through the crevices and fissures. But in reference to this opinion, it is proper to observe, that choke-damp occurs in workings of coal far removed from any limestone strata.

II. UNSTRATIFIED ROCKS.

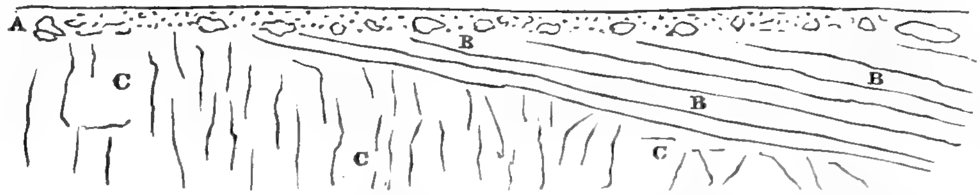
I proceed now to an account of the unstratified rocks of the district. 1. The trap-rocks of the district may be conveniently divided into hills and dykes.

(1.) There are no *hills* or amorphous masses of trap within the proper boundaries of this coal-field. They are all *beyond* the *crop* of the workable coal-seams and limestones. The only places where these masses of trap occur are the following :—*Lochend, Edinburgh Castle Rock, Calton Hill, Arthur Seat, Braid Hills, Pentland Hills, Blackrock* (near *Blackshiels*), *Barrows* (near *Gifford*), *Morham, Traprain*, and the *Garlton Hills.* The *greenstone and basalt* occur at Lochend, Salisbury Craigs, Black-rock, and Barrows. The Braid Hills, Pentland, Garlton Hills, and Traprain Law, are composed of *felspar porphyry.* On the Calton Hill and Arthur's Seat are enormous accumulations of *trap tufa.*

It is of course unnecessary for me to say one word of the trap-rocks at Lochend, Calton Hill, Arthur's Seat, Braid Hills, or the Pentlands. They are too well known to need any description here. Let me allude, however, to the Black-rock-basalt, which I do not think has been before noticed. It occurs about half a mile beyond the crop or outburst, as it is called, of the Crichton-dean limestone. It is there extensively quarried for road metal. The popular name of blue whinstone gives a tolerably correct idea of its appearance. What struck me as most interesting about this whinstone, is the form of its arrangement. It consists of large cylindrical masses, though of irregular shape, some of them five or six feet in diameter. These masses are associated together in the form of columns or round pillars, about ten or twelve feet in diameter, and reaching to a depth that has not yet been fathomed. The quarrymen work the rock by digging out of these pillars the individual masses of whinstone; and when they have been worked to a considerable depth, these spots present the appearance of wells—or rather of niches, as they are cut open on one side. The quarry having been extensively worked, there were, when I visited the quarry, eight or ten of these niches, which presented a singular appearance at a distance. These semicircular hollows are separated from each other by a few inches (not exceeding 18), the interstices being filled up with disintegrated basalt and argillaceous matter, which form veins. The structure of this rock is no doubt owing to imperfect and extensive crystallization, somewhat similar to what may be seen at Arthur's Seat. The following section may help to render more intelligible the description that has now been given. AB is a section of the hill. C, C′, C″ are the cavities out of which the trap has been quarried.



At the west end of the Garlton hills, there is a section presented in a quarry which is worthy of some notice. Over the solid rock are several strata of disintegrated trap, consisting of lilac-coloured clays, and small grained conglomerate, each of these strata being not more than a few feet in thickness. The whole section is about 12 or 15 feet deep, and 20 or 30 yards long. They slope at an angle of 10° or 12° from the Garlton hills. They are overlaid by the debris of gravel and boulders which every where cover the strata of the district. The following figure illustrates this description. A is a covering of gravel and boulders. B are the stratified deposits of disintegrated trap. C is the solid trap-rock on which these strata rest.



(2.) There are three or four dykes of trap in the district, all of them except one consisting of greenstone. The dykes are all situated *within* the limits of the coal-field proper; at least they traverse and intersect for the greatest part of their course the workable coal and lime strata; and if they run out beyond the limits, it is merely to join the hills of trap, which are on the immediate confines of the coal-field. The first of these dykes which I may refer to, runs in an east and west direction, about a mile south of Portobello. The line of it is crossed once or twice by the Brunstain Burn. It runs in a direction N. 60° W., and its course is indicated on the map by a double line, coloured *green*. It consists chiefly of clay, but containing also a good deal of felspar; and it contains other minerals common to trap-rocks of that description. Its texture is tough, and it has a greenish-brown colour. In one place not far from the new road leading from Dalkeith to Leith, it was formerly quarried for the roads. It is there 50 or 60 feet wide. But it gets gradually thinner towards the east, and terminates near Brunstain-house. It has not been traced farther west, I believe, than Niddry Mill, so that its known and ascertained course is not more than one and a half miles in length. It is extremely probable that it is connected with Arthur's Seat, the trap of which is in one part of the hill similar to it in texture.

Another dyke may be traced from Morison's-haven by Preston, Seaton, Redcoll, and so eastwards towards the Garlton hills, a distance of nine or ten miles. This dyke varies greatly in thickness. At Morison's-haven it appears to be some hundred feet thick; at Bankton (about three and a quarter miles east from this point) it is about 60 feet; at Mr CADELL's railway (where it is quarried for the roads) the north side or "veeze" of it is visible, but not the south side;—what is visible measures 96 feet. Near Long Niddry, where it has been also quarried, this dyke appears to be not more than 50 feet wide. This dyke runs in a direction E. by S.

It will be observed that, on the map accompanying this, the Morison's-haven dyke is represented as interrupted near Redcoll. I do not know whether it there takes a turn to the south,—an occurrence to which I know no analogous case,—or whether there is another dyke which runs from that point towards the Garlton hills, along a parallel line, which is situated half a mile to the south of the line of the Morison's-haven dyke. Certain it is, that a greenstone dyke, though harder, larger grained, and more crystalline, shows itself SW. of Coats-farm, and appears to run into the Garlton hills; for it is again seen at a point SE. of Coats, and about a mile from the former spot.

A third dyke makes its appearance at Cockenzie, and may be traced eastwards running under Port-Seaton, and for a little distance along the shore. There it is lost; and as in the case of the Morison's-haven dyke, another greenstone dyke takes on about 400 yards to the north. This second one runs by Boglehill, and then by Harelaw, and so on to Redhouse Castle, beyond which I have not traced it. The circumstances just referred to, suggest the notion that a dislocation of the dyke had, at some period, taken place, and that they had then been separated. But I merely throw this out for farther inquiry. The Cockenzie part of the dyke runs E. and W., and at its west end measures about 76 feet across. The Boglehill part of the dyke runs E.SE., and gets thinner towards the E. At Boglehill, it is between 50 and 60 feet wide; at Gordon-Spittle it is only 30 feet wide.

A greenstone dyke may be seen crossing the Water of Leith, a few yards to the east of St Bernard's Well, and running in a direction N.NW. It is only a few feet wide.

A greenstone dyke, probably connected with the one last mentioned, runs to the north of Craigentinny House, by Lochend, Quarryholes, Hillside, Marshall's Entry (Leith Walk), and Albany Street. It was worked formerly at all these places—and, as I am informed, was seen at the east end of Abercromby Place, in excavating for the foundations of the houses. It is between 50 and 60 feet wide. It runs in a direction N. 14° W., or very nearly in that of the one last mentioned. It seems to get thinner towards the west.

III. POINTS OF CONNECTION BETWEEN STRATIFIED AND UNSTRATIFIED ROCKS.

Having thus described generally the Stratified and the Unstratified Rocks as separate and distinct classes, it may be proper next to notice some circumstances which characterize them when they are in contact.

(1.) I need hardly, in the present advanced state of geological science, adduce any facts to shew the alteration in quality or internal structure of the stratified rocks, where they come in contact with trap. It is enough to say, that all the well known phenomena—of the coal having much of its bituminous qualities *expelled*,—of the sandstone, shale, and other strata being *hardened*, &c., when in contact with the trap-dykes, occur in the district. These effects were most particularly observed along the Niddry dyke.

(2.) There is one phenomenon, however, which is said to be common elsewhere in such circumstances, but which I have not observed, and which I rather think does not occur here at all. It is said that strata, when intersected by trap-dykes, are very much altered in their relative position; and that, as in the case of slips, the strata are "cast down" on one side of the dyke, or "cast up" on the other. This, however, is not the case with the three principal dykes above mentioned intersecting the coal-seams, viz. the Niddry dyke, the Morison's-haven dyke, the Cockenzie dyke.

The Niddry dyke, as already explained, runs for about two miles, and is in one part 60 feet thick. It produces no derangement of level in the strata. But it is important to observe, that, at Brunstain, where this dyke thins away to nothing, and at which point a slip or fissure runs towards the east as if in continuation of the dyke, the strata are deranged. They are down on the north side, or up on the south side, 15 fathoms. The strata, though they are neither cast up or thrown down at this dyke, suffer a change in their direction. It will be seen from the map that they, along the line of the dyke, form an angle with each other, the inner sides of which face the west.

The Morison's-haven dyke was bored through in 1836 by the late Mr GRIEVE, lessee of Preston colliery. He told me that, on working the coal up to the north side of the dyke, and driving an adit through it, he found the same seam of coal immediately opposite, shewing that there had been no derangement.

The only reason I have for believing that the same holds true with the Cockenzie dyke is, that SINCLAIR, in his Hydrostatics, makes the following statement in regard to it: "In the Earl of WINTON's ground at Cockeny, there is found a course of coals and freestone, dipping to the SE. in the Links; and upon the full sea-mark, there is a *tract or course of whin-rocks, lying E. and W.*, underneath which these coals and stones come through, without alteration of course, and are found within the sea-mark, with the same dip and rise upon the north side they had upon the south side of the said rocks.

The whin rocks within sea-mark which SINCLAIR alludes to, must be the Cockenzie dyke, and which he states does not alter the course or the dip of the coals.

I here conclude the First Part of my Memoir. In doing so, I leave entirely unhandled, and even untouched, many subjects which might have fallen within a geological description of the district, had I aimed at describing every thing possessing geological interest. I have, for example, taken scarcely any notice of the ironstone existing in the district, and alternating with the coals. I have omitted all notice of the vast, though partly explored, field of organic remains. My reasons for not attempting to describe these and other topics of interest, are chiefly two. (1) Because, to have treated of every subject with any degree of accuracy, would have been to compose a *work*, and not a memoir for this Society; and (2), Because the matters to which I have applied my attention are of themselves so important as to deserve a separate description, and are, moreover, so difficult of ascertainment, that they require to be investigated by one who does not allow himself to be distracted by other objects.

The only proper foundation for any attempt to explain the laws of nature, is an extensive collection and accurate classification of facts. If this is true in regard to all the physical sciences generally, it is peculiarly so with regard to Geology, in which most of the relations are extremely complicated, and not easily discovered or observed. Therefore it is that, in describing so minutely the subjects noticed in this first part of my memoir,—and above all, in assorting together in a convenient and visible form the facts therein described,—my object has been to lay a foundation, on which the explanations to be offered in the next part of the memoir can safely rest: for the same process of reasoning which has been so successfully employed in explaining the phenomena of Chemistry and Astronomy, must, if properly applied, prove no less successful in explaining the phenomena of Geology.

II. EXPLANATORY PART.

Having concluded my narrative of facts, descriptive of the rocks of the district, I proceed next to offer some remarks, with the view of explaining these facts. In doing so, however, I will not prevent myself from noticing any additional circumstances, necessary for illustrating these explanations.

Before beginning to reflect upon the various phenomena of the district, it is proper to have some idea of the geological epoch when the strata existing in it were formed, and to consider their relation to the neighbouring hilly ranges.

The coal-measures of the Lothians are bounded on the south by the Lammermuir Hills, which consist (as is well known) almost entirely of greywacke strata. I say *almost* entirely; for amongst them we find great masses of trap, as at St Abb's Head, Oldhamstocks, Cockburnlaw, Fassney, and Soutra Hill. At the three places last mentioned,—whole hills of granite exist,—a fact not generally known. Now, along the whole north flank of this range of hills, there occur the same description of strata as those which compose the district more immediately the subject of the present memoir. It is true that the coal-seams and lime-strata, where they approximate these hills, are scarcely if at all *workable.* It is only at *La Mancha, Middleton,* and one or two other places, that they become thick enough for that purpose. But there can be no doubt that the thin seams of coal and limestone worked there belong to the series of measures previously described; and there is scarcely any doubt that these thin seams of coal and lime can be recognised and identified as particular members of that series. Near *Blackshiels,* for example, there are two or three thin seams of coal, which lie under the *Crichton Dean* limestone. These coal-seams have been found under the same limestone at other places, as at *Middleton,* at *Trabroom* (in the parish of Gladsmuir) at *Alderstone,* at *Coalstone,* at *Moreham,* at *Amisfield,* at *Dunglass,* and various other localities in East-Lothian. Moreover, at many of these localities, the superincumbent limestone occurs, and is now, or was formerly extensively worked;—as at *East Salton, Traprain Law, Palmerston, Dunglass, Broxmouth,* and even as far north as North Berwick. In fact, this lowest limestone stratum, with its subjacent thin seams of coal, spread over and as it were undulate through the greater part of East Lothian. This at first sight is somewhat inconsistent with the fact, that these identical strata crop out on the east side of what I have called the Tyne basin; but this apparent anomaly disappears, when it is explained, that along the eastern margin of that basin there is an anticlinal line, formed by these strata dipping down again, though very gently to the east, and constituting, in fact, a third but very flat and extensive basin in that direction. This basin reaches even to the sea on the north-east and east;—and is in various parts much broken up and interrupted by eruptions of trap, of which the *Garlton Hills, Traprain Law, North Berwick Law, and Whitekirk Law,* are the principal. But, notwithstanding these interruptions and exceptions, I state it as a proposition generally true, that the same coal-measures which fill up the valleys of the Esk and the Tyne, stretch to the sea at *Aberlady, North Berwick, and Dunbar,* and skirt the northern flanks of the greywacke hills from the shore at *Dunglass* as far west as *La Mancha in Peeblesshire.*

Between these greywacke hills and the coal-measures, there lies an intermediate formation, which corresponds with, if it does not constitute, the old red sandstone formation. It consists of red slaty sandstones, some strata of red clay, and thick beds of a red conglomerate. This intermediate formation is on an average not more than 150 yards in thickness. The conglomerate is uniformly in the lowest part of the formation, resting immediately on the greywacke. It may be seen at a great many points close to the hills,—as, for example, at *Kingside Edge, Middleton, Blackshiels, Woodcot, Dunglass, and Thurston.* The rock consists of rounded pebbles, from the size of a walnut to that of a cocoa nut; they are imbedded in a clay basis, coloured and hardened with iron, and they consist chiefly of greywacke, though occasionally also of the peculiar kinds of trap which occur among the Lammermuir Hills.

It is not merely at the foot of these hills that the conglomerate occurs. It may be seen also at the opposite side of the coal-field, and apparently rising up from under it; as for example at Craigmillar and at Libberton, where it is in the same relative position, viz. under, and a considerable way under, all the coal and lime strata. I believe the same coarse conglomerate occurs in other places, along and under the western side of the coal-basin, as at the Pentland Hills, though I have not myself seen it there.

This *conglomerate* is (as I have said) the *lowest* member of the old red sandstone-formation. Its upper parts consist of red slaty sandstones, containing a good deal of mica. Their smooth surface often exhibits white round spots, formed, as I conceive, from a chemical change in the iron with which the stone is impregnated. In the centre of the spot will generally be found a filament of iron, derived, as I conceive, from the blanched part of the stone. These white spots are common in this old red sandstone-formation, as well as in the new red sandstone-formation lying over the coal-measures, and a portion of which occurs in Berwickshire and Roxburghshire. But I never saw any in the red sandstones of the coal-measures proper.

It is hardly necessary to explain, that, in this older formation, there is not the slightest appearance of coal or lime; and I may add, that, so far as I know, no organic remains have ever been found in these red sandstones.

The strata of this formation rise to the hills, and dip under the coal-measures. They are steepest where they are near the hills; and they seem to lie conformably with the coal-measures.

From the short outline which has now been given, it is obvious that the coal-measures of this district must have been formed, not only at a subsequent *epoch*, from that in which the greywacke strata of the Lammermuirs were formed, but under conditions totally different. Every thing leads to the conclusion that these greywacke strata, after their formation and consolidation, had been thrown up by volcanic agency; and that, after this period, there had been deposited on their flanks, first a series of red sandstones, and next the series of strata commonly termed coal-measures, which have been particularly described in this memoir. It may be proper to observe, that, on the south side of the Lammermuirs (viz. in Berwickshire and Roxburghshire), there is a deposit first of red sandstone, and secondly of coal-measures, possessing exactly the same general features which these formations have on the north side of the range,—so that it is more than probable that these sedimentary strata were formed on both sides of the range of hills, by the same agents, and under similar, though certainly not exactly the same circumstances.

What those circumstances were, can best be discovered from an examination of the strata themselves; for, if properly examined, they will be found to contain, to a considerable extent, internal evidence of their origin and history.

It is hardly necessary to enter upon any formal demonstration of the now generally received opinion, that most if not all the strata composing the old red sandstone and carboniferous formations must have been deposited in an aqueous medium. The beds of conglomerate, skirting the sides of the Lammermuir Hills, composed as they are entirely of greywacke, and occasionally trappean boulders and pebbles, cannot be explained in any other way than by supposing them to have been washed down from the adjoining hills, and to have accumulated along the margin of a sea or lake. In like manner, the sandstone strata which lie over these conglomerates, composed as they are of sand and occasionally fine gravel, necessarily point to a similar condition of things. This inference, obvious from the very nature and situation of the sandstone strata, becomes still more obvious, on considering some of the facts mentioned in the first part of this memoir. It was shewn by statistical details, that, of all the strata, none is so irregular or variable in thickness as sandstone. This is at once explained by the greater weight and specific gravity of its ingredients, which causes it to be deposited more quickly, than fine muddy sediment out of which the shales and limestones were formed. The latter can remain suspended for a longer period, and in fact will not sink to the bottom in any considerable quantity till the water becomes comparatively tranquil. In this way there is time allowed for the sediment being equally distributed through the aqueous medium before it reaches the bottom; so that, after it does reach the bottom, it forms beds or layers of tolerably uniform thickness. But the case is quite otherwise with siliceous sediment. It is transported but a short way before it falls to the bottom. There is no time for it therefore to be diffused equally over the district; and the consequence will be, that the beds or deposits of sandstone will, generally speaking, be of very variable and irregular thickness. Whilst, on this subject, I may be permitted to refer to the examples given in the first part of this memoir, of the sudden variations in the thickness of sandstone strata, and in particular to the account there given of that remarkable "Saddle-back," as the colliers term it, which occurs in the upper part of the coal-basin. It is a sandstone rock, which lies over a particular coal-seam. I can compare it to nothing except a sand-bank, such as is formed in our existing seas. It is at its base about 120 fathoms in width, and it has been traced running in a S.SE. direction for about three miles. It is of a semicircular form, the lower part or base of it being perfectly flat. The top is about 10 fathoms from the base. It is quite impossible to look at the position of this sandstone bed, and see the manner in which the various strata of shale, sandstone, and coal, come up to the sides of it, and then rise over it, diminished in thickness, but not fractured or deranged, without being convinced, *first*, that the sandstone rock has been formed at the bottom of an aqueous medium agitated by currents; and, *second*, that these other strata had been afterwards deposited in the same medium, when in a state of comparative tranquillity.

So far with regard to the manner in which these several strata were formed; and the reason why the sandstones should be more variable in their thickness than the other kinds of rock. But after these strata were formed, they would be liable to be worn down and occasionally hollowed out by the agency of currents; and, therefore, if the above theory as to the mode of their formation be true, we ought to find in the rocks of this district examples of such erosive action. This inference is verified by the fact; for, it may be remembered that, in the first part of this memoir, several examples were given of hollows in the strata, which were shewn to be filled or occupied by the particular stratum lying over it, and which, of course, exhibited at that spot a bulging out or protuberance below, whilst it was perfectly flat or level in its upper part.* It is not possible to explain such facts in any other way, than by supposing that these strata were deposited at the bottom of a body of water, which, in certain parts, or at certain periods, was still and tranquil, and which, in other parts, or at other periods, was agitated by currents.

I might, in reference to the same inquiry, have alluded to the vegetable remains, and especially the large trunks of trees found imbedded in the sandstone strata, and which must have been transported to that situation by some powerful agent. But it is quite unnecessary to do more than barely allude to this additional palpable proof of the existence of an aqueous medium, at the bottom of which these sandstone strata were deposited.

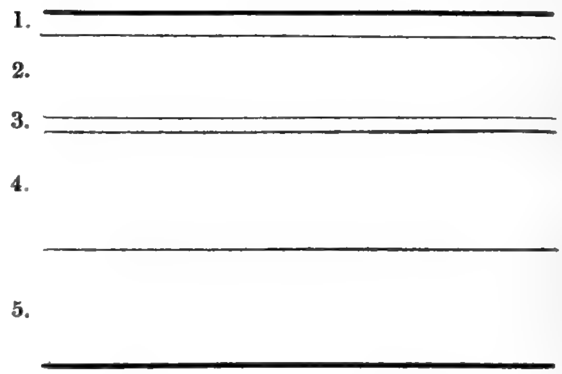
I might, in like manner, point to the shale and limestone strata of the district, as proving incontestibly the existence of a large body of water; for, that they were all deposited in water, no one can doubt, who but looks at the innumerable shells, zoophytes, fishes' teeth, and other exuviæ with which they abound.

I have not yet spoken of the formation of the coal-seams particularly and specially;—though, as being parts of a series, all the other members of which are proved to have been deposited at the bottom of a sea, it is a fair conclusion that they must have been formed by the same agent, though in circumstances somewhat different. There are some geologists, however, who maintain, that the vegetables which compose, or are found imbedded in the substance of coal, have actually grown and flourished on the very spot where we now find them; that the vegetable ingredients of coal have been accumulated, not at the bottom of the sea, but on the surface of dry land, either in the same manner as peat, or in extensive marshes. I am not now going to enter into all the details of this controversy. I wish only to mention one fact, which appears to me to go far to put an end to the controversy altogether. I allude to the discovery in the coal-seams of this district of fishes' teeth, spines, and scales. The discovery was first made by Lord GREENOCK about four years ago, and he exhibited a number of specimens to the British Association. These specimens, I observe from his Lordship's paper, as published in the Transactions of the British Association, were found in the shale or blaes which lies immediately above and in contact with the *Jewel Coal*, at Sir JOHN HOPE's colliery near New Hailes. But similar remains have also been found in the substance of the coal itself;—and not merely in the *Jewel Coal*, but in another seam of coal, at the same place, called the *Splint Seam*, which, at that colliery, is about 33 fathoms above the *Jewel Coal*. These teeth, scales, and spines, are generally about four inches down below the upper surface

* See page 261.

of this coal. The following figure will shew the exact part of the seam occupied by these interesting relics:

1, Is a thin seam of fine cubical coal half an inch thick.
2, Is a band of parrot coal from 2 to 4 inches thick.
3, Is the division between the parrot coal and the splint coal, and about one-fourth of an inch thick;—along this line are the fishes' teeth and scales, imbedded in coarse coal.
4, Coarse splint, $1\frac{1}{2}$ feet.
5, Good splint, $1\frac{1}{2}$ feet.



The fact now stated seems to be quite irreconcilable with the notion that the vegetable matter that was ultimately converted into coal, could have accumulated on dry land, or any where else than in an aqueous medium of considerable depth.

I do not at present allude to the inquiry, whether this body of water was salt or fresh,—or how the vegetable matter was transported. I wish here only to shew, that the coal-strata must have been, in common with the other members of the carboniferous series, formed by successive layers at the bottom of an aqueous medium of some kind or other. I have said that this body of water must have been of considerable depth: it is still more clear that it must have been of considerable extent. For as several individual members of the series have been traced throughout the whole of Mid-Lothian and East-Lothian, it is evident that it must have covered these counties at least, and probably washed the base of the present Lammermuir Hills, both on their north and on their south flanks.

The next question, in our attempt to explore the origin of the several kinds of strata constituting the coal-measures of this district, is,—From what sources were derived the elements or ingredients which compose these strata?

That they were all derived from one and the same locality, or even from the same quarter of the horizon, is extremely improbable. We can easily conceive that the greywacke hills of the Lammermuirs should have supplied the aluminous ingredients which compose the shales and clays of the district; and that the vegetation which covered them afforded, to a certain extent, materials for the deposits of coal. But it is obvious, that these hills could not have produced the enormous quantities of siliceous matter which compose the sandstone strata: for, in point of fact, silex enters, to a small extent, into the composition of these greywacke rocks; and, though it is probable that what is called the Old Red Sandstone has been, in a great measure, derived from the Lammermuir Hills, these red sandstones contain a much smaller proportion of silex in them than the sandstones of the coal-measures. Indeed, it is obvious, that, had there existed in the greywacke group as much siliceous as aluminous matter, the result of denudation, in these circumstances, would have been, not alternate layers of sandstone and shale, but a confused mixture of both in one and the same mass.

It is infinitely more probable that the elements of these several strata were derived from different quarters. Thus, it may be supposed that the siliceous matter was washed down from the primitive formations situated to the north and north-west, in Perthshire and Stirlingshire, whilst aluminous matter was washed down from the greywacke hills situated to the south; and that these several supplies were brought down at different periods, so as to give time for the deposition of one kind of sediment before the arrival of another kind. This hypothesis as to the particular districts from which the elements of the shales and sandstones have respectively come, is, of course, little better than conjecture, and is adduced merely in illustration of the possible explanation. It is, however, a confirmation of it, that the siliceous matter which pervades the district is, on the whole, much more abundant than the aluminous matter, and that the sandstone rocks are much thicker in the north-west part of the district than in the southern parts. From the same source, must of course have been derived that remarkable "saddle-back" of sandstone which I have more than once alluded to as running from New Craighall towards the south, and which has been deposited exactly in the direction to have been expected, if the supply of siliceous sediment came from the north.

It is proper, however, to observe, that this conjecture as to the arenaceous sediment having been brought from the west, is inconsistent with the position of the fossil trees imbedded in the sandstone rocks. In 1830, a tree was excavated from Craigleith quarry, 59 feet in length, having a diameter of 5 feet at its lower end, and 2 feet at its upper end. The tree dipped S. 70° E., at an average angle of 34°. The strata in which it was deposited dipped towards the E.NE., at an angle of about 13°. It seems probable, that the whole strata of this quarry have been raised by Corstorphine Hill;—on which account, we may assume, that the original dip of the tree was about 28°. It may be added, that the lower end had some appearances of roots,—and at that end, there was a sort of trough in the strata:—that is, they there suddenly dipped on each side of the tree towards the south and north.

In 1833, another fossil tree was discovered, about 300 yards to the west of the former. It is 32 feet in length;—but it has not yet been entirely excavated. The diameter of its lower end is about 3 feet, and of its upper end about 1½ foot. This tree dips S. 50° W., at an angle of about 46°. The strata in which it is deposited dip E. by N., at an angle of about 25°;—so that its original dip may be assumed at 32°. On several parts of the trunk, branches or the remains of branches are very apparent. It is only on the under side of the tree that any whole branches exist: there are none on the upper side, but there are several sockets where branches of large size have been, and from which they have been apparently torn off.

It will be observed, from the description now given, that both of these trees slope in nearly the same direction, having their heads towards the W.NW. If these trees were transported by currents, and were at length arrested by their roots, or sticking in silt or sediment, their upper ends, especially if any branches remained on them, would slope upwards in the direction of the current. If no other circumstance interfered, this would undoubtedly be the case; and if it appear that all or the great proportion of the fossil trees in the district have their tops towards the same quarter, they may be considered as affording a true and unequivocal indication of the direction of the current which transported them.

I have been unable to learn with any thing like precision the direction of the other fossil trees found at Craigleith. One was discovered very recently at Granton, the thickest end of which lay in a direction E.NE.

I have not said any thing of the origin of the limestone, a subject which seems as yet to baffle the skill of geologists. All are agreed that it was formed at the bottom of an aqueous medium, but, from what source the calcareous ingredients came, has not been discovered,—some imagining that it has been transported from a distance, like the sediment of shale and sandstone,—others that it has been suddenly produced by chemical agency of some sort. The difficulty of the former theory, in such a district as this, is to discover where the calcareous matter could have come from. There are not, near the district, any older limestones, by the degradation or attrition of which materials could have been provided for the creation of these carboniferous limestones; and, moreover, they thicken towards the Lammermuir Hills, among which there is not a particle of lime. It is indeed a fact of a very singular character, that the stratum of limestone which, in the north part of the district, does not exceed 4 or 5 feet in thickness, should regularly and uniformly thicken towards the south, and that, where it is close upon the greywacke range, it should reach a thickness of between 30 and 40 feet. In the former part of this memoir, I mentioned another fact, which I think ought here to be kept in view;—viz. that beds of shale, which in other parts of the district contain little or no lime, become towards the south "bastard limestones."

It appears to me, that these facts strongly support the theory of chemical agency. Water, when cool, can hold carbonate of lime in solution, provided there is an excess of carbonic acid. But, if heat be applied to the water, so as to drive off a part of the carbonic acid, a precipitate immediately takes place. Now, it is probable that the estuary which covered this district was warmer along the flanks of the hills than elsewhere, and for two reasons,—*one* is, that it was shallower, in consequence of which, the influence of the solar and atmospheric heat would there have more effect on it,—the *other reason* is, that, supposing the greywacke hills to have been elevated by subterranean heat, the water along their sides would be warmed, not only from being in contact with them, but also from being near the vents and fissures communicating with the source of heat. On this principle, the fact that the limestone stratum lying under the North Greens coal, is eight or ten times thicker near the Lammermuir Hills than at Duddingston, may be explained. But the theory now suggested may be applied, not merely to explain the particular phenomena here adverted to. It may serve to explain the deposition of limestone strata in all situations. We see, that during the period when the strata which compose this particular coal-field were being formed, there were altogether five or six deposits of lime. May this not have been caused by heat being communicated, from subterranean sources, to the aqueous medium holding the carbonate of lime in solution? At each accession of heat, there would be precipitated a stratum of lime, extending more or less over the whole district, in proportion to the general diffusion of the heat. This agent would be more efficient in its operations at first, *i. e.* before any great number of strata had accumulated; for, in proportion to that accumulation would be the *distance* of the aqueous medium from the subterranean heat, and the means of intercepting it. Accordingly, we find, that, in this district, the thickest deposits of lime are in the lowest parts of the basin, and the thinnest above; and further, that, in the upper half of the series, no lime strata exist at all.

In saying that, in the upper half of the series, there are no lime-rocks, we do not mean to say that the strata are entirely devoid of all calcareous matter. It is found that the sandstones and shales are in all parts of the series more or less impregnated with carbonate of lime. For example, in the Craigleith sandstone, a small proportion of this substance exists. It is a remarkable fact, that, in the fossil trees imbedded in the Craigleith sandstone, carbonate of lime should form more than one-half of their substance, and that oxide of iron and magnesia should exist also in a considerable proportion, whilst hardly any silex is to be found in them. It is very obvious, from these facts, that the sandy sediment had been deposited at the bottom of an estuary which held in chemical solution a large proportion of carbonate of lime, magnesia, and iron. The liquid containing these substances would soon make its way into the interior of the trees, though the grains of sand could not; so that, when the water evaporated, these carbonates would be left in the substance of the fossil.

The view above suggested, for explaining the greater thickness of the limestone strata near the Lammermuir range, might be employed to explain the large proportion of iron in the old red sandstone formation: for, if bicarbonate of iron was held in solution by the waters which covered the district, the higher temperature of the water along the flanks of the hills would drive off a part of the carbonic acid, and cause a precipitate of carbonate of iron, among the sediment accumulating in these parts.

With regard to the source from which the ingredients of the *coal* strata were derived, it is impossible to say any thing, without more extended observation. It certainly is a very remarkable circumstance, that *all* the coal-seams should be thickest in the north part of the district, and that they should all thin away towards the south; nay, that some of them should entirely disappear before they reach that limit. It is not an unfair conclusion from this fact, to hold, that there must have been a larger supply of vegetable matter in the north than in any other part of the district; and if this be so, it would seem to be a corollary, that the dry land which supplied, if not all, at least the largest quantity of vegetable matter, was also towards the north. I have already observed, that, judging from the accumulations of sandy sediment in the district, there was probably dry land to the N. and NW. Moreover, if it be true (and it does seem extremely probable) that the Fife coals form part of the same deposit to which the Lothian coals belonged, the theory now suggested receives strong confirmation; because there the coal-seams are still thicker than at Gilmerton, Niddry, and Wallyford. Mr Landale, in his valuable paper on the Fife coal-field, states, that one of the coal-seams at Dysart is 21 feet thick, and that there are three others, each of which exceeds 10 feet in thickness. In Mid-Lothian, the thickest seam is 14 feet thick; and the next in point of thickness is less than 10 feet.

There is still another subject connected with the origin of these several strata deserving of attention. How have the elements that compose them been transported? Have they been transported by means of rivers? This notion does not appear inconsistent with the conditions presented by the shales and sandstones, and, on the contrary, it is absolutely necessary, in order to answer some of these conditions, implying, as they do, the existence of powerful currents. How else could the enormous trees found in the Craigleith sandstone, at New Craighall, and at the Roman Camp, have been transported? Trees of such weight and size could not have been carried off from their native sites except by an agent of considerable power; and it is worthy of remark, that the strata in which they have been found, are invariably sandstone, which, as already remarked, indicates generally the prevalence of agitated waters. In what manner the supposed rivers became charged with such accumulations of sedimentary matter, and loaded with the spoils of primeval forests, it is difficult to imagine,—except on the assumption, that, by occasional, or periodical inundations, they overflowed their ordinary banks.

It is not quite so easy to conceive in what manner *the vegetables* that compose the actual *coal*-seams were collected and drifted down. We see from the tables and sections of this coal-field, that it was only at particular periods, during the formation of the whole basin, that these vegetable accumulations took place. Are we to suppose that, at these periods, a more extensive inundation than usual occurred, whereby a larger quantity of vegetable matter was carried off? This would imply a degree of violence, inconsistent with the perfect integrity of the plants (many of which are very delicate) preserved in the coal-seams. The circumstance that these vegetable debris would be mixed and entangled with sand and mud, whilst, in the actual coal-seams they are free from all such admixture, presents no difficulty,—because they would continue to float long enough to get quit of the soil attached to them. Besides, there might, even after deposition, be a separation of the earthy from the vegetable matter. But, would the quantity of vegetable matter floated down in this way, be sufficient, when it sunk to the bottom, to overspread the whole district,—(a district, be it remembered, in one direction fourteen miles in extent), so as thereafter to form one individual seam of coal? Look, for example, at the North Greens seam. It crops out for eight or ten miles, along the south side of the Pentlands,—and then, at Carlops, it turns round towards the SE., skirting the foot of the Lammermuir Hills towards the east. We must suppose that every part of the supposed estuary was entirely covered with a mass of floating vegetables; and not only so, but that this mass became gradually, uniformly, and regularly thinner towards the SW., S., and SE. I confess it is not easy to conceive such a condition of things;—for, unless it can be assumed that the sea in which this widely extended mass floated, was calm, and free from currents, the continuity of the mass, and its uniformity of thickness, must have been destroyed.

A good deal, therefore, depends on the extent and character of the aqueous medium in which the coal vegetables were floated, and at the bottom of which they were deposited. If it was a small and shallow lake, the waters of which were still and tranquil, there would be little difficulty in the problem. Or, if it was an arm of the sea, narrow and land-locked, so that its waters could not be agitated by the swell of the ocean or by extensive currents, the difficulties would not be insurmountable. But this could not have been the character and condition of the waters, at the bottom of which the strata of the East-Lothian and Mid-Lothian coal-fields were deposited. We have seen, that whilst towards the west they washed the base of the Pentland Hills,* towards the south they reached the Lammermuir Hills, and stretched alongst their range even as far as Dunglass, (a distance of about thirty miles), and covered the whole of the present counties of East and Mid Lothian. Moreover, into this expanse of waters we see that

* This is on the supposition that the Pentland Hills had been ejected and formed before the epoch of the coal-measures. If they were ejected afterwards, then the coal-seams must have extended much farther towards the west than they do now. I admit that it is by no means easy to determine whether the Pentland Hills were elevated before or after the deposition of the carboniferous rocks. On this point see some observations in the notes explanatory of the Map in the Appendix.

rivers flowed, charged with the spoils of the dry land;—and that these rivers were of great depth and magnitude, is obvious from the number and size of the trees which they transported.

It is impossible, therefore, to assume, that the waters in which the coaly vegetables floated and sunk were free from currents,—and currents of very considerable force. It is true that these currents may have been more powerful at one time than at another. When any great inundation took place, whereby immense quantities of vegetable matter were swept from the plains and marshes, the currents would be greatest;—and then there would be immediately a deposition, first of sandstones and next of shales. It would be some time before the vegetable masses would sink; and undoubtedly the waters would then have attained a more quiescent state. But still, unless it be supposed that the rivers were at times altogether dried up, there must have been currents to interfere greatly with the equal distribution of the vegetable matter.

Such would be the case, even on the supposition that the waters which covered the district were entirely fresh-water, and not subject to the action of the *tides*. But this would be a supposition far more favourable than the facts warrant. The existence of shells and zoophytes, undoubtedly *marine*, in beds of shale* and limestone, which occur in the lower half of the deposit, proves incontestibly, that the waters were at that period entirely salt, and therefore probably subject to oceanic tides. This circumstance, therefore, must be taken into account in considering the whole question.

I may here observe, that whilst the waters which covered the district were, during the deposition of the lower half of the strata, of decidedly marine character, they were latterly, in all probability, mixed with a larger proportion, if they did not entirely consist, of fresh water. It will be remembered, that it is in the upper series of coals that the two species of *Unio* occur, forming a bed extending for many miles. This change of character in the waters, is precisely what would be expected, if there were rivers which, either incessantly or periodically, spread over the bottom of the estuary large supplies of sedimentary matter. The bottom, as it rose in level, would gradually push back the sea, and thus alter the proportions of salt and fresh water, till little of the former remained. From this, another effect would follow, viz. the greater influence of currents, arising from the river-floods, so that in the upper part of the deposit we ought to find greater irregularity in the thickness of all the strata. This inference agrees

* In the shale which forms the roof the Rough or Kailblades coal at Bryants (situated about 25 fathoms above the North Greens coal), I have found a species of *Lingula* in great abundance. It appears to belong to a species undescribed. It resembles most the *Lingula Beanii*. (Phillip's Yorkshire, i. 128.) In the shale which forms the roof of a coal-seam near Rutherford Inn, (in the parish of Linton), I have found innumerable remains of the *Producta costata* (Phillips), with the spines well preserved. The coal-seam is double,—the upper part being 16 inches thick.

with the fact;—it is in the *upper* series of coals, that the sandstone "saddleback" of New Craighall, and the diamond coal (the most irregular of all), occur.

These views may, at first sight, appear inconsistent with the fact supposed to have been demonstrated by Dr Hibbert, that the limestone of Burdiehouse and the strata contiguous to it, lying at the very bottom of the coal-basin,* were deposited in waters nearly if not entirely fresh. But there is no real inconsistency. Burdiehouse is nearer to the Pentland Hills than any other part of the basin. It was therefore nearer the shore of the ancient sea, than the Gilmerton limestone was;—and it is not difficult to understand, how there may have been a greater admixture of fresh-water at the former place, than at the latter.

I remarked, in the first part of this Memoir, that, beneath every seam of coal, there is invariably a bed, more or less thick, of clay. This is perfectly consistent with the notion, that the coal-seams owe their origin to accumulations of vegetables uprooted and carried off, having attached to them a quantity of the soil on which they grew.

But whatever be the way in which the vegetables composing the coal strata have been brought, it is exceedingly probable, from the internal structure and organization of coal, that, after its deposition, the vegetable matter has been influenced by chemical affinities;—and this circumstance may have to a certain extent assisted, in creating a uniformity of thickness in the different strata. In one and the same seam of coal, it often, nay, it most generally happens, that there exist several different kinds of coal. For example, in the "Great Seam," there is (1), the rhomboidal, cubical, or cherry coal; there is (2), the splint or slaty coal; and there is (3), the conchoidal or parrot coal;—and specimens of these several varieties may be got in the same hand specimen. Now, all these possess more or less a crystalline structure; and, what is more, each of them has a peculiar crystalline structure of its own, each being separated from the other by a distinct line of demarcation, called, in the language of colliers, a "parting."

The peculiarity of crystalline structure which characterizes each kind of coal, depends (as was explained in the first part of this Memoir) on certain joints or fissures which intersect the coal, and which intersect the coal at different angles in each variety.

There must, of course, have been some important difference in the constitution and condition of the vegetable matter which produced these different varieties of coal, and gave to each a peculiar crystalline structure. Accordingly, it appears, from the analysis of Dr Thomson, that each kind of coal has a different organization. His analysis shewed, that the following were the proportions of elementary substances in the different kinds of coal.

* See Appendix E.

	Carbon.	Hydrogen.	Oxygen.	Nitrogen.		
Splint-coal, . .	75.00	6.25	12.50	6.25	=	100
Cherry or Rough-coal, .	74.45	12.40	2.93	10.22	=	100
Cannel or Parrot-coal,*	64.72	21.56	0.00	13.72	=	100

It is very remarkable to observe the different proportions of hydrogen in these several kinds of coal. It is enough for my present purpose to shew, by reference to the elements as well as the crystalline structure of coal, that each variety is perfectly different in its constitution and organization.

Now, how is this difference to be accounted for? Will it be said that it may have been occasioned by differences in the character of the *vegetables* which compose the coal? This notion was lately started by Mr Hutton of Newcastle,—and nothing is more likely. But these different vegetables were of course not originally deposited, in separate layers. They must all have been blended together, when they settled down and formed a pulpy deposit, at the bottom of the estuary in which they had been floating. How, then, did they afterwards come to separate into distinct seams? What agent put the elements of the vegetable mass in motion, so as to make them form new combinations? Would subterranean heat have that effect? On this subject, it would be very desirable to have experiments to refer to, instead of having to offer merely explanations which are little better than conjectures. At the same time, there are many circumstances which *a priori* support the view just thrown out. We know that, shortly after the period when these carboniferous strata were deposited, there was a great evolution of subterranean heat; and it is impossible to doubt, that in rising up through the sedimentary strata, it would effect important changes in their organization and structure. For example, it would cause them to contract in size or volume, by expelling from them much of the water with which they were impregnated; and they would not contract, without having cracks and fissures formed in them, whereby they would of course acquire the outlines of a rude crystallization. Moreover, the same agent may explain the formation in these fissures, of the veins of carbonate of lime, iron, and magnesia, described in the first part of this memoir. I have already alluded to the great probability, that, at this period, the sea in which these strata were deposited, held many of these substances in chemical solution,—as it still holds some of them, to a small extent. In that case, the immediate effect would be, whenever heat reached the vegetable deposit, to expel a portion of the carbonic acid, perhaps also to evaporate a portion of the water which was previously in the fissures; and thus leave in them films of carbonate of lime, iron, and magnesia.

I need hardly add, that the views now thrown out, would explain the occur-

* In the Appendix F, will be found a statement of some experiments recently made, which shew how various are the proportions of hydrogen in different kinds of even the same sort of coal, viz. parrot-coal.

rence of these substances also in all the other stratified deposits, whether in fissures or in drusy cavities—inasmuch as all these deposits must have been saturated with the water in which they settled down.

Before leaving this part of the subject, let me, in a single sentence, advert to the origin of the hydrogen which forms so important an ingredient in most coals. I say in most coals, for it does not exist in all kinds of coal. For example, anthracite or Kilkenny coal wants it entirely. It exists, however, in all the varieties of coal which occur in the Lothians, and in the proportions previously stated. This gas is the carburetted hydrogen, which in some collieries proves so fatal by explosions, and is called " fire-damp" by the miners. Some suppose that it is generated in old wastes by the decomposition of water, the hydrogen of which unites with the carbon of the coal. But it has recently been discovered by Mr Hutton of Newcastle, that the gas exists in the substance of the coal itself, being contained in small cells, only discoverable by the microscope. He is of opinion, that it may exist in these cells even in a liquid form, in consequence of the great pressure to which it is subjected. It is in this way that the blasts are accounted for, which occasionally take place in the English and Ayrshire collieries;—for by the working of the coal, the pressure is removed, and the hydrogen immediately passes from a liquid into a gaseous form.

Hydrogen gas is therefore an original constituent element or ingredient, of the coal strata in which it occurs; and it is not generated by external causes, as the decomposition of water. If the latter theory were true, this gas should occur in all kinds of coal,—but it does not.

That the hydrogen gas contained in the cells of the coal has been derived from the gums and resins of the vegetable matter which formed the substance of the coal, is not only probable, but obviously true. But I would venture to express a doubt, whether the fire-damp of coal mines may not, in many cases at least, come from a totally different source. It is well known that this gas is evolved in many parts of the globe, where no coal exists. Farther, and what is more to the point, it is evolved in this very district from subterranean sources. This is the case at Prestongrange. It comes up through the fissures which intersect the strata at that place, where they are in contact with basalt and greenstone. The quantity is so considerable, that, when the coal is worked near the trap-dyke, safety-lamps must be used. It is a strong corroboration of the view now submitted, that in no other part of the Lothians is this gas known, at least in the form of fire-damp. Whilst, on the other hand, in Fifeshire, Stirlingshire, and Ayrshire, where the strata are riddled by trap-dykes, this inflammable gas is very abundant. These views suggest a possible origin for some of the hydrogen gas, with which the coal itself is impregnated; for if it was evolved in large quantities from Nature's subterranean laboratories, before the vegetable deposits had become hardened and consolidated, much of the gas might be retained by them.

It was mentioned in the former part of this Memoir, that both in the "North Greens" and "Great Seam," the parrot band is thickest at and near Niddry, and that it thins away, both towards the north and the south, till at length it disappears, and its place is occupied by a bed of shale. This fact may be accounted for, by supposing, that the vegetable matter was more abundantly supplied in one part than in another.

2. I have thus attempted to throw out some general views, as to the way and manner in which the strata of the district were originally deposited and formed. The *next subject of inquiry* is the *history* of these strata, with reference to the changes and convulsions they subsequently underwent.

This branch of the subject is no less extensive, and hardly less dark, than the one just treated of. But it is perhaps possible to catch a few gleams of truth, regarding the most striking and obvious of these changes.

That these strata were deposited in positions exactly horizontal, is extremely unlikely. The deposits would, near the shore of the ancient sea, slope from the land at a greater or less angle. It is believed that, if the inclination to the horizon does not exceed 20° or 30°, sand and mud deposited on it will form regular layers. Now, the strata on the south-west, south, and south-east quarters of the district, slope from the hills, but they do not slope at a greater angle than 10° to 15°; and, in most places, their inclination is only 5° or 6° to the horizon. The amount of their slope is, therefore, no proof that *they* have been elevated since their deposition. On the contrary, it affords some evidence that they now are, as they were originally deposited. This observation applies to all that part of the district which extends from the sea-shore to the north of *Gladsmuir*, round by *Penstone, Pentcaitland, Ormiston, Cranstone, Middleton, La Mancha, Coaley Burn*, and the *Bents.*

But the case is widely different on the west side of the basin. There, in very many places, the strata slope down at an angle of no less than 80°, and at some points they are exactly vertical. This fact alone affords irrefragable evidence, that, after their deposition, some prodigious force was applied, whereby these strata were tilted up, and forced into a new and unnatural position. It is hardly necessary to say, that the numerous hills of trap which occur on the west side of the district, are quite sufficient to have caused the elevation now alluded to. In the sea, there have been outbursts at *Inchkeith* and *Inchcolm*, (not to mention smaller islands): on the land, there have been outbursts at *Lochend, Calton Hill, Castle Rock, Arthur Seat*, the *Braid Hills*, and indeed along the whole of the western side of the district.

We are thus brought to the undeniable conclusion, that, after these limestones, coal-seams, and other sedimentary strata, were deposited and formed, an epoch of subterranean convulsion arrived, which was ultimately characterized by the eruption through these strata, of enormous masses of molten lava. It is true that Arthur Seat, and the other trap-hills in that neighbourhood, are beyond the precincts of the present coal-field; but there is the clearest evidence, that before the sedimentary strata had been thus burst through and shattered, they had extended over all the district now occupied by these trap hills. This is not the proper place for enumerating the facts which shew this; and, therefore, I will merely state the results.

(1.) Seams of coal, ironstone, besides various other strata, such as usually occur in the Dalkeith coal-basin, are found to the north and west of these trap hills; and, generally speaking, they all rise towards these hills. For example, on the east side of the *Calton Hill*, beds of conglomerate or puddingstone, which are the very lowest members of the series, if they do not belong to the old red sandstone formation, are to be seen dipping towards the east, or away from the hill; and they increase in dip as they approach the top of the hill. To the east of the *Castle rock* the strata dip in a similar way. On the north side of this rock, the strata dip *north*,—as may be seen in the Prince's Street gardens,—as was seen when Hanover Street was built,* and as was seen during this winter (1837–8) in Castle Street.† In the Water of Leith, above and below St Bernard's Well, beds of shale occur—dipping north; and some years ago, coal was worked in Mr RAEBURN's property a little to the east of St Bernard's Crescent. In the rivers and burns near Colinton and Slateford, many strata of shale and sandstone, including thin seams of coal, occur dipping to the west.

(2.) A still more unequivocal proof that the trap hills of Edinburgh and its neighbourhood were ejected, after the formation of the coal strata, is afforded by fragments of these strata found on the tops of these hills, enveloped in the trap. On the SE. part of *Arthur Seat* (and at a height of from 400 to 500 feet above the sedimentary rocks), there occur masses of sandstone, conglomerate, and limestone, which have been torn from the basin, and carried up by the trap. On the top and at the west side of *Craiglockhart Hill*, there are imbedded masses of sandstone.‡ In the *Calton Hill* pieces of glance-coal have been found, which are supposed to have been taken up by the trap in its passage through some coal seams.

* The strata here consisted of sandstone, shale, and limestone. A mass of greenstone had intruded itself among them, and formed several fissures, which were filled with trap. This fact I became acquainted with by a sketch, taken by JOHN CLERK of Eldin, the intimate friend of the celebrated Dr HUTTON. This sketch is now in the possession of Mr CLERK's son, WILLIAM CLERK, Esq. Advocate; and he has besides it several others, also taken by his father, which are of high geological interest. On my suggesting that he would confer a great benefit on science if he would allow these sketches to be published, he expressed his willingness to do so, and stated that he would be happy if the Royal Society of Edinburgh thought them deserving of their notice.

† At this spot, situated in North Castle Street, felspar porphyry, of a white colour, and containing iron-pyrites, was found.

‡ I state this fact on the authority of Professor FORBES.

Under these circumstances, it is obvious what extraordinary force and violence the trap must have exerted, before it burst through the sedimentary deposits that overlaid it. These deposits attain the thickness of nearly 1000 fathoms, and how much more there may be below the North Greens coal (to which that measurement extends), it is difficult to say. There was not merely the *weight* of these accumulated deposits to be overcome before the molten matter could get vent;—there was also the *tenacity* arising from their partial consolidation, to be overcome. It is impossible, therefore, to imagine that the outburst could have taken place, without there having been previously stupendous upheavings of the strata, at and around this part of the district. The pressure would of course not be confined to one spot. The igneous matter which eventually came up, and came up not in one place, but at several places far removed from each other, and in enormous quantities,—shews, that it must have extended beneath these sedimentary strata to a considerable distance. The consequence would be, that during the prevalence of the subterranean pressure, these strata not being every where of equal tenacity or weight, would suffer extensive upheavings and oscillations. The strata of the entire district, would be, in a manner, floated or buoyed up upon the surface of the subjacent volcanic matter, and would undergo tremendous fractures and dislocations.

When the outburst of *Arthur Seat* and the trap in its vicinity took place, let us consider what would be some of the most obvious effects on the strata to the east of it.

(1.) In the first place, there would be a great hollow produced below the sedimentary strata, in that part where the trap previously existed. Here, it is important to remark, that a great portion of the trap which flowed out, appears to have come—not straight up from beneath, but rather in a slanting direction from the eastward. It is difficult to account for several of the phenomena of Salisbury Craigs (especially where the strata have trap interposed between them), except on that supposition. Now, what would be the consequence of an immense hollow being produced under the coal-measures to the east of Arthur Seat? There would evidently be a sinking of the sedimentary strata. The tilting or turning up of the edges of the strata would not be the only effect;—there would be a general sinking of the deposit *en masse*, in that part at least which was immediately over the hollow caused by the outburst of Arthur Seat, Blackford Hill, &c.

Before going farther, let us see, whether these effects tally with what is the fact. It has been explained, in the first part of this memoir, that the Esk coal-basin is greatly deeper and steeper than the Tyne basin. The Great Seam of coal runs through both, and the respective levels of that seam, in the centre or trough of each basin, may serve to illustrate in some degree the point now adverted to.* The Great Seam is—near Cockenzie, about 20 fathoms below the sea-level; whilst

* See in illustration of what is here stated, the sections on plate XV.

at Fisherrow, it is about 500 fathoms below the sea-level, being a difference of no less than 480 fathoms within the short interval of three miles. If the coal had sloped from Cockenzie gradually to the *west* throughout these three miles,—this difference of level might have been explained in another way. But it will be remembered, that it *rises* from Cockenzie to *Prestonpans and Tranent*, and that it is from that locality, or rather from one still farther west, that the Great Seam begins to dip into the Esk basin. I imagine, therefore, that if the Esk coal-basin has not actually been created, solely and exclusively, in consequence of a hollow having been formed under it, it has at all events been thereby made vastly deeper and steeper than it would otherwise have been.

(2.) The sinking would of course be greatest where the sedimentary strata were nearest the volcanic outburst. This also is remarkably confirmed by the fact. The centre or trough of the Esk basin runs from Fisherrow in a line about SW., which is no where very distant from the course of the North Esk river. Now the basin is far deeper at the north end of this line at Fisherrow Harbour, than towards the south, as, for example, at Roslin and Lasswade;—and why? Evidently because at the former place, it is within *two* miles of Arthur Seat,—whilst at the latter, it is more than *six* miles from the Braid, Blackford, and other trap hills, skirting the SW. limits of the coal-field. It is hardly necessary to remark, that the same circumstance accounts for the excessive steepness, or rather the verticality, of the strata at Edmonstone, Niddry, and Duddingstone, —and their comparatively less inclination at Gilmerton, Loanhead, Dryden, and other points in that direction.

(3.) In considering what would be the effects of an outburst from under the strata, it is proper not to confine our regards to the sinking and change of dip on the west side, but also to the production of similar phenomena on the east side of the basin. These would of course not take place to the same extent. Whereever the limit was, to which the subterranean lava reached,—beyond that limit there would be no sinking; and the strata immediately within the limit, would slope down there, less rapidly than at the opposite side, in the immediate vicinity of the eruption.

Any one who has followed this reasoning, must see, that it readily explains the origin of the ridge of high ground, which runs from the Frith of Forth at Prestonpans to the Roman Camp by Carberry and Fuffet. That ridge forms (as previously mentioned) not exactly a straight, but a curved line, the inner parts of which face Arthur Seat. Moreover, the reasoning just sketched, explains the fact, that alongst the inner sides of that ridge, the strata dip down less rapidly than the strata on the west side of the Esk basin,—and more rapidly than the strata on either side of the Tyne basin.

(4.) It is obvious, that the sinking down of the strata, to such an extent as I conceive those of the Esk basin to have sunk,—whilst the rest of the district to the east remained fixed and stable, could not have happened without having produced fractures along the line of the ridge just described.

This deduction is also consistent with the fact; for it has been explained in the first part of this memoir, that at Prestonpans the Great Seam of coal which had once been continuous is broken across, and there is a gap or chasm between its ridges of several hundred yards. This gap increases very considerably to the south. At Chalkieside and at Fuffet the *limestone* which lies below the North Greens—and therefore very far below the "great seam," may be seen broken across, and dipping in opposite directions. At this last spot, there is about fifty yards of interval betwixt the edges of the fracture.

(5.) All the effects now deduced, would follow simply from the sinking of the strata by their own weight. But the production of these effects would be greatly aided, by the play of another power, which must have operated to a certain extent. I allude to the *lateral pressure* of the igneous rocks during their eruption. They must have squeezed the whole basin towards the east, and so either have pushed the central parts of the basin down,—or else have caused the strata on the opposite or eastern side of the basin to rise up and form a ridge. It is possible, that the formation of the Tyne basin may actually be owing to this very effect, because it is easy to see, that a ridge would thereby be produced to which the strata on each side would rise. If on the east and south side of that ridge, the strata did not dip down towards the Tyne, at a steeper angle than 20° or 30°,—lateral pressure would not be necessary to explain the facts. But at Cousland they dip towards the SE. at an angle of about 60°,—and at Blinkbonny they dip to the S. at an angle of about 40°. These facts cannot be accounted for by deposition merely.

About the middle of this ridge,—and in particular, between Fordel and the Roman Camp, there are several minor troughs, which form, on the map, loops in the line of outcrop, of the coal and limestone. These may be accounted for, by supposing, that there were in some places, two anticlinal lines of elevation. The strata between these lines would, of course, be formed into small basins.

The strata are found occasionally along the Roman Camp ridge, in a soft or sandy state. At Bryants, a coal seam, which is known usually to be strong and solid, is so friable and shattered, as to be unfit for working. A thick sandstone stratum adjoining it, was found to be in the same state. They had been crushed, by the enormous lateral pressure. The pressure must have been against the *edges or ends* of the strata, to have produced the effects observed. A pressure on the *surface* of the beds, would have only consolidated, and not broken them.

(6.) Having thus endeavoured to shew, what would be the shape and condition of the coal-basin in its several parts, arising from the outburst of the igneous rocks of the districts, I proceed to consider other effects of a more local, though no less interesting, description.

I allude now to the formation of *dykes* and *slips*.

It is obvious, that consolidated strata subjected to the subterranean and lateral violence above explained, must have suffered very numerous and very extensive cracks and dislocations. If they subsided at all, their mere subsidence would alone be sufficient to occasion such effects;—and observe, where these fractures would commence. When in the act of bending, the fracture would necessarily commence at the external or lower side of the arch;—and thus it follows, that both slips and dykes would originate from below, and not from the upper part of the basin. It is clear, therefore, that any slips or dykes which are seen at the surface must be unfathomable, that is to say, they must reach to the very lowest of the sedimentary strata; and accordingly (as mentioned in the first part of the memoir) no instance exists where, in going down from the surface, a slip is found to stop or cease. On the other hand, one instance was mentioned of a slip, which does not rise so high as the surface. This is easily accountable on the principles just noticed.

I may here take notice of another phenomenon, which appears to me to depend on the same principles as the slip just alluded to. In a pit called the Mucklits pit, in Sir JOHN HOPE's workings at New Craighall, the coal-seam there (the *Splint*) is intersected by frequent dykes or "gullets"* of clay—which, however, rise no higher than the *roof*, consisting there of a micaceous sandstone. The pavement of the coal is clay, in quality exactly the same with the gullets. These clay dykes run for considerable distances, and vary in width from a few feet to 15 fathoms. They have all the appearance of being a portion of the pavement squeezed up into the substance of the coal; and this idea is strongly confirmed by the fact, that they make their appearance exactly in that part of the basin where the splint coal is most curved and fractured,—that is to say, in the very trough of the basin where the strata rise to the NW. and to the SE. They are known in no other part of the district. The coal in bending there has, on account of its brittleness, cracked, and as the cracks would commence at the lower part of the seam, the clay of the pavement, in consequence of the enormous pressure, instantly rose up, so as to fill and widen the cracks. The clay could not, however, rise higher than the roof, because the sandstone which lies above the coal, from its slaty and micaceous qualities, would bend without cracking.

(7.) If the formation of slips and dykes is attributable to the violence of the subterranean and lateral action, which preceded and accompanied the eruption of the trap-hills, we should naturally expect to find them *most numerous* near the eruption; and there also the slips and dykes should produce the *greatest derangement* or dislocation of strata.

This inference is consistent with the fact. On inspecting the accompanying map it will be seen, that *all* the *dykes*, and the *greatest number* of the slips, are

* This is one of the terms given by the pitmen to these clay-dykes—another term is "lunker."

in the north part of the district. Further, the table of slips shews, that the most formidable slips—that is to say those which run farthest and produce the largest dislocations, are also in the north part of the district. There are 98 slips which run in a north and west direction;—and only 21 which run in a north and east direction.

(8.) Following out the same reasoning, it is not difficult to account for the general direction of the dykes and slips that intersect the district,—when their several positions are examined.

Reasoning *a priori*, it is evident that a crack or fracture would originate at and run from the point, where the force or strain was the greatest. If, on the west side of the basin, there was a lateral pressure, which did not act every where with perfect equality,—one part of the whole mass would be pushed more to the east than another part, and the result would be that the cracks would run in an east and west direction.

This result is susceptible of mathematical demonstration. Mr HOPKINS of Cambridge has, in the very able memoir lately published by him, afforded it. More striking proofs of the correctness of that demonstration, as well as of the truth of many of his deductions, could hardly be wished for, than are exhibited in the district which I am now describing.

It will be seen on looking at the map, that the dykes and slips all point towards the particular trap-hill or hills nearest them, and which, from being nearest, were probably most instrumental in elevating and dislocating that part of the basin intersected by these dykes and slips. The Niddry dyke, for example, which runs about W.NW. bears on Arthur Seat. The dyke at St Bernard's Well, which is on the opposite side of that hill, also bears on it, running in a direction S.SE. Proceeding farther south, we find the slips cease to bear upon *Arthur Seat*, and that they point upon other hills of trap—such as the *Braid* and *Blackford Hills*—they having been the means of elevating *these* portions of the basin. In short for seven or eight miles along the west side of the district, a series of slips occur, all bearing on the trappean masses nearest to them, and which it is reasonable to suppose, were instrumental in elevating the strata which these slips intersect.

It will be observed, that along this side of the basin the slips are, generally speaking, parallel. The reason is obvious. The force which acted upon these seven or eight miles of the basin, did not proceed from one central point or focus; it acted continuously along the whole of that western side. If it had acted from a single point, instead of a succession of points, the slips must have converged on the common focus from which they originated.

This last proposition is made very apparent, by attending to the slips which intersect the strata of the Roman Camp, which (as already explained) runs from Tranent, and divides the basin of the Tyne from the basin of the Esk. This ridge does not run farther west than Stobshill, which is on the south-west side of the Roman Camp,—there the strata dip SW.—whilst at Bryants they dip NW., and at Edgehead (on the opposite side of the hill) they dip SE. The same strata, therefore, absolutely surround this hill in three parts out of four, dipping successively towards three points of the horizon. The North Greens crops out near the top of the hill on all sides, and at several places the limestone which underlies that coal, may be seen coming up from opposite sides. Now, on examining the direction of the slips running through the strata of the Roman Camp hill, it is found, that they all more or less converge to the top of the hill;—that is, they, generally speaking, coincide with the dip and rise of the strata. If the hill had been formed by a protrusion of trap in the central parts, so as to occasion a pressure towards the circumference, the facts just stated might be accounted for. The central force acting laterally on the adjoining strata, would also act unequally in consequence of the unequal tenacity of these strata. Dislocations would be produced by this cause. But we have shewn it to be possible, nay probable, that the Roman Camp ridge may have been produced, not by a protrusion of trap (of which there is not a vestige either on the surface or in the underground workings,)—but partly by the subsidence of adjoining strata towards the north, and partly by a thrust of them upwards from the west. This last movement would equally produce dislocations running towards the ridge. Subsidence alone would cause them to assume a different direction, for in that case the tendency of the force being to separate the strata, the fractures would be parallel with the line of crop. This is very clearly shewn by Mr Hopkins in his valuable memoir above alluded to.

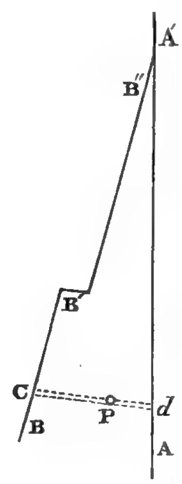
In whichever way, therefore, the quaquaversal dip of the strata at the Roman Camp has been produced,—whether by an elevation of the centre by subterranean trap as Dr Hibbert supposes—or by a push upwards from a more distant quarter, it is evident that the dislocating forces would act in such a manner, as to cause the slips to converge more or less towards a common centre.

After these explanations, it must be very apparent why all the dykes, and nine-tenths of the whole slips, run not far from a direction NW. and SE. The chief eruptions of trap which have elevated and dislocated the strata of the district lies to the NW. of it, and of course the dislocations have acquired that direction.

(9.) In connection with this branch of the subject, viz. the direction of the slips, I would remark, that though in general the slips of the district are all either parallel to each other, or converge towards a common centre, there are a few (and some of them are marked on the map) at right angles to each other. This arises (as Mr Hopkins distinctly shews) from the effect of different forces. Those which run parallel with the crop of the strata, shew a divellent force; those which are coincident with the dip (and these are the most numerous), shew unequal lateral pressure.

The operation of these two forces would, when combined, produce an almost inconceivable number of dislocations. This will appear the more evident if we observe what must be the consequences of single dislocations, from whatever cause produced. Suppose that a fissure takes place, intersecting all the strata for a mile in horizontal extent, and to a depth of more than 100 yards. The strata on one side or the other of this slip would be pushed up or would sink down, according to the nature of the force which produced it. But it is difficult to conceive, that under such circumstances, this would be the only dislocation produced. Others would take place in the district, some of which would probably intersect and unite. The slips marked on the map 112 and 113, are an example of this. It will be seen from an inspection of the table, that the strata on the outside of these slips have sunk down, leaving the triangular parts comprehended between them standing up. Suppose that the parts comprehended between that triangle and the slips 61 and 62, had sunk down simultaneously. Unless the slips 61 and 62 are prolonged indefinitely,—there must have been a cross fracture produced somewhere,—beyond which the sinking did not extend. Then there must in like manner have been a fracture intersecting 112 and 113; so that, in general, every dislocation would be accompanied by, because it would occasion, several others.

There are very many examples in the district, of slips changing their direction. Two of these, near each other, are marked on the Map, and relative Table, on both of which they are numbered 73. Another instance may be mentioned, which was lately observed on opening a pit on the Edmonstone property, called on the plans *Pit* 23. After sinking the pit P, a drift Pd was made towards the NE. When at a distance of 10 yards, a slip A A′ was discovered running N. 35° W., and throwing the strata down on the NE. side from 15 to 20 feet. Another drift Pc was made from the pit bottom towards the SW., when at a distance of 22 yards, another slip B B′ B″ was discovered running N. 50° W. This slip, at the distance of about 34 yards from the pit, took a turn B′, and ran N. 45° E. for a short way, after which it ran on in nearly the same direction as before. At B″ (about 154 yards from the pit) the strata were found to be downcast on the NE. side 12 feet. At B the amount of downcast is 15 feet. The slip B B′ B″ unites with A A′ a little to the NW. This last-mentioned slip is known to run a considerable way farther in that direction. The strata which these two slips intersect dip towards the SE.



It is probable, that the slip B B′ B″ was formed subsequently to the slip AA′. As the amount of dislocation produced by the former, is towards the SE., or trough of the basin, that slip may have been occasioned by a subsidence of the parts situated towards the NE. side of B B′ B″. The deviation of this last slip at

B out of its regular course, has been possibly caused by its proximity to the slip A A′, which had shattered or weakened the strata at B′; but the progressive direction of the original force had been such as to prevent the crack from making a greater deviation, and compelled it to resume its former course towards B″.

(10.) Another important subject of inquiry, regards the amount of vertical derangement of the strata caused by the slips.

It is obvious, that when the continuity of the strata was broken by a fissure, reaching from the lowermost up to the very highest of the sediment-strata, there would be a tendency in the strata to sink down on the one side or the other of the fissure. Is it possible by any fixed principles to judge on which side of a slip these strata *would* sink down? We have seen that nearly nine-tenths of the whole slips in this district run in a north and west direction. The table I have constructed shews, that whilst there are fifty-two slips which throw down the strata $861\frac{1}{2}$ fathoms on the *north* side, there are thirty-seven slips which throw down on the *south* side, and that to the extent of only 402 fathoms. The result on the whole is, that the number of fathoms that the strata are thrown down towards the N., is more than double the number of fathoms that are thrown down towards the S. Can any explanation be afforded for *these* facts? On this subject I throw out the following views.

It has been explained, that two causes have operated in producing dislocations of the strata—*subsidence* and *lateral pressure.* Now we have shewn, that the subsidence must have been greatest towards the north, because subterranean action and volcanic eruption were most prevalent there. Hence when dislocations took place across the strata, the strata would, generally speaking, sink more on the north sides of the slips than on the south sides.

I may here observe, that in the Fife coal-field, described by Mr Landale in his interesting account of it (published in the Highland Society's Transactions), an opposite rule prevails. Mr Landale says that the slips there "are innumerable; and with one exception, all the large ones throw *up* to the *north;* and point out to us in the most decisive manner, the very steps by which the county has been raised to its present level."* Now, in the Lothians, the throw *up*, as has just been shewn, has been most frequently on the south side of the slips. I have, however, spoken not so much of the side on which the strata are thrown *up*, as the side on which they are (to use the same form of expression) thrown *down;* and applying that symbol of expression to the Fife coal-field, then it would appear that the large slips mostly all throw down on the side next to the Firth of Forth, in the same way as happens in regard to the largest of the Lothian slips. In the Lothian district, it appears to me that most of the facts can be explained by supposing that the strata after being broken across, slipped down by their own weight on one side or other of the fracture; and it is probable, that the

* No. 33, p. 346.

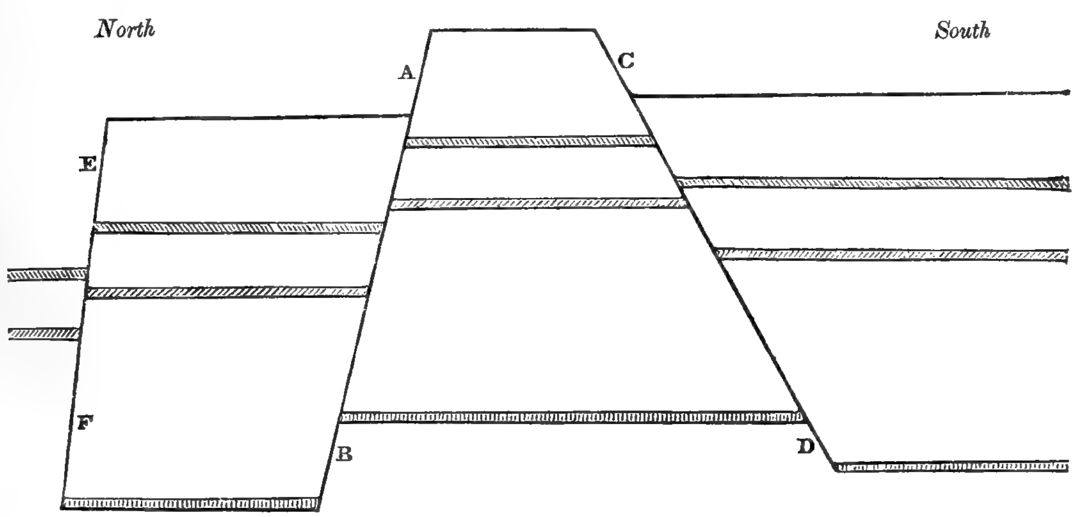
same theory may be sufficient to account for most of the phenomena in Fife; though it is quite possible (as Mr LANDALE supposes), in regard to some slips which are in the immediate neighbourhood of trap-hills, that the strata were actually lifted up on one side, whilst they stood fast on the other.

There is another important fact connected with the derangement of strata by slips, which appears to be established by the table in the Appendix. In the Lothian basin, the *amount* of *derangement* or displacement of the strata is found, generally speaking, to increase towards the *dip* of the strata. The reason is probably to be found in the circumstance, that the intensity of the force caused by *subsidence* was greater than that caused by *elevation*. The latter force would act at or near the sides of the basin, where the trap hills occur; and all the slips produced by that force, would exhibit a larger amount of dislocation towards the crop of the strata. The former force would act chiefly in the central parts of the basin, and the fissures caused by it would increase in amount of dislocation towards the centre.

Many of the phenomena, described in the previous part of this Memoir, are accountable, on the supposition either of an elevating force acting alone, or a subsiding force alone;—as, for example, the occurrence of ruts and scratches on the sides of slips. But these appearances in the Sheriffhall slip, (which run not in a *vertical* but in a *slanting* direction, forming an angle of 20° or 30° with the horizon), show, that this force acted in a manner neither precisely vertical nor precisely horizontal, and that they were undoubtedly caused by lateral pressure. We have shewn, that some of the slips were formed by an elevating force, and others by the force of subsidence. The Sheriffhall slip was probably produced by the latter force;—for the strata on the north or lowest side *are near the slip*, all tilted up, and much shattered.

There are many other interesting phenomena connected with slips, which I am constrained to pass over. It would be detaining the Society too much with mere details, to dwell longer on the subject. I cannot leave it, however, without adverting to one phenomenon, which distinguishes the slips from the dykes of this district. I formerly adverted to the fact, that along the *trap-dykes* there is no derangement of the strata on either side; and that it is only along slips, that the strata are thrown down or cast up. This fact is the more curious, if it be true that slips and dykes have both been produced by the same cause, viz. violent subterranean action. This action, we have seen, was occasioned, or at least accompanied, by great accumulations of trap in a state of fusion, which were pressed upwards against the superincumbent sedimentary rocks. One consequence of the enormous weight and pressure which the igneous matter had to sustain, must have been to force it up—not only into the soft strata of clay and shale, and thus form layers or beds between the harder strata,—but also into the vertical cracks and chinks formed in the strata by their elevation and subsidence. By these means, the trap would in several places, be able to find its way even to the very surface of these strata, though in most instances, it would not reach that height before cooling and becoming thereby arrested in its course. Now, it is obvious that the trap when forced up into cracks and openings in the strata, would act as wedges on the adjoining strata, and prevent them from slipping down, as they might otherwise have done. The broadest or thickest part of the wedge would be below,—so that it would occupy the position, and assume the form, best calculated to produce the effect referred to. This explanation is strongly confirmed by what is observed along the Niddry dyke, and especially at that end of it where it thins away, and at last becomes a mere slip. Exactly in that part of its course, viz. *where the dyke turns into a slip,—a derangement of the strata commences.* Along the slip,—into which the trap had not flowed, the strata are down on the north side, 15 fathoms below the corresponding strata on the opposite side.

There is one other phenomenon, which I must notice before I altogether take leave of the subject of slips. In the first part of my memoir, I alluded to the very curious circumstance, that if a slip dipped or sloped towards any quarter of the horizon (instead of being exactly vertical), it was invariably found, that the strata (if deranged in position) were thrown down on that side towards which the slip dipped. The slip, for example, last mentioned, namely, the prolongation of the Niddry dyke, dips or slopes towards the north; and it is on that side that the strata are lowest, or have sunk down. If it had sloped towards the south, the strata would have been lowest on that side. It appears to me, that this phenomenon is susceptible of the following explanation; but I offer it with great diffidence, knowing that no satisfactory explanation has been reached by those who are far better qualified to treat of these matters:



Let AB, CD, EF, represent three slips, respectively inclined from the vertical plane. The strata, by these slips, must have been (as above explained) fractured from the very lowest member of the series. Suppose, that, in the first instance, the slip AB was produced,—among strata previously unfractured, and floating on a mass of liquid volcanic matter, violently pressed up—and suppose farther, that this fracture incline or slope to the north. Let us see how the strata, after being thus severed, would be supported on each side of the slip. On the north side, it is obvious, that the lower part, or basis of the strata, became less in breadth than the top, so that their foundation was narrowed;—and the question occurs, whether that foundation was as able as before to bear the weight of the superincumbent strata. It is true that the truncated edges of the strata on the north side of the slip, were resting on the surface of the slip; and this new support might *partly*, though not entirely, compensate the other element of weakness just referred to; but the slip would cease to afford any support whatever, if there was a *continuous moving*, or *undulation of the strata* on either side, arising from unequal or lateral pressure. In that case, the strata on the north side would obviously have a tendency to slide down the upper surface of the slip,—and would continue to sink, until the compression was such as to prevent any farther sinking.

In regard to the south or under side of the slip, it is obvious that the case would be entirely different. So far from the foundation or lower part of the strata being narrowed, it would be widened; and so far from the superincumbent weight being increased, it would be greatly lessened. The strata on that side of the slip, would therefore have no tendency to sink.

We have supposed the slip AB to slope a little from the vertical. If it is exactly vertical, it is evident that no tendency to sink would be occasioned more on one side than on another;—and this is found to agree with observation.

Suppose, then, that another slip takes place to the south of the slip AB, viz. through the strata which remained firm, and sloping to the south, it is evident, that the stratified mass between the old slip and the new one CD, will have no tendency to move. It will then become pyramidal in shape, and acquire several qualities calculated to render it firm and stationary. On the other hand, the strata to the south of the slip, will acquire a tendency to sink down on that side,—for the reasons previously explained.

Suppose that a third slip takes place EF (to the north of the original one), through the strata which sunk down on that side,—and that it slopes to the north. The effect of this new dislocation will, of course, be to throw the strata on the *north* side of it still more down on the north, whilst the strata between this slip and AB can acquire no such tendency; and even if the slip EF were to take place, so, that the base of the figure between it and AB became less than the top, whereby a tendency to sink would be again produced, still it could not sink so far as the strata immediately north of it. If the slip EF had sloped towards the south, so that the stratified mass between it and AB acquired the form of a cone with the base uppermost, it is evident that, whilst it acquired a great additional tendency to sink, the strata on the north side would remain firm.

One important inference is fairly deducible from the foregoing views, if they be correct. Wherever the slips are found sloping towards each other in their under parts, there must have been a sinking of the strata or district between them. Let us apply this rule to the slips of the Fife and Lothian coal-fields. It has been mentioned that all the slips in Fife slope to the south,—whilst most of the slips in Mid-Lothian slope to the north. This would tend to shew, that there has been a prodigious sinking of the surface between these two shores;—and probably this may explain in part the formation of that immense hollow, running in an east and west direction, which is filled by the waters of the Frith of Forth.

It is now time to bring this first branch of my memoir to a close. Let me only add in conclusion, that I think I have at least proved the truth of the observation made at the outset, that, interesting as this part of the island is, on account of its unstratified rocks,—it is no less interesting on account of the phenomena which characterize its sedimentary deposits;—and I trust, that the present attempt to investigate these phenomena, if not attended by any direct benefit to science, will have, at all events, the effect of stimulating other geologists of greater experience and more leisure, to confirm or correct the descriptions I have given, and the opinions I have ventured to express.

II. *On the Superficial Deposits of the District.*

By "superficial deposits," I mean the extensive beds or layers of gravel, sand, clay, and other substances which cover the rocks stratified and unstratified, of the district,—and which intervene between these rocks and the existing vegetable soil.

It is hardly necessary to allude to the great importance of this branch of the subject. It is important with reference to the particular deposits which are the immediate subject of inquiry; and not merely in a scientific, but even in a practical point of view. It is important also, on account of the light which it may reflect, on the origin and formation of older deposits. If, as many geologists suppose, all sedimentary strata, of whatever epoch and character, have been formed by agents similar to those now in operation, the best method of discovering what these agents were, is by studying the phenomena most recently produced by them, and which therefore are the most legible indices of these agents.

This, however, is a branch of the subject, much more difficult than might have been anticipated.

No attempt has hitherto been made, that I am aware of, to describe or even to examine the superficial deposits of this district,—or indeed of any of the adjoining parts of Scotland. There are two short papers in the Transactions of the Wernerian Society, one of which states the different clays found at Blair-Drummond, and in the neighbourhood of Stirling; the other of which mentions the discovery of an elephant's tusk in cutting (for the Union Canal) through what the author terms the "old alluvial cover." These are the only publications I have heard of, which throw light on the various deposits that cover the rocks of the south of Scotland. Indeed, the subject is not one naturally susceptible of accurate or extensive *investigation*;—and even when investigated, it is not very susceptible of a *description* that is precise, or that can be readily apprehended. In endeavouring to examine these deposits, the geologist finds few places where they are visible; and unless they are seen at the very moment they happen to be opened up, they soon, in consequence of their loose and friable nature, become concealed under a mass of rubbish. In this respect, therefore, the subject presents greater difficulties than the study of the stratified rocks,—the croppings of which are to be seen, and may be easily examined, in old quarries, in the channels of rivers and burns, and even along the sides of roads and ditches.

The other difficulty which the geologist has here to contend with, is the want of appropriate terms to convey ideas of what he has seen;—and what is no less perplexing, there is the existence of several terms used in common language to signify some of these deposits, which terms, however, are of uncertain and variable import. So long as both of these difficulties remain, any distinct description or information on this subject, must be next to impossible;—and the opinions of geologists themselves must continue fluctuating, and be at perpetual variance. All phrases, such as "*old alluvial cover*,"—"*till*,"—"*recent alluvial cover*,"—"*diluvial debris*," are dangerous, when used in description, unless at the same time the thing described be otherwise very specially characterized. General phrases are no doubt useful and necessary, in order to avoid repetition; but the great desideratum is a distinct and copious description of the characters and contents of the deposits themselves.

In describing the different accumulations now referred to, it will lead to precision to follow a certain arrangement or classification. A very convenient one is suggested by the order in which they occur, in respect of position; certain of these deposits, with well marked characters, being found throughout the district, always in the same relative positions. I think it possible to identify and individualize at least seven formations, each of which has separate characters in respect of texture, contents, and appearance,—and each of which belongs probably to different epochs. I will now enumerate them, beginning from the surface; and in doing so, I will, for the sake of convenience, designate them by particular terms.

(1.) The existing soil, supporting vegetation.
(2.) Upper covering of gravel and boulders.
(3.) Deposit of sand and shells.
(4.) Beds of fine sand.
(5.) Beds of fine clay.

(6.) Coarse gravel or stoney clay.
(7.) Lowest boulder clay.
(8.) Beds of sand and gravel.

I shall begin with the lowest of the deposits now enumerated.

1. *Beds of Sand and Gravel.*—This deposit has been observed in several parts of the district, covering the edges of the stratified rocks. It prevails extensively in that part of the district situated between Dalkeith and Cowpits. Some years ago, coal was worked at the latter place; and in sinking various pits through the superficial clay and gravel, a bed of sand lying immediately on the rocks, was invariably passed through, which, being full of water, occasioned great practical difficulties, and even risks, to the work-people. A few months ago, a similar bed of sand was met with on the Duke of BUCCLEUCH'S estate, near Dalkeith, in sinking an engine-pit to work the coal. The pit had been formed through the boulder-clay, on reaching the bottom of which a bed of sand was encountered, which suddenly gave way, and laid the building in ruins. It was found necessary to form a new pit at a different place,—above the level to which this particular deposit reaches,—which appears to be about 200 feet above the sea. At the place where the first pit was put down, the sand was 9 feet thick, and between it and the rocks there was a mixture of sand and fine gravel 7 feet thick.

At Joppa likewise, (at the east end of the village, near the shore), the clay is separated from the subjacent coal-measures, by a bed of sand 5 or 6 feet thick. This sand-bed was found in the borings made for a particular coal-seam there, called the Splint Coal. The sand-bed covered this seam;—fragments of the coal were found in the sand, to the distance of 10 yards from the crop or outburst of the seam. It is not unimportant to observe, that, in the sand-bed, these fragments were all situated to the *west* of the coal-seam. Some fragments were also found at the bottom of the superjacent boulder-clay;—these were situated mostly to the *east* of the coal-seam.

At Leith, and in the manufactory lately occupied by Mr BURSTALL, a well was sunk through the boulder-clay 45 feet. A bed of sand and fine gravel was then reached, from which water immediately gushed up,—shewing that the bed was probably of considerable extent.

2. *The Lowest Boulder Clay.*—This deposit consists of a very hard coarse clay, of a colour generally blue or black, and having sometimes in it a shade of brown. Its texture is coarse and gritty. No laminæ are visible in it, such as may be seen in tranquil deposits of clay, and in most deposits of sand. It is difficult to work or excavate it, being quite impervious to the spade, and requiring the heavy pick to loosen it. It comes off in fragments of irregular shape, which on no side present any smooth, even, or regular surface.

This deposit of clay has interspersed through it, in most places, an immense quantity of stones. They vary in size, from small gravel to blocks many tons in weight. These stones seldom lie precisely in contact;—they are often separated by intervals of several feet. They do not present any regularity of position in the clay;—and are not even collected in heaps. The heaviest are not always at or near the bottom, nor are the lightest always at or near the top:—the lightest and heaviest occupy indiscriminately all parts of the bed.

The blocks of stone found in this clay are most usually round-shaped, and perfectly smooth in their surface. There are, however, two remarkable exceptions to this rule in this district, which are worth mentioning. In widening the road between Piershill barracks and Portobello a few years ago, the Road Trustees came upon an enormous mass of trap;—the lower part of it, was imbedded in the boulder-clay, the upper part in sand. The mass was sharp and jagged in its outline, as if it had not been transported far. It was quarried for road metal, and about ten cart-loads, or nine tons of stones, were got from it. Another block of the same description was found on Craigentinny farm, near Fillieside, which yielded so much as fifty cart-loads of stones. These masses were stated to have been a dark-coloured *whin*.

These are the only two cases I know, where the fragments of rock imbedded in this deposit of clay were not found rounded and smooth. I should add, however, that, at Cowpits and some other places, I have seen a layer of angular fragments at the bottom of the deposit;—this was, where the clay was incumbent on, and in contact with, the stratified rocks. The fragments were most generally sandstone, and appeared to have been derived from the strata immediately subjacent.

Among the rounded blocks, I have found (near the shore of the Firth of Forth), Basalt, Granite, Mica-slate,* coarse compact Conglomerate, coarse Sandstone, Quartz-rock, Limestone, compact Felspar of a deep red colour, besides ten or twelve varieties of greenstone. Though these boulders are generally smooth, some of them have ruts or scratches on their upper sides, and which have been apparently produced by the passage over them of harder bodies. I have more particularly observed these ruts or scratches on blocks of limestone, sandstone, and greenstone. It is an object of some importance to ascertain the direction of these ruts,—but it is at very few places in the district where this can be ascertained. The direction of the ruts can be very distinctly seen, along the shore at Joppa near Portobello, and at Seafield near Leith. They appear at both places to range between W. and W.SW. by compass, but the most general

* In a bye-road which runs to the east of North Leith church, there was, in 1837, a block of mica-slate about 4 feet in diameter. The author of a useful little work, entitled "Excursions illustrative of the Geology and Natural History of the neighbourhood of Edinburgh," 1835, states, that he found a block of mica-slate with garnets in it two miles south of Dalkeith. (p. 70.)

direction is W$\frac{1}{2}$S. A great many boulders have been lately dug out of this deposit, in the excavations for the Newhaven and Edinburgh Railway. The direction of the scratches on them is W$\frac{1}{2}$N.

It is hardly necessary to observe, that these boulders belong chiefly to rocks which, with one exception, exist only in very distant parts of Scotland. The Greenstone boulders, of course, may have been derived from the trap hills in the vicinity of Edinburgh,—though more probably from the Ochil range. From this quarter also, most probably, the coarse conglomerate boulders have come. The pebbles in this conglomerate consist of Greywacke, Quartz-rock, Lydian-stone, and Felspar. One of these boulders, taken out of the clay at Joppa, is now lying in front of Mr Grieve's house, in Grove Street, Musselburgh. It is hardly necessary to add, that the granitic and mica-slate boulders must have been likewise transported from the west.

This lower deposit of hard coarse clay which I have been describing, covers a very large extent of country. Along the southern shore of the Firth of Forth, I have traced it from the river Cramond to near Prestonpans. It is visible at a great many points along this shore,—as, for example, at the mouth of the Cramond, where it forms the east bank of the river,—to the east of Caroline Park, where it was cut through in forming the new road to Granton Harbour,—at Newhaven, where it is cut for the railway,—at Seafield, and at Joppa-pans, at both of which places it is exposed by the ocean. But whilst visible only at the places now mentioned, there is no doubt that this boulder-clay forms a continuous bed from Cramond to Magdalen Bridge. At Leith, in boring for water, it was penetrated to the depth of 80 feet. The whole of the shore between these points is seen at low-water strewed with immense quantities of rounded blocks, which, by the cliff becoming undermined, have fallen out of the clay on the beach, and remain there unmoved by the recurring tides, and by the waves or even the storms of the ocean,—thereby testifying the prodigious force of the agent which was able to transport them from their native sites.

This same deposit of clay may be seen at other points along the coast, though unpossessed of the large boulders just alluded to. It is this clay, I think, which is worked at the brick-fields situated at Drumore gate, and at Prestongrange. It is a coarse gritty clay, having interspersed through it small pebbles, and occasionally a few thin seams of gravel. It is there about 18 feet thick.

In describing the previous deposit, I mentioned that, at Joppa, near the shore, this boulder-clay is separated from the rocks by a layer of sand. At Joppa quarry, nothing intervenes between it and the rocks. If there had been any older superficial deposit there, it must have been washed away by the boulder-clay; for it is seen in this quarry covering the edges of the strata, and forming a bed about 8 feet thick.

With regard to the extension of this lower boulder-clay to the south, and through the interior of the country, it is of course impossible to speak with the same degree of precision. It has been gone through, however, in making borings and sinking shafts for coal, in various parts of the Esk valley,—as at New Craig Hall, Shaw-fair, Cairnie, Niddry, Monkton, Craighall, Millerhill, Sheriffhall, Cowpits, Dalkeith, Eldin, Newbattle, and Bryants. It may be well seen in the channel of the burn which flows past Straiton Mill. There are numerous boulders of encrinal limestone in the clay there, the scratches on which are NW. by compass.

At the village of Ford, in a burn which there joins the Tyne from the north, a bed of clay about thirty feet thick full of boulders may be seen, and possessing in all other respects the same characters which this deposit has elsewhere. It is, however, deserving of observation, that though this boulder-clay exists on the north and south, and I believe I may add the east flanks of the Roman Camp,* it does not seem to occur above a certain level on these flanks. In none of the Cowden or Chalkieside quarries, for example, does this boulder-clay exist, covering the outcroppings of the strata that are there bored. These are between 350 and 450 feet above the sea. It has been proved, in the various borings recently made by the Duke of Buccleuch on the same side of the hill, and which are about 140 feet above the sea. The boulder-clay is wanting also at Fullarton and Monk-Loudon (situated about eight miles SW. of Dalkeith), which are elevated about 850 feet above the sea. I might mention various other spots where the deposit does not exist, and where the rocks are covered in general only by a deposit of small gravel, to be afterwards noticed. These cases can, in my opinion, be explained only on the supposition of the boulder-clay having been abraded and washed away by currents of water flowing over it, after its deposition.

The part of the district where I have seen the boulder-clay highest above the sea, is between Carlops and West Linton. It covers the rocks in that district for many miles, and forms a deposit which is on an average about 960 feet above the sea. I have noticed in it there, boulders of Basalt, Coal-Sandstone, Limestone, Greywacke, and Felspar-Porphyry. There is, near Rutherfurd House, a block of Limestone, about a ton in weight, and not much worn at the edges. No Limestone of the same description occurs nearer than four miles: and there are only two places where it is known to exist; the one at Baddensgill, situated about N.NW. from Rutherfurd House; and the other near Fairneyhaugh, situated W. by N. from it. The blocks of coal-sandstone have most probably been transported from Cairnsmuir, a hill among the Pentlands, situated to the W.NW. In speaking of Carlops, I should notice the existence of a valley near it, to the west, which extends for about 1½ miles, and runs in an E. and W. direction. In the middle of this valley are several pyramidal masses of rock, so isolated and bared to the west, as to suggest strongly the notion, that the valley has been scooped out by a rush of water from that direction.

In the burn which flows between Fala and Woodcot, this boulder-clay can

* It was cut through in improving the Edinburgh road, about two years ago, at Kippilaw, on the north side of the Roman Camp, and near Fordell, on the SE. side of the hill.

be traced up to within a few hundred yards of the Edinburgh road. At the point where it stops, it is about 700 feet above the sea.

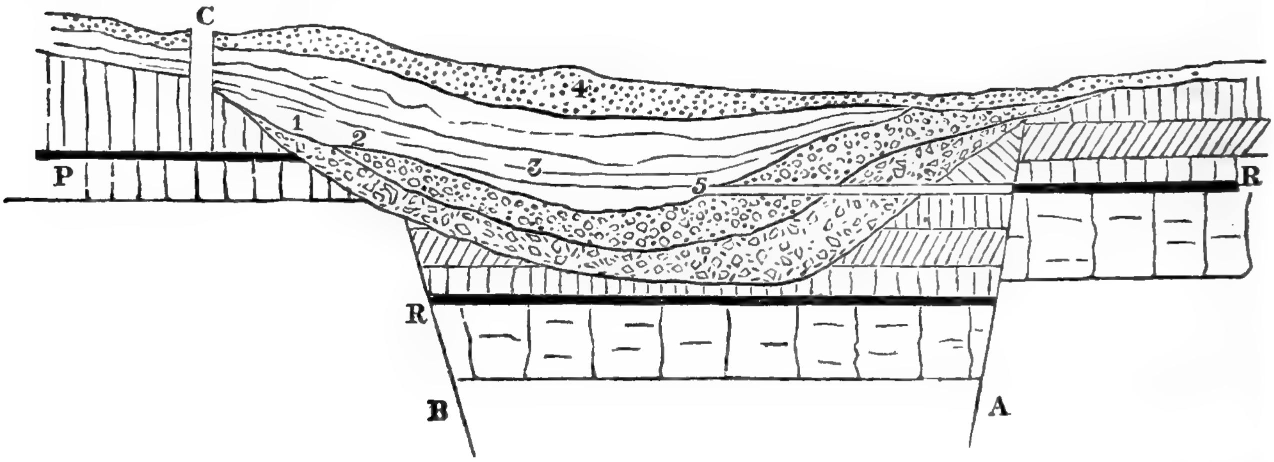
This boulder-clay extends into East-Lothian, and may be seen in different parts of the Tyne valley; as, for example, at Ormiston and Yester. At West Garlton, about $1\frac{1}{2}$ miles north from Haddington, there is a bed of clay which has been worked for bricks. The upper and workable part is eight feet thick. I rather think it belongs to the deposit now described.

Towards the west of Edinburgh, boulder-clay covers the country. Near Redhall, it may be seen lying upon the sandstone rock quarried there, to the depth of 20 feet, and containing large blocks of greenstone and basalt. One of these blocks is not less than 40 cubic feet in size.

With regard to the depth or thickness of this lowest deposit, my inquiries have not been so successful as to enable me to afford much information. Along the sea-shore it is about 45 feet at Leith, and it is said more than 60 feet at Portobello. In the quarry at Cowpits it is only 8 feet thick. In the borings lately made by the Duke of BUCCLEUCH to the SE. of Dalkeith, it varies from 9 to 15 feet. Between Carlops and West Linton it is from 4 to 20 feet thick.

There are some places, however, in Mid-Lothian where the depth is much greater, arising from a circumstance very remarkable. At Niddry, where the "great seam" of coal was worked, the depth of all the superficial deposits was found to be between 60 and 70 feet, of which about 30 feet was clay full of boulders. Here, it was ascertained, there had been a scooping out of the rocks, to a certain depth, and that the excavation was filled with the clay. The excavation cuts across the crop of the strata, and near Niddry it is about 100 yards wide. It runs in a NE. direction, and has been proved at several places in that direction. It was proved at the Wisp, where the "great seam" was worked. It was proved about 1000 yards to the east of this, by certain borings. It was also proved at New Craighall, which is about two miles east from the Wisp. The depth of the boulder-clay at this last point was 48 feet, overlaid by other 48 feet of different deposits, to be afterwards described. The width of the excavation at New Craighall is about 200 yards, being nearly double what it is at Niddry. The sides or walls of this excavation do not appear to be vertical. At the bottom of this channel, the boulders are said to be entirely of blue whin, and so close as to be almost touching. The above particulars I learnt from the late JOHN GRIEVE of Musselburgh, who was coal-manager for Sir JOHN HOPE, and who, in describing this excavation to me, stated that it was in every particular like the course of the river Esk at Hawthornden; and that he had no doubt but that this was an ancient river course, which had been choked up with the boulder-clay. I do not mention this now for the purpose of theorizing, and far less for the purpose of offering Mr GRIEVE's theory as the true one. I notice it only for the purpose of conveying an idea of the nature and extent of the excavation in question. I thought at first that it might have been owing to a fracture across the strata, of which there are many in this district, and in consequence of which the strata being shattered along a particular line, would be the more easily carried away. On asking Mr GRIEVE whether any levels or mines had been run under this supposed river course, and whether any slips had been met with under or near it, he replied that the coal had been worked under it, and that no slips occurred there. He stated in addition, that the coal had, in one or two places, been worked up to the sides of the excavation, and where it suddenly ended.

There are several other places where analogous phenomena occur; and as they are extremely curious, I may be pardoned for describing them.

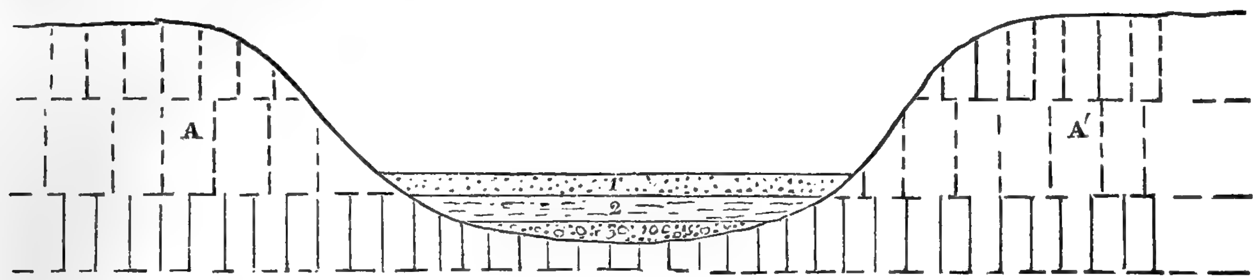


At Bryants (the property of the Marquis of LOTHIAN), the annexed figure represents an excavation in the rocks;—A is a slip which throws down the rough coal RR about 12 fathoms to the NE. B is a slip which throws up the metals so high, that the parrot or North Greens coal P is brought near the surface, where it is worked by the pit C. A level, No. 5 in the above figure, was driven through the slip A from the rough coal R, which first went through clay containing angular fragments of sandstone, No. 1 in the figure; it then went through another bed of clay containing nothing but rounded blocks of whinstone, No. 2 in the figure; and lastly, it entered a bed of sand and mud, No. 3, which choked the level, and caused the farther prosecution of it to be abandoned. The depth of this hollow or excavation from the adjoining surface of the rocks is about 90 feet; the width of it is 200 yards. It appears to run in a direction W.NW. and E.SE. How far it extends in that direction, has not yet been ascertained. A bed of gravel, No. 4, was found immediately beneath the surface. Bryants colliery, where these appearances occur, is situated about 380 feet above the sea.*

At Barleydean, between Carrington and Rosewell, there is a narrow valley, in the centre of which the engine-pit is situated. The width of the valley when measured along the surface of the ground, is 227 feet at the engine-pit, and its sides rise to a height of about 60 feet above the mouth of the pit. The rocks in

* Mr GIBSON, the coal overseer of the Marquis of LOTHIAN, who communicated to me the above facts, states, that many more particulars regarding his "*gash*" or excavation, as he calls it, will be ascertained in the course of a short time, after a mine now driving has been completed.

the bottom of the valley are covered by superficial deposits, which have a united thickness of 22 feet. The upper deposit consists of clay, having interspersed through it small water-worn stones. The nature of the lower deposit has not been ascertained so exactly; but it is thought, that the upper part of it consists of sand and mud, whilst the under part consists of clay, having interspersed through it angular fragments of sandstone. The following section, drawn to a scale, illustrates the description just given—A represents a very thick bed, or series of beds, consisting of soft red sandstone; they are here nearly horizontal.



No. 1 is the upper covering of (boulder ?) clay; No. 2 is the bed of sand or mud; No. 3 is the clay with angular fragments of sandstone. The total depth of the excavation is between 80 and 90 feet. It runs in a direction about W.NW. and E.SE., and it has been traced for several hundred yards.

I have never heard of any organic remains having been found in this lower Boulder-Clay, except on one occasion. I allude to the discovery of an elephant's tusk on the Clifton Hall estate, when the Union Canal was being formed. Mr Bald states that this tusk was found in what he terms the old alluvial cover, and which, from what he says of it, appears to be the same as the deposit now under consideration. The tusk was 39 inches long by 13 in circumference.

3. The next deposit to be described is the Coarse Gravel, or Stony Clay.

It is much more sandy and gravelly in its texture than the boulder-clay. It is not nearly so hard, or so difficult to be worked; and consequently it is not, like the boulder-clay, impervious to water. Its colour is different from that of the former deposit, being not of a black or blue, but of a light brown or straw-colour. There is in this deposit the same absence of laminæ as in the former one; and there is the same want of regular arrangement in the fragments of stone interspersed through it. These fragments are neither so large nor so rounded as in the boulder-clay. I have never seen any more than half a ton in weight. The species of rocks from which these fragments are derived are also somewhat different. I have seen no mica-slate blocks in it. With this exception, the species of rocks occurring in it, are much the same as in the boulder-clay.

This gravel or stoney deposit rests generally on the boulder-clay, though sometimes it rests immediately on the subjacent rocks. It may be seen on the new road to Granton Harbour, resting on the boulder-clay. There it is from 10 to 12 feet thick. At Niddry, and all over the district to the east, it also rests on the boulder-clay. At New Craighall it is about 40 feet thick. At the Duke of Buccleuch's new engine-pit it is 23 feet thick.

At the places just mentioned, this stoney clay is lying upon the boulder-clay. I may next mention some places where it is lying immediately on the rocks. In that part of the Leith and Dalkeith Railway which is to the south of the turnpike road leading from Brunston to Duddingstone, the strata of shale are seen nearly vertical dipping SE., and overtopped by a bed of stoney clay 15 feet thick. The boulder-clay appears to have been denudated at this point, for a little to the north it takes on again, under the stoney clay.

4. *Beds of fine clay*, form the next member in the series of deposits in this district.

The clay is generally fine, free from stones or gravel, and laminated horizontally. It is of various colours, being sometimes dark yellow, sometimes light brown, sometimes dark brown, sometimes having a shade of blue in it. From Harden Green to the Duke of Buccleuch's new brick-work, two miles east of Dalkeith, this clay appears to form one continuous bed. It is throughout of a brownish-yellow colour, and is overlaid by sand. Beneath the clay at these places is what I have termed the *gravel or stoney clay*. The same bank or bed continues to the SW., and is worked at Redheugh on the Arniston property. How far it stretches towards the NE. is not known. Near the Duke's Kennel it is more than 15 feet thick. At the Cowpits, where the coal-engine was, the bed was (as Mr Grieve informed me) between 20 and 30 feet thick. It does not appear to exist in the old quarry situated to the north.

It is at Portobello that this particular clay is most extensively worked. It is now worked in two or three several places there, at the north end of the town. It was formerly worked in another place more to the south of any of the present brick-fields. It was also worked (about fifteen years ago) a mile to the west of Portobello, on the south side of the Edinburgh road.

At Portobello, the clay which is worked in the brickfields, consists of two beds, the lowermost of which is more clammy or (to use the expression of the workmen) *stronger* and *fatter* than the other. Both these beds or layers dip at the Edinburgh road towards the SW. with an angle of about 10°. This basin of clay extends eastwards nearly as far as Joppa, where it rises up and thins off. Toward the SW. it stretches to Duddingston Mill (where it was once worked), and there is reason to believe that it extends through the grounds of Duddingston House, and even as far as the Inch. I have reason to believe, that a similar bed of clay exists in the Cowgate of Edinburgh, for I have heard, that in digging out the foundations of the South Bridge of Edinburgh, a thick bed of cockles was discovered.

The uppermost bed is worked on the west side of the road near Mr Bailey's glass-work. It is of a light blue colour, and easily cut with the spade. Its thickness on the south side of the work, is about 25 feet; it gets thinner towards the NE.,—and at the Edinburgh road, it crops out altogether.

The second bed of clay which supports the one just described, is worked in the fields between the Edinburgh road and the sea. Near the shore it is only 7 or 8 feet thick, but its surface has evidently been lowered there, by the operation of subsequently emerging causes. Near the Edinburgh road, it is 35 feet thick. It appears, however, to get thicker towards the SW., for in sinking a well at Mr Bailey's manufactory on the south side of the road, about 80 feet of clay was gone through before water was reached. The water gushed up from a thin bed of sand. This lower deposit of clay is supposed to be there, from 55 to 60 feet thick.

It has been ascertained that a bed of gravel lies beneath this fine clay at Portobello. I have not been able to discover whether this gravel bed belongs to that member of the series which I have termed the gravel or stoney clay;—it would appear that about 10 or 12 feet from the top of the lower bed, there is another layer of fine gravel in the clay about 14 inches thick.

Both the upper and the lower bed of clay at Portobello is finely laminated. The layers are not more than one-sixth of an inch in thickness,—in some places, scarcely thicker than the leaves of a book;—and they separate very easily. This arises from there being a thin film of very pure and attenuated mud and sometimes sand, between the *laminæ.*

This brick clay is thicker at Portobello than any other place I know of in the district. At Harden Green it is only 8 feet thick. To the east of Dalkeith it varies from 3 to 13 feet. Near Sheriffhall engine it is 13 feet thick. It appears to be thickest in those situations where it occupies the lowest level. I do not think it exists at a higher level than 150 feet above the sea.

I have a strong suspicion that the particular stratum of clay I am now describing is contemporaneous with the Carse clay of Falkirk and Stirling. This Carse clay is partially described by Mr Blackadder in the 5th volume of the Wernerian Society's Transactions. My reasons for this opinion, are founded not merely on the great similarity in the texture of the clay in both districts, but also on the fact of its being covered, as in this district, by layers of sand and gravel. The sand, according to Mr Blackadder's account, in the eastern part of the district (as about Falkirk) contains pieces of coal,—but towards the west, as at Stirling, it is entirely free from them, and contains pebbles of mica-slate.

In regard to the existence of this bed of clay in East-Lothian, I have obtained as yet but scanty information. I have found a very similar deposit near the mouth of the Tyne, and extending round the margin of Belhaven Bay. In Belhaven Bay, this clay is now worked to the depth of 9 or 10 feet. A section of the same bed at Hedderwick, where it is cut through by a burn, shewed a depth of 15 feet.

In this deposit of clay, there have been found at Portobello organic remains of various kinds. Several large trees were dug out. Unfortunately they were not kept, but on inquiry among the work people, I learn that one of them was 20 feet in length, and about 7 feet in circumference. It had a few branches near the top. This was on the top of the clay, where it had formed for itself a sort of bed. Its top was lying towards the S.SE. After exposure to the air for some time, the tree cracked in all directions, shrivelled up, and then crumbled to pieces. The work-people consider that this one and all the trees which have been found, are *oak*. In the course of last year I extracted from the clay in two of the brick-works at Portobello several fibres or fragments of wood. They resemble the roots and branches of a dicotyledonous tree, though of what kind, I am unable to say. They were from 2 to 4 feet below the surface of the clay.

I learned, also, that hazel-nuts had been frequently found in the Portobello clay. They were got on the surface of the bed, and in what is called a parting, which lies between that bed and the superincumbent sand.

I observe from Mr Blackadder's paper, above referred to, that wood, hazel-nuts, and the leaves of trees, have been found in the carse clay of Falkirk.

Shells occur,—but very sparingly, in the Portobello clay. I regret much that I have never seen any perfect specimen, though I have often sought and inquired for them. I am therefore obliged, for the following statements, to rely on the information of the tacksman of the brickwork, Mr Henderson, who seemed, however, an intelligent man. He stated, that the shells were small, of a white colour, and not unlike cockles. They were very tender, he said, and crumbled to pieces when handled. He has found them with both valves adhering, and closed. These shells, it rather appears, do not form a layer, but are indiscriminately interspersed through the clay. They have been found to the depth of twenty feet from the surface. The fragments of shells, which I have seen in the clay at Portobello, were so imperfect, and so few, that it was impossible to know what they were.

I was informed also, that in Morton's brick-field at Portobello, which is not now worked, he has found bones nearly as thick as a man's thigh. They were such, he says, as excited the surprise of the workmen. That such bones were found, is not unlikely, when it is remembered it was in the Carse clay that whale bones were found, at Airthrey, at Blairdrummond, and at Dunmore. The whale bones at these places were from twenty to thirty feet above the present level of high-water mark. Near Camelon, there were found the bones of a seal, in a bed of clay ninety feet above the sea; and in a bed of sand beneath them, were shells of the razor or spout-fish. In Drummond's Agricultural Museum at Stirling, are preserved the head and antlers of a red deer, which, in

July 1837, were dug out of the Carse clay at Stirling, between ten and eleven feet beneath the surface of it.

5. *Beds of fine sand* form the next deposit in the ascending series.

The sand is in general very pure, that is to say, free from admixture with clay. In external appearance it is white, and resembles strongly sea-sand. It is laminated, the laminæ being generally not horizontal, but forming angles varying as much as 10° or 15° from the horizon.

The sand frequently contains gravel, both in single pebbles, and in thin patches or layers. I have never seen any, so large as a cocoa-nut. They are all water-worn. At Harden Green, there is a bed of this sand, about three feet thick, lying on the stony clay, and in it I found a fragment of flesh-coloured felspar, derived apparently from the Pentland Hills. Sometimes there are fragments of shale and coal. A piece of coal, about half the size of my fist, I took out of a sand-pit between New Hailes and Fisherrow. In the same sand-pit, there were numerous particles of coally matter between the laminæ.

To this deposit belong the beds of sand and mud which have been described as occurring at Barleydean—about 500 feet above the sea. At Blackshiels (about one-fourth mile north from the inn, on the Edinburgh road), similar deposits are visible. Their height above the sea at this last-mentioned spot is about 700 feet.

To this particular deposit, I think, belong those immense accumulations of sand which lie to the north of Edinburgh. Most of those whom I now address, must have frequently seen the deep sand-pits near Inverleith Place, St Mary's Church, Claremont Crescent, the Old Botanic Garden, and on the south side of the Horticultural Garden. These sand-pits shew beds of sand exceeding thirty feet in thickness at least, and how much more I do not know. In this part of the district, there appear to be two or three separate banks of sand. One of them runs westward from the Old Botanic Garden towards St Mary's Church through the nursery gardens. At the east end of this bank, viz. near Leith Walk, the laminæ in it rise at a small angle to the west. At the other end, viz. near St Mary's Church, the layers rise gently towards the east. The laminæ are in some places made very distinct, by particles of coal or shale lying betwixt them.

Another of these sandbanks runs through Inverleith Terrace, and across the Water of Leith to Redbraes Villa, where it was formerly extensively wrought. A third runs still farther to the north of the Water of Leith, and has been lately cut through for the Edinburgh and Newhaven Railway. It there presents a section of about forty feet in depth. The layers of sand form arches, rising toward the top of the bank, and forming on the north side an angle of 40° with the horizon.* This bank (as well as the others) runs in a direction nearly E. and W., and may be

* A section of this bank is given on page 72 hereof, where there is a description of the upper deposit of small gravel which covers the sand.

traced to the westward nearly a mile. Its width (where it is cut through) is about eighty yards. There are numerous slips or fissures in the sand, which break the continuity of the layers, and produce in miniature exactly the same phenomena which have been produced by slips in the subjacent rocks. The layers of sand, where dislocated, are invariably lowermost on the upper side of the slip, as they are found to be in the rocks of the coal-measures.

There appear to be several of these sand-banks between the Water of Leith and the Frith of Forth, all running parallel to each other, and forming undulations of surface in that particular surface, which are sufficiently remarkable.

This same deposit of sand occurs also in *Inveresk Church-yard*,—at *Wardie* (where it is eight feet thick),—*New Hailes* (where it is eighteen feet thick), at *Cowpits* (where, in the engine-pit, it was found to be about eighteen feet), and near the Duke of Buccleuch's new brick-work, where it is at least eight feet thick. Immense banks of it are also to be seen in the parishes of Lasswade and Roslin. To the west of Loanhead a bank of sand was gone through in sinking a coal-pit, and found to be thirty-two feet deep. It rests there on boulder-clay about five feet thick. These ridges run, generally speaking, in an east and west direction, and extend in that direction many hundred yards. Their breadth on an average does not exceed 100 yards. The Loanhead sand-bank can be traced for nearly a mile,—forming a continuous ridge or rampart, very visible from the Edinburgh and Peebles road. On the south side of the Esk at Springfield, sand-banks equally extensive, and still more deep, occur.

I have found and heard of no organic remains in the sand belonging to this part of the series.

6. I proceed now to describe another member of the series, which I have termed a deposit of *sand and shells*. This deposit is distinguished from all those previously described, by this circumstance, that it is not situated much above the present level of the sea: it forms a margin or fringe along the south and north shores of the Frith, and also along the shore of the ocean in East-Lothian. Many persons, in travelling along that shore, must have been frequently struck with a high bank, which almost every where presents itself at a greater or less distance from the beach. This bank varies in height from thirty to sixty feet above the ground which slopes from its base to the sea; and its *base* is also from thirty to forty feet above high-water mark. The bank may be very distinctly traced along the shore at Granton, Wardie, Newhaven, Seafield, Joppa, New Hailes, Inveresk, Prestonpans, Seton, Aberlady, and at intervals round the coast to Belhaven. At some of these places, the bank is nearly three-fourths of a mile from the present shore, at others, it is close upon the shore. At all of them, the intervening space is occupied by a layer or deposit of sand or shells. It is this deposit which I am now about to describe.

It consists of a fine sand at the following places, viz. Caroline Park, Portobello, Musselburgh, and Seton, varying in thickness from two to eight feet. The layers are in general horizontal, or else dipping towards the sea, in accordance with the subjacent clay on which it rests. At all the places just mentioned, there is at the bottom of the bed, a quantity of boulders, some of which are of enormous size. They are generally greenstone and basalt, though I have remarked also blocks of limestone. At Portobello, these blocks are dug out and removed, in order to get at the brick-clay, the surface of which is covered by them. Among these blocks are quantities of marine shells, all apparently belonging to the same species now existing in the Frith of Forth.

But this deposit does not every where consist of *sand*. Accumulations of very fine gravel are occasionally met with, mixed up with prodigious quantities of marine shells. The shells are generally broken and shattered, in the same way as they usually are on a sea-beach. I do not say that all the shells which exist in this deposit, are to be found in the present sea;—because I have not yet collected a sufficient number to enable me to say so. It is quite possible, that among these shells in this particular deposit, as in the one previously described, some species may exist which are now extinct.*

I have walked along the whole shore, from St Abb's Head round by Dunbar, North Berwick, Aberlady, Cockenzie, and Newhaven, to Queensferry, and moreover traversed the greater part of the Carse district from Falkirk to beyond Stirling. I did this, in order to collect facts which might throw light on the important geological question, whether any proofs exist in this part of the island of a recent change in the respective levels of sea and land. When I commenced this investigation, my object chiefly was to discover the localities of sea-shells, and their exact height above the present level of the sea. I had not gone far, before I was much struck by the occurrence of the high bank formerly alluded to, running everywhere nearly parallel with the existing shore;—but it was not till a later period of my investigations, that I perceived the connection of that bank with the subject I was examining, and the far greater importance of ascertaining the height of its *base* above the sea, than that of the shelly bed occurring between it and the sea.

In general, this bank is distant from the present line of high-water mark about 300 yards; whilst, at other places, it is close upon the sea, and occcasionally ceases altogether. I avoid at present offering any explanations of these facts. I would only here add, that the bank is everywhere most inland, in those parts of the coast where the sea is shallow, and where, consequently, it recedes at every tide to a considerable distance. Along the coast near North Berwick, where the

* Mr Smith of Jordanhill is of opinion, that several of the shells he has found in the superficial deposits in the west of Scotland belong to extinct species—to the number of twelve or thirteen.

submarine surface dips rapidly from the shore, the old bank is at very few places at all perceptible.

With reference to the height above the sea at which these shells occur, it may be proper to observe, that there is much irregularity. This is, of course, owing partly to the impossibility of discovering the maximum height everywhere along the coast. At the same time, I am far from saying that these shells form everywhere a continuous bed along the coast. Sometimes they cover the whole space intervening between the shore and the old bank; sometimes they can be traced only half-way to the bank from the shore; at other places, they do not occur at all, the whole intervening space being covered by sand without shells.

This deposit of sand and shells, mixed with gravel and boulders, is undoubtedly an old beach, which had been formed by the sea, when its level was considerably higher than it is at present. I have heard some persons say, that the shells may have been *blown* up to the level they now occupy. This notion is quite irreconcilable with one fact, which I may here mention. On the coast to the south of Dunbar, there is a small bay called Skateraw. The shelly deposit there contains numerous fragments of limestone, derived most probably from the strata of limestone which occur in the immediate vicinity to the north and west. Most of these limestone fragments are bored with *Pholladæ*, and I found the shells in the stone. Moreover some of these fragments had smooth surfaces, and on some of them I found numerous specimens of *Serpulæ* and *Patella vulgaris* sticking. These shells are now, therefore, in the exact situation where they lived. At this place, the shelly deposit is 13 feet above high-water mark.

The height of this shelly deposit above the sea, is less on the shore of the open sea (as at Skateraw and Dunglass), than it is in the upper parts of the Frith of Forth. At the former, it is no more than 12 or 13 feet above high-water mark; at the latter, it is about 30 feet.

In Dirleton Common, there is an isolated rock of greenstone about 300 yards from the shore. It rises to the height of about 60 feet above high-water mark, and it is about 300 yards in circumference. I found the *Patella vulgaris* in great abundance on every part of this rock, to the height of 39 feet above high-water mark. At the same time, it is right to mention, that I did not find any shell actually adhering to the rock. But I have a strong impression, that on clearing away the sandy soil which covers the rock, this important discovery would be made.

It is proper to add, that, in this shelly deposit, other organic remains have been occasionally found. I was informed, that at Dunbar, and in the churchyard of Aberlady, the horns of a species of deer were found. In the Dunglass old beach, I picked out some bones, two of which I shewed to Dr KNOX, who obligingly examined them for me, and reported them to be those of a small species of ox.

With reference to the *old bank* above referred to, I would add some remarks, as to the materials of which it is composed, and the height of its base above the sea, before leaving this branch of the subject.

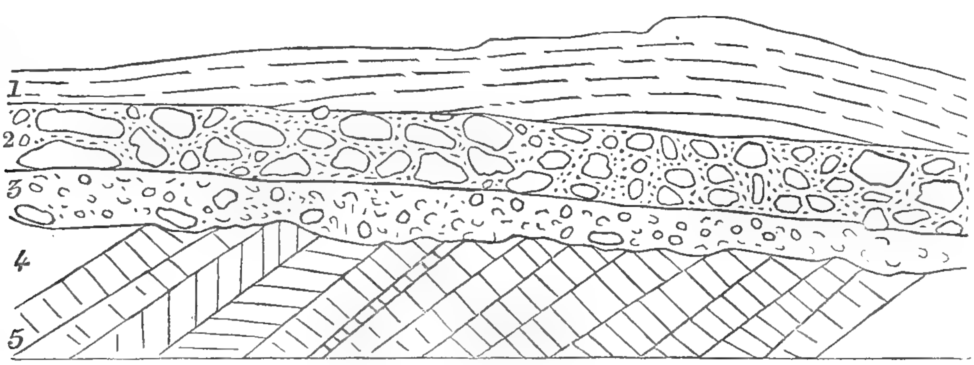
In some places, this bank has been formed on the boulder-clay;—there it is always highest and steepest, because the clay is hard and strong, and is not easily affected by atmospheric influences. In this respect, it is more enduring even than rock, for it is impervious to water, and therefore is not acted on either by dryness or moisture, nor is it affected by variations of temperature. At some places, however, the bank is low, and indeed scarcely discernible,—as at Portobello. There it consists of the fine brick-clay, which is not so strong or stiff as the boulder-clay, and can be more easily worn down by ordinary atmospheric action.

I wish I could speak with certainty and precision regarding the height of the base of this bank above the sea. I cannot yet do so; though I expect ere long to have a table completed, affording this information for every part of the coast. I may state generally, that, whilst its average height is between 30 and 40 feet along its whole extent, I have reason to think it rises towards the west. I think it is 15 or 20 feet higher at Falkirk and Bannockburn than it is at Aberlady.

7. I proceed next to describe the *upper covering of gravel and boulders*, which has been deposited even more recently, than the beds of sand which contain fragments of existing sea-shells.

Before alluding to the contents of this deposit, or the extent of country it covers, I should wish to notice two or three localities, an inspection of which must satisfy any one, not only that it exists, but that it is also more recent than the deposit of sand and shells of which I have just concluded the description.

About a mile to the west of Gosford, the seat of the Earl of Wemyss, there is, near the shore, a section that may be represented by the following figure:



1. is a quantity of sand, probably blown.
2. is a stratum of boulders and gravel. The boulders are from 3 to 8 inches in diameter. They are rounded. They consist chiefly of greenstone, and of a peculiar kind, which occurs *in situ* on the beach about a mile to the west; it is about 2 feet thick.
3. is the stratum of shells mixed with pebbles and gravel. The shells are much broken. They consist of oysters, wilks, lempits, &c. The lowest part of this bed is $7\frac{1}{2}$ feet above high-water mark.
4. are strata of shale, &c. Their base is washed by the sea at high-water.
5. the line of high water.

At Catscraig, on the shore of East-Lothian, about three miles south of Dunbar, the shelly deposit may be seen covered by a layer of gravel and boulders, altogether similar to that which is represented in the foregoing figure.

As this deposit thus lies above the deposit of sand and shells last described, it is hardly necessary to observe, that, in certain situations, it may be expected to be seen covering all the older deposits successively.

It may be seen at a great many places covering the layers of *fine sand* which forms No. 5 of the series. It may be seen, for example, in the cut lately made for the Newhaven Railway, between Easter and Wester Warriston,—as represented in the following section:—

Fig. 1.

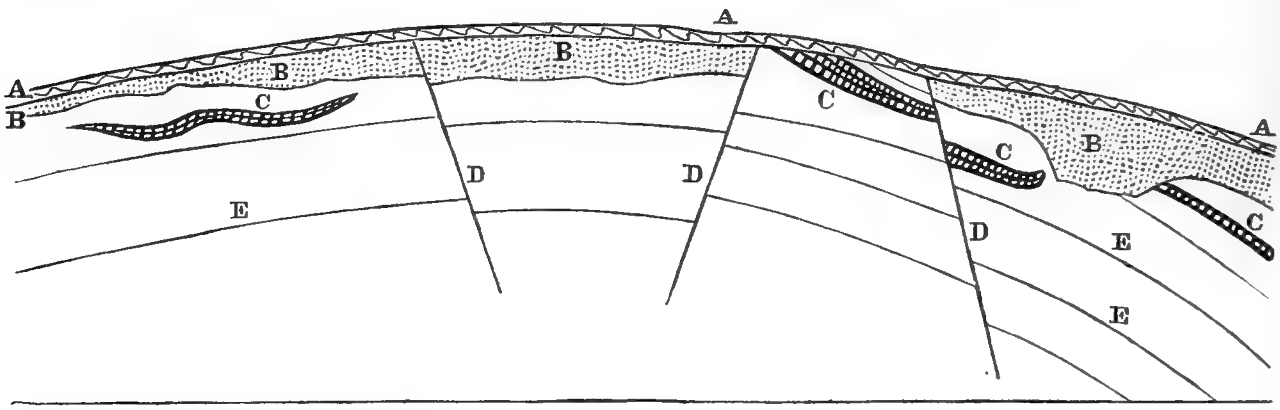
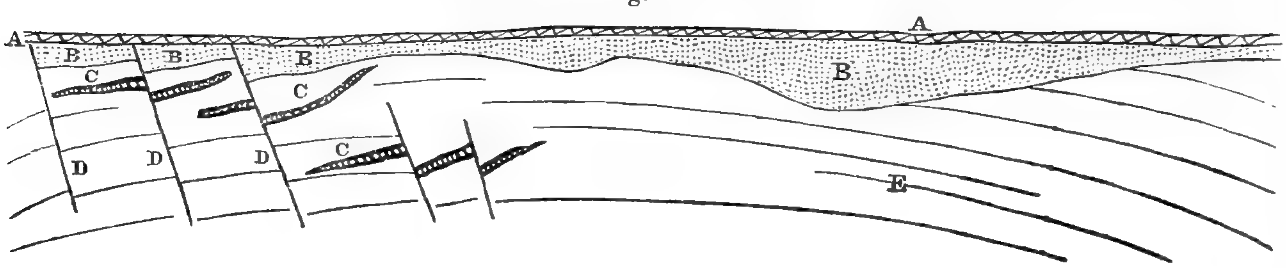


Fig. 2.



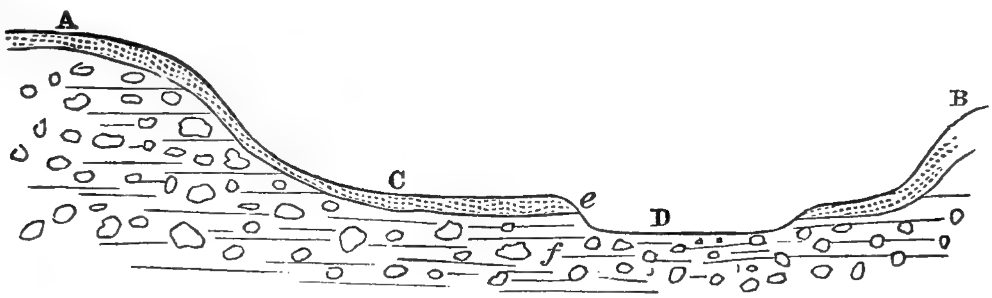
Figs. 1 and 2 are intended to represent sections of the two sides of the cut, —fig. 1 being the west side and fig. 2 the east. A, represents the existing soil or mould of vegetation;—B, layers of gravel belonging to the series now described, and lying upon beds of sand which belong to No. 5 of the series. C, represent layers or *laminæ* composed of small fragments of coal, none exceeding a man's hand in size. The layers are not more than 3 inches thick. D, are slips or fissures which intersect the gravel and the sand. E, are the *laminæ* of the sand-bed, which, as mentioned in a previous part of this paper, are, near the sides of the bank, inclined to the horizon at an angle of about 40°.

The depth of the sections represented by these figures is about 40 feet, and the length 200 feet.

In a cut made for the Fisherrow Railway, near Newhailes, I noticed a similar bed of small gravel, overlying the sand.

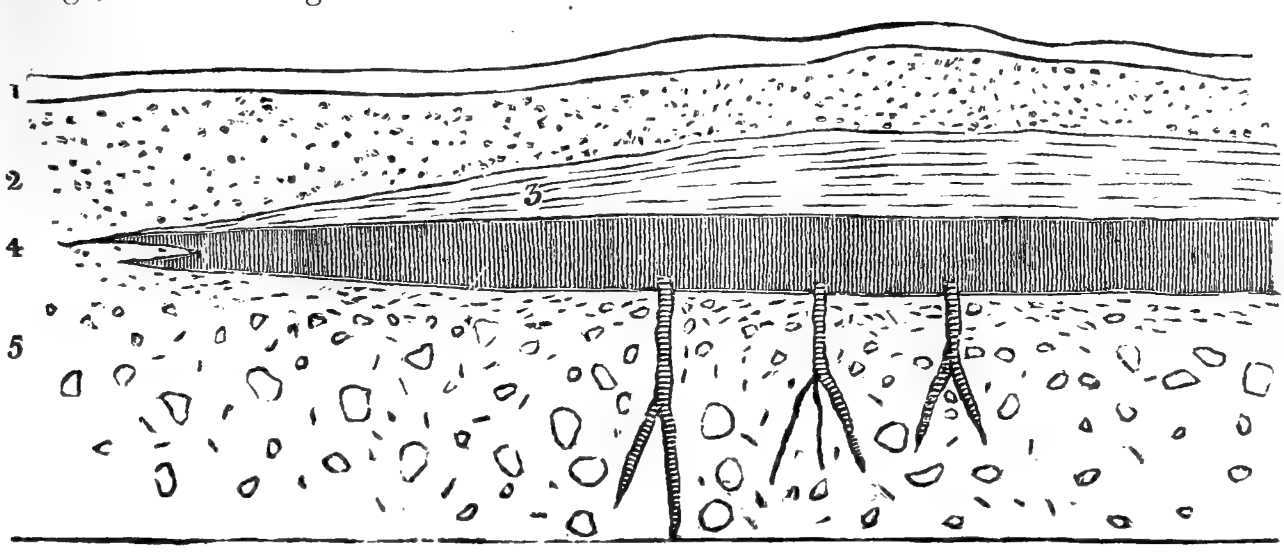
This deposit of gravel may be seen covering *the boulder-clay* in the old quarry of Cowpits, in the Water of Leith, at the Dean Bridge in a quarry there, and for some distance above that quarry in the channel of the river. It may also be seen at a variety of places, the level of which is much above that of the boulder-clay, and resting immediately on the subjacent rocks,—as, for example, at Cousland, Chalkieside, Garlton Hills, Fullarton, colliers' houses near Carlops, Langlaw and Whitehouse quarries. The two last-mentioned quarries are not far from the summit of the Roman Camp, and are probably about 200 feet below it. There is a quarry nearer the summit of that hill, where I think it does not exist. My present impression is, that this upper covering of gravel exists in our district as high as 900 feet above the sea.

This gravel appears to me to have been spread over the country, after the present inequalities of hill and valley had been formed. It exists in most places between the sea-shore and the old bank formerly spoken of. It exists even on the ground sloping down to rivers. At the Water of Leith, about half a mile above the Dean Bridge, the following section presents itself:—



A is a steep bank, on the south side, sloping to the river, about 40 feet high.
B is a steep bank, on the north side of the river.
C is low flat ground, between the river and the bank.
D is the channel of the river.
f is the boulder-clay.
e is the upper deposit of gravel now treated of.

On the west bank of Bilston-burn, and about half a mile north of Pentland village, the following section occurs:



No. 1. is the existing soil.

No. 2. is a covering of small gravel, being the deposit referred to in this part of the memoir. It is, in the above section, from 3 to 5 feet thick.

No. 3. is a bed of sand from 3 to 4 feet thick.

No. 4. is a bed of black peat from 1 to 2 feet thick. How far it extends towards the right of the figure I could not discover: on the left, it appears to have been destroyed by supervening causes, as it is there crushed, and a good deal mixed up with sand and gravel.

No. 5. is boulder-clay, obviously belonging to the deposit previously described under that name. The depth of the clay cannot be ascertained, but it is at least 8 or 10 feet thick. The imbedded boulders are not large. The upper part is gravelly.

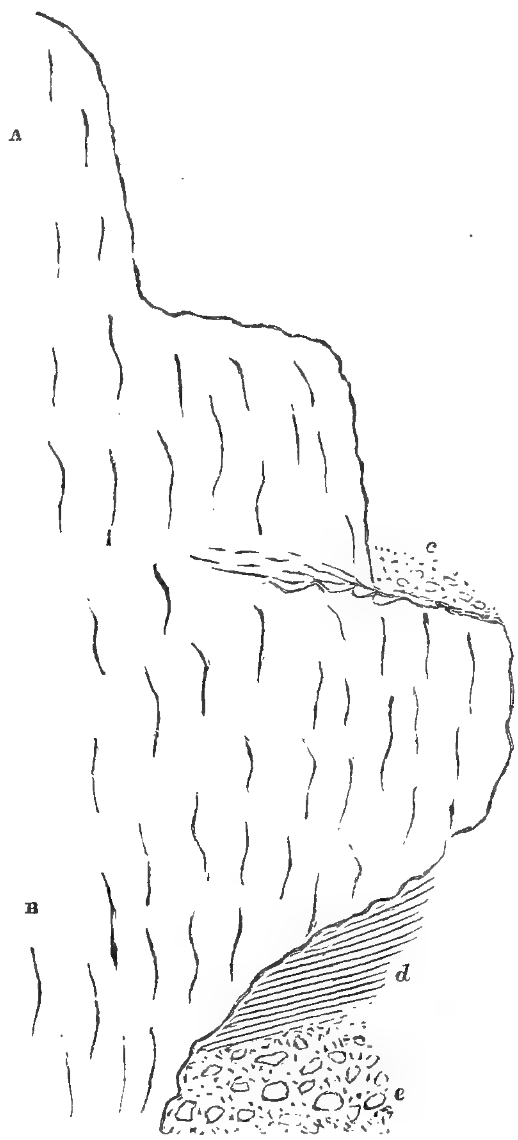
In this last and lowest deposit are numerous roots of trees, which shoot down from the peat, but which do not rise to the surface of the peat. It is obvious that the trees have grown in the clay,—if not before the peaty stratum was formed, at all events before the bed of sand and the layer of gravel were deposited. The spot now referred to is about 520 feet above the sea.

About a quarter of a mile above Straiton Mill, roots of large trees, apparently hazel, are also to be seen in the boulder-clay,—covered by a deposit of yellow gravel about 3 feet thick. This spot is 470 feet above the sea.

With regard to the nature of this gravel, I may observe, that the fragments are in general small and much rounded. They are seldom so large as a cocoa-nut. They consist mostly of the debris of secondary rocks, such as sandstones and limestones. I have found fragments of greywacke also in them. The usual colour of the deposit is yellow, from a quantity of ferruginous matter contained in it, and which may be derived from the sandstone pebbles it contains.

The *depth* of this deposit is in some places considerable. Near Carlops it is 10 feet thick. Near Carrington and Temple, it forms rounded hillocks from 20 to 50 feet high. It sometimes occurs also in the form of ridges, which are either straight or circular. One of these ridges has frequently arrested my attention, in travelling along the road between Dunbar and Haddington. It is on the south side of the road, and about three miles from Haddington. On the east side of the Dalkeith and Edinburgh road, some remarkable ridges of this upper gravel make their appearance. They there go by the name of *Kaims*, and are thought by many to be artificial. They run for about a mile, and make a considerable bend in their course. Their steepest side is towards the east. A somewhat similar ridge I have noticed on the estate of Mr DUNDAS of Arniston, near Outerstown. It is on one side (the north-east) about 40 feet above the adjoining plain, on the other side about 20 feet; and its width at the top is 30 feet. This ridge runs for about a mile; its direction is south-east and north-west, or parallel with the Moorfoot Hills, which are distant from it not more than half a mile. Similar ridges occur at Fullarton and Monk-Loudon,—and are apparently continuations of the Outerstown ridge just mentioned. They vary in height from 30 to 50 feet.

There is a remarkable accumulation of sand and gravel in the quarry of Blackford Hill, which I think must be referred to the deposit I am now describing. AB is the cliff or precipitous face of Blackford Hill, which has been extensively quarried. It is about 130 feet in height. The rock is chiefly clinkstone,—in some parts a fine-grained greenstone. At the base of the cliff, *e*, there is an accumulation of gravel, consisting chiefly of felspar, but containing also pieces of coal-sandstone, not much rounded. Above the gravel is a bed of sand, *d*, which is in contact with the overhanging face of the rock. The upper part of the sand, next to the rock, contains numerous pieces of shale and coal. It is proper to add, that, on clearing away the sand, I found the face of the cliff very much rutted and scratched. The direction of the scratches is nearly east and west. About 30 feet up the cliff, there is another deposit of gravel, *c*, including pieces of coal-sandstone. I learnt from a labourer who had worked in the quarry for fifteen years, that this deposit of sand and gravel, when it was discovered, extended in an east and west direction,—*i. e.* along the face of the cliff about 120 yards, and in a north and south direction, or from the face of the cliff about 50 yards.



The base of this cliff is about 320 feet above the sea.

It appears to me, that the gravel and sand in the above locality, must have been brought from the eastward.

II. *Inferences or explanations suggested by the foregoing phenomena.*

I have now detailed all the facts, which have come under my observation, regarding the superficial deposits of the Lothians. Though I have attempted to present these facts in a certain form or order of arrangement, I have in doing so endeavoured to abstain as much as possible, even in my own mind, from theorizing regarding them. The advantage of following this course is, that it not only enables *myself*, by reviewing the whole phenomena together, to draw inferences of a general nature, but that it also enables *others* to build on facts alone, any theories or opinions different from mine.

I proceed then to state very briefly, the inferences which appear to me to be warranted by the facts described in the first part of this memoir.

Looking generally to the whole series of superficial deposits enumerated, every one will admit, that they shew the existence of large bodies of water, sometimes still and tranquil, sometimes in violent agitation and movement. Another conclusion is, that these bodies of water must have stood at a level far above that of the present ocean.

It is, however, proper to look at the subject more in detail.

1. That the stratified rocks of this district, when they were broken up, in the manner explained in the previous part of this memoir, were covered by water, is evident from many circumstances. (1.) There is no reason to suppose that the water, at the bottom of which they were originally deposited, had withdrawn before this period. (2.) The occurrence of debris, and especially of clay, in fissures intersecting the strata, cannot be explained otherwise than by supposing, that water impregnated with sediment, had flowed into the fissures. It is not merely in *slips* that such sedimentary matter is found, but also in the *joints* which intersect individual strata. In the limestone quarries of the district, the circumstances now alluded to are very strikingly exhibited. In almost all of them, the slips and joints which intersect the rocks, are found filled with a dark yellow clay or mud, of fine consistency. As these fissures are occasionally more than a foot wide, and many feet in depth, the quantity of clay is in these cases very abundant, and implies a very considerable period for its deposition. The limestone quarries where I have noticed this clay in greatest abundance, are at East Salton, Fullarton, Monkloudon, and Burnbank near Carlops. Now the three places last mentioned, are situated about 900 feet above the present level of the sea; so that the waters which introduced the sedimentary matter into the fissures that intersect the stratified rocks of the district, must have been the waters of a sea which stood above the top of Arthur Seat. (3.) That there was such a body of water, at the epoch when the strata were dislocated, is placed beyond all doubt, by the occurrence in the district of extensive beds of sand and gravel immediately over the rocks. These (as previously explained) have been traced for many miles, lying beneath the boulder clay. That it forms no part of the boulder clay, and must have been deposited before it, is evident not merely from the great difference in the nature of its materials, and the mode in which they are arranged,—but also from several other circumstances, one of which may be noticed. The agent (whatever it was) which brought the boulder-clay, has transported it *from* the west. It is impossible to discover any boulder or other member of that deposit which has had even the appearance of a movement *towards* the west. But it is not so with the beds of sand below the boulder clay. It was

mentioned in the first part of this memoir, that the sand covering a workable coal-seam at Joppa had been transported towards the *west*, carrying with it fragments of the coal;—whilst fragments of the same coal were found in the lower part of the boulder-clay there, considerably to the *eastward*. The fragments in the clay had of course been washed out from the sandy bed beneath, by the passage of the boulder-clay over it.

There is therefore complete evidence of there having existed, when the stratified rocks were broken up, a deep and extensive body of water, at the bottom of which there were deposited those beds of sand and gravel, which form the lowest of the superficial deposits covering the district. Nor is it difficult to perceive, from what source, the materials of this lowest deposit were derived. That, immediately after the elevation and rupture of the strata, their shattered remains must have been washed away in enormous quantities, is evident on the slightest reflection. The simple fact, that—even in the greatest slips known in the district—the rocks on each side of them are at one and the same level, proves that, on one side at least, there must have been prodigious abrasion. One of the slips at Loanhead caused the strata to sink down 360 feet,—leaving of course on the other side of the slip a lofty precipice of that height. One of the slips at Blinkbonny must in the same way have produced a precipice of 480 feet. The great Sheriffhall slip caused a precipice, which in one place would be no less than 500 feet,—which is as high as Arthur Seat is above the adjoining district. It is true that these precipitous fronts of shattered strata would not overhang the abyss formed on one side of them, but would slope back from it. There were other circumstances, however, quite sufficient to insure their speedy demolition. In the first place, the dislocation of its strata, and the ponderous rubbing of their broken edges against one another, when the sinking or elevation took place, would tend greatly to loosen the tenacity of the materials. In the next place, we must recollect the excessive multitude of slips, by which the entire coal-field was shattered. Indeed, so numerous are they, that there is scarcely an acre of it not absolutely reticulated by slips. It is evident, therefore, that the entire district, immediately after being so fractured and fissured, would present the spectacle of numerous ridges and precipices, little calculated, from their size, form, or internal condition, to resist the abrading action of tides and currents. We see, accordingly, that they were all worn down, and so completely, that every trace on the surface of the previous convulsions and dislocations became obliterated and effaced. The strata which had composed these submarine ridges were entirely washed away, and converted into mere debris;—and these again were ultimately worn down into gravel, sand, and mud.

Such I conceive to have been the origin of the lowest superficial deposit, which I have described in a former part of this memoir. It is true that this deposit has not been traced in all parts of the district,—which it ought to have entirely covered according to the above explanation. But it will be remembered, that much of this lower deposit may have been washed away by *supervening* agents; and indeed it is evident, from what was observed at Joppa, that the boulder-clay did operate in this manner.

2. The *Boulder-Clay* in different parts of the district, exists (as has been previously mentioned) at the height of nearly 960 feet above the present level of the sea. The body of water at the bottom of which this clay was deposited, must have greatly exceeded this depth. This inference would be warranted by the single consideration of the weight and size of the boulders imbedded in the clay;—but it is still more forcibly suggested, by the fact, that some of the hills over which these boulders were transported, are 1200 feet above the sea.

That this body of water was, when these boulders were transported, in a state of violent and extensive movement, must be obvious even to the most superficia observer. Some persons have endeavoured to account for the phenomena, by supposing a *wave* or partial body of water to have passed over the country. But it must have been a *wave* twenty or thirty miles long in one direction;—for it is one and the same deposit which is found on the coast of Fife, and in the county of Mid-Lothian, both in its northern and its southern parts, and therefore must have been one body of water which flowed over all that district. The force and violence of the aqueous movement is demonstrated, by the enormous quantity of the materials transported by it,—by the fact of the clay in which they are imbedded not exhibiting any laminæ,—by the imbedded boulders not being arranged according to size or weight,—by the ruts or scratches both on these boulders and on the subjacent rocks,—and by those deep excavations or scoopings in the rocks, several of which I particularly described. That the same body of moving waters which thus flowed over the district, extended to distant parts of the country, is farther indicated by the character of the boulders, *most* of which are round-shaped from having been rolled far, and *many* of which belong to rocks that do not occur nearer than fifty miles. The circumstance last alluded to, shews also the direction of the rush of waters. The mica-slate blocks *must* have come from the westward, because it is only towards the west that this species of rock exists. This inference is confirmed by the ruts or scratches, which all lie in the same direction;—and it is confirmed by yet another circumstance. The place where the boulders are largest, and where they exist in greatest abundance, is on the east side of the hills,—as, for example, along the shore between Seafield and Joppa. The fragments of rock there, are from ten to fifty tons in weight. They have evidently rested there, in consequence of being sheltered by Arthur Seat, the Calton Hill, and other rocky eminences, from a western rush of waters.

In the sandstone quarry at Joppa, may be seen a very striking proof, not only of the violence which accompanied the transport of this boulder clay, but also of the direction in which it went. It has been mentioned, that the edges of the strata are there covered by the boulder clay. The strata dip at an angle of about 60° to the E.SE. On the west side of the quarry is a stratum of shale, which runs along the whole length of the quarry for several hundred yards. This stratum, along its edge or outcrop where it is in contact with the clay, is for a considerable distance raised up from its natural slope of 60°, to almost a vertical position. There is a line of crack visible, occasioned by its having been bent or forced into that position;—which crack runs parallel with the edge or crop of the stratum, and is distant from it several feet. That the shale has been pushed eastwards into this position by the superincumbent boulder-clay, I think there can be no doubt.

The remarks now offered correspond entirely with the inferences of a very distinguished member of this Society, the late Sir JAMES HALL. He it was who first pointed out the phenomena in this neighbourhood, and particularly to the west of Edinburgh, of what has been termed "crag and tail." He discovered various ruts and scratches on different parts of Corstorphine Hill; and ascertained their average direction to be W.SW. by compass. It was his opinion, that the body of water which flowed over Corstorphine Hill, carrying with it the enormous boulders which were left on its east side, could not have been much less than 1000 feet deep.

My own observations in other parts of the district, quite confirm Sir JAMES HALL'S opinion. I have shewn that the boulder-clay, with its imbedded blocks, reaches in several places to the height of 960 feet above the present sea. It cannot be imagined, that the water which transported these blocks did not reach a higher level. On the contrary, the water must have been very deep, in order to have possessed the force necessary to transport materials so weighty and so extensive. If therefore, we find these materials reaching to a height of 960 feet, the water which transported them must have been at least 1700 feet above the sea,—a depth sufficient to have covered most of the Pentland and Lammermuir hills.

It is not difficult to understand, where the imbedded blocks of stone may have come from. But it may form a question, where the enormous mass of clay could have come from. That it was derived from the westward, is rendered probable from the fact of the boulders imbedded in it having been brought from the westward. But farther than this, I do not think we can yet safely go in quest of the true explanation. I have several times heard a member of this Society, possessed of great practical knowledge, express opinions on this subject, in which, however, I cannot concur. He thinks that the immense volume of boulder clay, or "old alluvial cover," as he has termed it, may be accounted for by the slips which have broken up the subjacent coal-measures. No doubt there must have been, as just explained, an immense mass of debris occasioned by these slips. But these materials never could have produced the *boulder-clay* of the district. They would necessarily produce a deposit of a totally different character; and moreover, they could not have been spread over the district except by a body of water, much more tranquil than that which existed during the deposition of the boulder-clay. The nature of the materials forming the deposit beneath the boulder-clay, and the manner in which they have been deposited, equally attest the comparative tranquillity of the waters at that period,—the period which immediately succeeded the elevation and dislocation of the stratified rocks. On the other hand, it is most manifest that, at the period when the boulder-clay was deposited, the waters were in a state of violent and extensive commotion. It is just what might be expected to result from a cataclysm, which washed over a country and bared it of its soil to an enormous extent;—and this idea is a good deal strengthened, by the fact of an elephant's tusk having been found in the boulder-clay. For this discovery shews, that when the boulder-clay was deposited, a country inhabited by animals existed to the westward, the soil of which was therefore such, as to support a luxuriant vegetation. It is true that no vegetable remains have as yet been noticed in this boulder-clay; but, considering the extraordinary resistance which attended the transport and deposition of this clay, it is not likely that many vegetables would be preserved in it, or indeed that there should be even any *animal* remains in it, except such as were of the hardest materials.

3. After the epoch of the boulder-clay, the waters which then covered this district, appear not to have been agitated by any currents of an extraordinary description. The deposit which lies above the boulder-clay, contains no blocks of stone that required great force of water to transport them. On the contrary, they are comparatively small and angular in their shape, and may all have been derived from the neighbouring district. But that there were currents in the water at this time is proved, (1.) by there being such a deposit; and, (2.) by the subjacent boulder-clay having been in some places entirely washed away. The materials thus abraded from the boulder and stony clay, may have supplied, in part at least, the beds of clay and sand, which form the deposit next in order.

4. When we examine the character and contents of these sandy and argillaceous deposits, we see proofs of much greater tranquillity in the waters, than would have existed during the immediately preceding period. The accumulations of clay and sand which prevail over considerable portions of the district, shew that the submarine currents had greatly lessened in force. I have mentioned, that the pieces of rock found in this particular deposit are generally very small, and that the beds of clay and of sand are usually laminated. This is a state of things which clearly justifies the above inference.

At this period, it would appear that the level of the waters had probably fallen below what it had previously been, for the sand and clay which marks the epoch now referred to, is not found at a higher level than 700 feet above the present sea. This would lead to the supposition, of there having been an elevation of the land at a period immediately antecedent to the deposit now referred to.

I have alluded to the remains, both vegetable and animal, imbedded in the clay which form the lower part of this deposit. All the vegetable remains are terrestrial, and are such as to indicate the existence of a river or rivers of considerable size. The animal remains are marine;—the shells, such as are usually found in great estuaries, where there is a commixture of salt and fresh water. It seems probable, therefore, from a consideration exclusively of these remains, that during this particular epoch, when clay and sand were deposited, there existed a bay or arm of the sea into which fresh water flowed. But as the waters then stood at least 700 feet higher than they now do, that arm of the sea must have stretched across towards Glasgow, in which case Arthur Seat, and most of the Pentland hills, were islands, as Inchkeith and Inchgarvie now are. Into this sea, would of course be poured immense quantities of muddy sediment derived from the boulder-clay, and which would be deposited in all the hollows of the boulder and stony clay. But at any given place, it would not be at every moment deposited in exactly the same quantity. The tides would in this respect have a considerable influence. As the largest supplies of the muddy sediment would be afforded by rivers, the deposition would be suspended, or at all events diminished, by every influx of the tide—when probably a slight sprinkling of sand would be thrown upon the muddy deposit which took place during the ebb-tide. In this way, we can understand how the laminæ visible in the Portobello brick-clay were formed;—and if this inference be correct, then each layer of clay would denote the period of one tide or half a day. As each layer is, on an average, about one-sixth of an inch thick, 120 feet (which is supposed to be the thickness of the fine clay at Portobello) would denote a period of twelve years as the interval of its deposition.

If from any cause, the waters ceased to be supplied with muddy sediment, then sand alone would be deposited. It would appear, that for a certain period muddy sediment had been supplied in this district,—after which sand, the natural product of oceanic waters, was accumulated in large banks over the clay. The problem then is, to account for a supply of clay or mud by the rivers in unusual quantity, and for a certain period only. Perhaps a solution of the problem may be found in the elevation of the district which took place immediately before this period. The bottom of the sea as it became exposed to the united influence of rivers and rains, being unprotected by a vegetable covering, would afford an abundant supply of muddy sediment, and would continue to afford this supply until vegetation had consolidated the soil, and the rivers had acquired for themselves permanent channels.

Notwithstanding these extraordinary physical revolutions, and the fact that the sea was at least 700 feet higher than it now is, it would seem that this district was in many respects much the same as it is at present. The character of its vegetation, and the description of animals which inhabit it, could not have been very different from what they are in our own epoch. We see from the remains of oak trees and hazel nuts found in the Portobello clay, that there were forests in the country, and that these forests contained trees of the same sort as those which we behold flourishing. We see from the remains of deer, of whales, of the grampus and the seal which were discovered in this deposit, in the immediately adjoining districts, that the same sort of animals which exist now, moved then on the land and in the waters.

5. The next epoch in the history of these superficial deposits shews, that another change subsequently took place in the relative level of sea and land, by which the waters were brought down to a level much nearer that which they now occupy. I allude to the period, during which there was formed the old bank, that has been described in the former part of this memoir, as running nearly parallel to the present shore, and the base of which is between 30 and 40 feet above the level of the sea at high-water mark. That this old bank is an ancient sea-cliff cannot be doubted by any one who looks at its parallelism with the shore,—the uniformity in the height of its base above the sea, and the occurrence of marine shells or sand almost every where between its base and the sea. That cliff, it is true, is in some places so worn down, as not to be now traceable; and almost every where it has a slope which is possessed by no existing sea-cliff, the base of which is washed by the waves. But these circumstances so far from being incongruous with the opinion above stated, serve in the strongest way to confirm its accuracy.

6. The sea then stood only between 30 and 40 feet above its present level,—so that, after the epoch of the deposit last described, there must have been an elevation of the land, to the extent of nearly 700 feet. But after the formation of this old *sea-bank* just mentioned, an opposite change of levels still more prodigious must have taken place. For we have seen that, after this period,—nay, after the formation of many of our existing valleys,—and after the surface of the country was covered with vegetation,—there was spread over this district, what I have termed an upper covering of gravel and boulders, and *that* to the height of no less than 900 feet above the sea. If this be the fact, it necessarily suggests a revolution of a most stupendous character, for it is difficult to explain the phenomena in any other way, than by supposing that the whole district had sunk, so as to be submerged for a time beneath the waters of the ocean,—and that it afterwards rose up to the level which it has ever since preserved. I confess that this is a supposition which would require to be confirmed by very strong evidence indeed, before it can be assented to. It appears to me, however, to be an inference, and the *only* inference, deducible from the facts described in a previous part of this memoir;—and therefore that it must be admitted and credited, notwithstanding our natural and very salutary aversion

to credit so stupendous a change. At the same time it should be observed, that this particular revolution would form only one of a series of revolutions indicated by the successive deposits existing in the district;—and that however unprepared we may have been, some years ago, for crediting the recent occurrence of extensive changes, we have now far less scruples in giving our assent to the evidence of it, when we consider, that at this very moment there are whole continents gradually sinking, and others gradually rising from beneath the waters of the ocean.

In conclusion, and with reference to the geology of the district generally, I may observe, that it is impossible for any one who is familiar with the outward features of it not to perceive, that even they bear, in a considerable degree, the visible impress of the subterranean changes and convulsions which I have, in the foregoing memoir, attempted to trace. I have stated, for example, that there are *two coal basins*, formed by the position into which the stratified rocks have been thrown. If this were the case, we should expect to find, that the lowest parts of the whole district would be along the centre or trough of these basins. This is found to be the fact. The trough of the principal basin commences at Fisherrow, and runs up by Dalkeith, Roslin, and Pennicuick, to Carlops. Now, these are known to be the points of lowest level in that part of the country; and accordingly, it is by these places that the principal river there, viz. the Esk, is found to flow. The other basin is, in like manner, for a considerable distance watered by the Tyne. It is hardly necessary to add, that it is owing to the same cause, that the deepest and most extensive superficial deposits of sand and clay, which overspread the district, occur in the central parts of the Esk basin.

I might advert to many other proofs,—as, for example, the existence of the Roman Camp hill, and the high ridge which runs to Tranent,—in illustration of the proposition that the present configuration of the country has been greatly affected by even the most ancient geological changes. But I pass on, to notice the effect which more recent geological changes have also had in this respect. It is not possible to go to any part of the district, without witnessing deep and indelible traces of that violent rush of waters, which bared all the western faces of our hills,—left on their eastern flanks a heap of rubbish, including boulders many tons in weight,—and scooped out hollows where the current was contracted and confined. I do not refer now merely to excavations discovered in mining operations:—I refer to others visible to the naked eye and the most casual observer, and which have ever since remained filled with large bodies of water. That the hollows now occupied by Duddingston Loch and Lochend,—and the low ground on the north and south sides of the Castle Rock of Edinburgh, till lately the receptacles of considerable accumulations of water,—that these localities bear the impress of that impetuous torrent, which left traces of its violence on the adjacent rocks, it is impossible to doubt. These rocks, we see, were able to stand the shock, and turn aside the waters, charged though they were with blocks of enormous weight. The increased force and rapidity of the stream occasioned by these impediments, would cause a greater scooping out to take place along their sides; and hence the reason why the ground at the west, as well as north and south sides of most of our trap hills, is lower than it is any where else.

The examples now given are sufficient to show, that even the outward form and present surface of the district, attest the occurrence of several of those mighty convulsions, which, from time to time, changed so completely the condition of all things in this part of the globe.

APPENDIX A.—p. 260.

The tabular chart referred to in the Memoir, represents, at one view, all the workable coal-seams and limestone strata in East-Lothian and Mid-Lothian, the names of the places where they are known to exist, with their thicknesses and mutual vertical distances at these places.

The first column in the table, is occupied with a list of the coal-seams and limestone strata, arranged in the order occupied by them in a supposed section of the basin, from top to bottom. Where the coal-seams are known by particular names, these names are stated,—with such a distinguishing mark as to shew the place where this appellation obtains. This first column consists of seventy-three lines, of which sixty-six are occupied by coal-seams, and seven by limestone strata.

The other columns in the table are devoted to a statement of the thicknesses in feet and inches, and the mutual distances in fathoms and feet,—of the strata mentioned in the first column, at various places specified in these columns.

The structure of the table will, however, be best understood, by making from it a few extracts; and these will be so chosen, as to illustrate and verify the statements in the text.

Thickness of "Great Seam" *of Coal in Esk Basin.*

West Side.										East Side.									
Joppa Shore.	Brunstain.	Niddry.	Woolmet.	Drum.	Gilmerton.	Loanhead.	Dryden.	Brughlee.	Newhall.	Preston-grange.	Wallyford.	Cousland.	Cowden.	Newmills Level.	Bryants.	Sinkquar-ry.	Arniston.	Stobhill.	Vogrie.
7 ft.	6½	8½	8	8	9	8	7½	8	1?	7	9.2	7	8.8	3.4	9	5	7	8?	4

Thickness of "Great Seam" *in Tyne Basin*

Preston Pit.	Tranent.	Elphin-stone.	Pearlstone.	Clouts.	West Win-dygowl.	Blindwells.	Sandee Quarry.	Blink-bonny.
7 ft.	9	9 to 10	6?	9	7.8	6 or 7	9	6½

Thickness of "North Greens" *Coal in Esk Basin.*

West Side.											East Side.				
Joppa.	Brunstain.	Niddry.	Drum.	Moredun.	Gilmerton.	Loanhead.	Dryden.	Brughlee.	Glencorse.	Carlops.	Fordel.	Fuffet.	Kippilaw.	Bryants.	Arniston.
3½	4.8	4.3	5	5	5.3	4½	4½	4.4	1.10	1.6	2.3	2.5	3	3.3	2.2

Thickness of "NORTH GREENS" *in Tyne Basin.*

West Side.						E. & S. Sides.		
Oxenford.	Edgehead.	Blink-bonny.	Stoddart's Pit (Cockenzie).	Penstone.	Cranston.	Middleton.	Side Quarry.	Monk-Loudon.
2	1.10	1.8	2?	2	2	1.9	1.10	1.8

These general statements are useful, as they serve to shew in what parts of the district, the vegetable materials from which the coal-seams were formed, the different coal-seams were most accumulated;—and where, on the other hand, they were more scantily supplied. On reviewing the above extracts, it is seen, that the two coal-seams to which they refer, are thickest at or near a line drawn from Gilmerton to the south of Tranent, and that towards the southern limits of the district these coal-seams thin away till they cease altogether, or at least become so thin as to be unworkable. This inference is confirmed, by an examination of all the other coal-seams, recorded in the tabular chart above referred to.

It is interesting to mark, not merely the variations in the *general thickness* of the coal-seams in different parts of the district, but also the variations in the nature and thickness of the *bands* which compose the seam. It is not so easy to procure information, so minute and precise as that now alluded to;—and therefore it is, that the following table is less extensive than those above quoted. But it is nevertheless sufficient to afford an insight into a very interesting subject.

The analysis which has now been given of the two principal coal-seams in the district, shews, that they are not homogeneous in their composition, but that they consist, or are made up of layers, the materials of which are extremely different. It is evident also from the nature of these materials, that they could not have been deposited and spread out over the district simultaneously. The several layers or bands of shale, sandstone, clay, and vegetables, must have been deposited separately and successively, so that, in fact, an ordinary coal-seam, when analysed, presents a miniature section of the various strata which compose an entire coal-field. This analogy holds true, in yet another respect. Not only does it appear, that the bands composing the coal-seams have been separately and successively deposited,—but moreover the causes which brought about this deposition, must have operated over extensive areas,—and been subjected to little or no disturbance. The fact, that bands of shale, not more than a few inches thick, scarcely vary in thickness, over an area several miles in extent, shews an almost incredible degree of stillness and placidity in the carboniferous waters. Any variation in thickness which they do exhibit, must have arisen, generally speaking, not from disturbing causes, but from a variation in the supply of sediment;—and hence we see, that the bands characterizing particular coal-seams get thinner in some places, and entirely disappear in others.

Component Parts of " North Greens*" Coal at the following places—*

Joppa.	Gilmerton.	Glencorse.	East Bryants.	Bryants, (Mansfield Pit.)	Blinkbonny.	Arniston.	Kipplaw.	Brughlee.	Loanhead.
Ft.	Ft.	Ft.	Ft.	Ft.	Ft.	Ft.	Ft.	Ft.	Ft.
Shale, &c. roof.	Shale, &c. roof, 30	Shale, &c. . 1.10				Blue shale, . 3.4 Ironstone, . 0.3			Shale, &c. roof, } 51
Parrot coal, 0.4 Rough Coal 3.2	Parrot coal, 1.1 Rough coal, 4	Parrot coal, 0.4 Shale, . . 0.6 Coal, . . . 0.2 Shale, . . 1.0 Coal, . . . 0.4	Coal, . . . 1.5 Parrot coal, 1.0 Coal, . . . 0.8	Coal, . . 0.7 Parrot coal, 0.6 Coal, . . . 1.0	Coal, . . . 0.4 Parrot coal, 0.5½ Coal, . . 0.10½	Parrot coal (in 3 bands), } 0.10 Rough coal, 0.10	Parrot coal, 0.8 Coal, . . 2.4	Parrot coal, 0.10	Parrot coal, 0.0
Shale, . . 3.6 Rough coal, 1.8 Sandstone pavement, } 90	Sandst. pavement, &c. } 180					Blue shale, . 0.4 Sandstone, . 0.8			

Component Parts of " Great Seam*" of Coal at the following places—*

Joppa.	Preston Pit.	Tranent.	Cowden.	Niddry.	Bryants.	Lingerwood.	Stobhill.	Loanhead.	Drum.	Gilmerton.	Brughlee.
Ft.	Ft.	Ft.	Ft.	Ft.	Ft.	Ft.	Ft.	Ft.	Ft.	Ft.	Ft.
Shale, roof, 15						?		Shale and sandst. roof } 23		Shale & sandst. roof, } 36	
Rough coal, 0.3 Parrot coal, 1.0 Coal, R. & S. 2.3 Fire-clay, 0.6 Coal, R. & S. 3.0	Rough coal, 1.0 Fire-clay, 2.0 Rough coal, 1.2 Soft shale, 0.4 Rough coal, 3.4 Bad parrot, 1.2	Rough coal, 1.0 F. clay, ½ to 4.0 Rough coal, 1.2 Soft shale, 0.4 Rough coal, 3.4 Bad parrot, 1.2	Rough C. 1.6 Shale, . 0.1½ Rough C. 1.10½ Shale, . 0.1 Rough C. 3.0 Parrot C. 1.1½ Rough C. 0.11½	Rough coal, 0.6 Parrot coal, 2.8 Coal, R. & S. 2.0 Fire-clay, 0.6 Rough coal, 3.0	Coal, . . 1.5 Do. . . 0.11 Checkstone, 0.6 Main coal, 1.9 Rough coal, 0.4½ Parrot do. 0.3 Duff ? . 1.3	Coal, . . 2.6 Black shale, 7.0 Coal, . . 7.0	Coal, . 2.3½ Midstone, 0.3 Coal, . 3.6½	Rough coal, 5½ Sandstone, 1.0 Rough coal, 1.6	Parrot, 2.0	Parrot, 0.0	Parrot, 0.3
Shale, . . 1.6 Sandstone, 40		Shale, and sandstone pavement, } 27					Shale and sandstone pavement, } 17½	Shale and sandstone pavement, } 23		Shale & sandst. pavem. } 21	

I proceed next to furnish some information taken principally from the tabular chart referred to in the text, in regard to the *Limestone* strata in the basin. These are :—

1. The *Coldcoats Limestone*, which is about 350 fathoms from the top of the basin, or about the middle of it.
2. *Another Limestone*, about 50 fathoms below the first.
3. *Another Limestone*, about 150 fathoms below the first.
4. *Another Limestone*, about 22 fathoms above the North Greens Coal.
5. *Another Limestone*, about 15 fathoms above ditto.
6. The *Gilmerton Limestone*, about 14 fathoms below ditto.
7. *Burdie House Limestone*, lying near the bottom of the basin.

Though it is undoubted, that these several strata of limestone exist, and that they extend over a large portion of the district, there is some difficulty in identifying each stratum at all the places where they shew themselves. This difficulty applies, however, only to the three limestones, marked above 4, 5, and 6, which are so near in respect of position in the basin, and so similar in respect of thickness, that they are on this account apt to be confounded ;—and to say the truth, I have a suspicion that No. 4 and 5 may be one and the same seam. It is very possible, therefore, that the columns in the following table, enumerating the localities where these three limestones exist, there are several mistakes ;—but it will be immediately shewn that these mistakes do not invalidate the particular inferences to be deduced from the table.

Coldcoats Limestone No. 1.	Ft.
Joppa,	3
Magdalen Pans,	3
Gilmerton,	2.11
Coldcoats,	3.4
Bryants,	2.9
Newmills level,	2.5

Limestone No. 2.	
Joppa,	3
Magdalen Pans,	2
Bryants,	3
Cowden,	3.6
Dryden ?	2
Newmills level,	2.11
Dykeneuk Bridge (1 mile S. of Penicuik),	4

Limestone No. 3.	
Woolmet,	3
Gilmerton,	2.7
Loanhead,	2
Prestongrange,	2.4
Wallyford,	1.8
Newmills level,	4.6
Bryants,	2 ?
Arniston,	3
Stobhill,	2.7

Limestone No. 4.	Ft.
Easter Duddingston,	8
Gilmerton,	6
Loanhead,	8 to 10
Venture-fair (2 miles S. of Penicuik),	4 ?
Masterton Mains,	15
Cousland Dean,	15
Lawfield,	16
Fullerton,	19
Esperston,	14
Middleton,	18
Crichton Dean,	20
Spilmersford,	13
Burnbank,	20
Bents,	12 ?
Side,	22
Turnydykes,	30
Edgehead ?	18
Fuffet ?	15 to 18
Catwell ?	30
Lanridge,	13
Red Coll,	12
Trabroun,	12
Jerusalem,	10 to 15
Harelaw,	12 to 14

Limestone No. 7. ?	
Burdiehouse,	27
Carlops,	3

Limestone No. 5.	Ft.
Niddry Mill,	6
Gilmerton,	3
Bilston Burn,	6
Loanhead,	10½
Glencorse,	5
Dossie,	12
Old Fordel,	13
Cousland,	12
Monk-Loudon,	18
Middleton,	12
Burnbank,	10
Limeston Green,	9
Whitfield,	9
Rutherfurd,	9
Arniston ?	14

Limestone No. 6.	
Easter Duddingstone farm-house,	2.10 to 3
Niddry Mill,	4
Eastfield,	7½
Moredun,	5½
Gilmerton,	10 ?
Bilston Burn,	6
Dryden,	
East Saltoun,	18
Blinkbonny,	15

It appears from the above tabular statement,—

(1.) That those limestone strata which are the thickest, are situated at or near the bottom of the basin ;—and that the thinnest are in the middle of the basin, or about half-way down from the top.

(2.) That those limestones which vary *least* in thickness, are situated about the middle of the basin ;—and that the amount of their variation over the entire district, does not exceed a few inches.

(3.) That those limestones which vary *most* in thickness are situated at or near the very bottom of the basin ; and that the amount of variation is so great, as that in some parts the same stratum attains a thickness six times greater than that which it possesses in other parts.

(4.) That whilst those lower limestones indicate a great increase of thickness as they approach the Lammermuir and Pentland Hills,—the upper limestones (being far from hills,) indicate no tendency to thicken in any particular direction.

This inference is an important one, and therefore requires an attentive examination of the above statistical details. It is apprehended that they fully warrant that inference.

It may be thought that the Burdiehouse limestone, which is stated in the table as occurring at Carlops, and being so much thinner there,—is inconsistent with this inference. Independently of the doubt, whether the particular limestone at Carlops now referred to, is really the Burdiehouse bed, I may observe that the fact of this stratum being as near the hills at the one place as at the other, would be a sufficient answer to the objection. Another is suggested by the consideration, that there may have been a greater supply of sedimentary matter at Burdiehouse than there was at Carlops ;—and it has been elsewhere explained, that there is an essential difference between the origin and formation of the Burdiehouse limestone, and that of all the other limestones lying over it.

It will be obvious, that the correctness of *the above inferences* would be by no means impugned,—though mistakes should be discovered in the table, as to the particular *places* where the lower limestones exist. For *they all* indiscriminately shew the same tendency to thicken as they approach the hills.

Before concluding my extracts from, and remarks on the table of coal and lime strata, I may mention, that attempts have been sometimes made, to calculate the quantity of coal existing in particular districts, and thus to anticipate the period of its total exhaustion. These calculations are necessarily very vague and uncertain,—arising chiefly from the impossibility of knowing, whether the coal is in precisely the same condition in the unexplored parts of the basin, as it is in those places where it is worked. But any calculations, however imperfect, afford some degree of *approximation* to the truth,—which cannot mislead if viewed only as an approximation.

From the tabular chart of coal-seams above referred to, it appears, that if all those that are worked or workable, with several too thin to be worked, are taken into account,—there would be a total thickness of coal amounting to 188 feet. This statement, I observe, coincides pretty nearly with an estimate made by Mr Bald. He has very lately calculated the quantity of coal in the Marquis of Abercorn's estate at Easter Duddingstone,—through which the two *lower* series of coals run. The *lowest* he calls the Duddingstone Group,—comprising all between the " North Greens" and the " Wood Coal ;"—the *middle* series he calls the Joppa Group,—comprising all above the Wood Coal, as far up as the " Golden" Coal ;—the *uppermost* series in the basin, he denominates the Brunstain Group. The two former groups, Mr Bald calculates, contains (in the Marquis of Abercorn's estate) a total thickness of 108 feet. I observe from my tabular chart, that the uppermost series presents an average total thickness of 75 feet,—which, added to Mr Bald's estimate of the total thickness of the other two, would make the thickness of coal in the *entire* basin 183 feet,—being only 5 feet less than the result of my own calculations.

Let it be assumed, then, that the thickness of all the known workable seams is 183 feet. If there was one coal-seam of this thickness, it would be easy, after marking on a map the line of its outcrop, to calculate the extent of it, and the quantity of coal contained in it. But where this thickness is made up by seams, some of which are at the very bottom of the basin, and the others at the top, in consequence of which the extent of the latter bears but a small proportion to that of the former, the calculation is attended with difficulty. Even, however, if we could accurately ascertain the quantity of coal in each individual stratum, it would be wrong to include the whole as *available* or *attainable* fuel. Several of these coal-seams, and the best of them, are at some places at such a depth from the surface, as not to be capable of being reached by any means that are either known or likely to be invented. This remark is particularly applicable to the "North Greens" Coal, which affords the largest supplies of parrot coal, and which it is probable is, all along the trough of the Esk basin, not less than from 500 to 800 fathoms below the surface. It may be safely said, that all the coal of this and other seams which are more than 200 fathoms below the surface, is entirely unattainable.

According to this view, the uppermost or Brunstain series of coal would all be brought within our calculation, as its lowest beds are not nearly so deep as 200 fathoms from the surface. They do not stretch farther south than the great 80 fathom slip, which runs under Sheriffhall pigeon-house and Dalkeith Palace,—having been all washed away on the south side of that slip. They extend, therefore, in a N. and S. direction about four miles, and in an E. and W. direction about three miles, so that, if they had been all horizontal, there would have been twelve square miles of coal. As they are basin-shaped, some addition should be made on that account;—but on the other hand, as several of them do not run so far as others, a still greater deduction on that account must be made. On the whole, then, it may be not unfairly assumed, that the Brunstain or uppermost series comprises 75 feet or 25 yards of coal, extending over ten square miles. Now 36 cubic yards of coal weighs about 32 tons; and as there are 3,097,600 square yards in a square mile, a square mile of coal, one yard thick, would contain rather more than 2,800,000 tons of coal;—and a square mile of coal 25 yards thick would contain about 71 millions of tons;—so that the Brunstain group of coals must contain not less than 710 millions of tons.

From the previous explanations, it is obvious, that it would be much more difficult to calculate the quantity of available coal in the two lower groups of coals. Suppose them all to constitute one seam, 36 yards thick, and situated about half-way between the two,—it would, at the deepest part of the Esk basin, viz. at Fisherrow, be about 500 fathoms below the surface, and in the southern parts of the district more than 200 fathoms. It may, therefore, be roughly estimated, that about one-half of these lower coal-seams are altogether unattainable;—and that instead of calculating their extent at about 100 square miles, we should estimate it at not more than 50 square miles. One square mile of coal 36 yards thick would produce 102 millions of tons, and therefore the two lowermost groups may be supposed to contain at least 5000 millions of tons.

I believe that in such calculations one-half is generally deducted for waste and for deteriorated coal;—so that the total quantity of marketable and attainable coals in the Lothians may be estimated at about 3000 millions of tons. But from this a deduction must be made for what has been already worked out. What this quantity may be it is very difficult to say;—but assuming it to be one-fourth, there would be left 2250 millions of tons. The annual home consumption of coal in Great Britain is at present about 30 millions of tons,—so that there is enough in the East-Lothian and Mid-Lothian coal-fields to supply the whole nation, for seventy-five years.

APPENDIX B.—P. 268.

Table of Backs and Cutters.

Names of Places.	Backs.	Cutters.	Quality of Rock.	Dip of Strata.	Angle of Dip.	Angles formed by Back and Cutter.
Turnydykes Quarry, . . .	North		Lime strata	North	3° or 4°	
New Craighall, ...	N. 37° W.	N. 40° E.	Coal	E.SE	5° or 6°	obt. 103° ac. 77°
Monktonhall?	NE.?					
Mucklits Pit, ...	N. 35° to 40° W.	N. 35° E.	Coal	NW. by W.	8°	obt. 107° ac. 73°
Gilmerton, ...	N. 48° W.?	N. 42° E.?	Coal	S. 47° E.	32°	obt. 90° ac. 90°
Ditto, ...	N. 38° W.		Ditto	S.SE.	33°	
Ditto, ...	N. 30° W.		Ditto			
Ditto, ...	N. 56° W.	N.	Ditto			obt. 124° ac. 56°
Ditto, ...	N. 67° W.		Ditto	SE. ½ S.		
Ditto, ...	N. 79° W.	N.	Ditto			obt. 101° ac. 79°
Cowden new Colliery, . .	North	W.NW.?	Great seam of Coal	NW.	24°	obt. 124° ac. 56°
Easter Cowden Quarry, .	North	W.NW.	Sandstone	S.SW.	10°	obt. 124° ac. 56°
Wester Cowden Quarry, .	NW.	North.	Sandstone	North	10°	ob. t. 135° ac. 45°
Dalkeith Park, one-fourth mile East of Stables, . . .	N.NW.		Sandstone	W.SW.	8° or 10°	
Dalkeith Park, at North Esk, (Ladyseat), ...	W.NW.	E.NE.	Coal	North	3°	obt. 112° ac. 68°
On Esk, below Smeaton farm-house, ...	N. by W.	NE.	Sandstone	W.NW.	2° or 3°	obt. 124° ac. 56°
On Tyne, one-half mile below Turnydykes, ...	W. by N.	N. ½ E.	Lime	North	2°	obt. 95° ac. 85°
Edgehead, ...	NW.	N.NE.	Coal	South	8°	obt. 113° ac. 67°
Bryans?	NW.	N.NE.	Coal	NW.	12°	obt. 113° ac. 67°
Deep Sykehead, ...	NW. by W.	N. by E.	Sandstone	E. by N.	2°	obt. 113° ac. 67°
Burnbank, ...	North	N.NE.	Lime	SE.	5°	obt. 117° ac. 23°

TABLE—*continued.*

Names of Places.	Backs.	Cutters.	Quality of Rock.	Dip of Strata.	Angle of Dip.	Angles formed by Back and Cutter.
Joppa Quarry, ...	SW.	East	Sandstone	SE. ½ E.	55°	obt. 135° ac. 45°
Joppa Shore, ...	SW.	SE. ½ E.	Fire-clay	SE.	45°	obt. 95° ac. 85°
Tranent, ...	N.	N. 50° E.	Coal	N. 50° E.	8°?	obt. 130° ac. 50°
Millerhill, ...	N. 30° E.	N. 30° W.	Coal	SE.	3°	obt. 120° ac. 60°
Esperston Quarry, . .	North	NW.	Limestone	N.	3° or 4°	obt. 135° ac. 45°
Side Quarry, ...	N. 10° E.	NW. by N.	Limestone	N.	3°	obt. 135° ac. 45°
Braidwood Quarry, . . .	N. 3° W.		Sandstone	N.	2°	
Roslin Powder Mills, . .	NE. ½ N.	NW.	Sandstone	SE. ½ S.	5°	obt. 95° ac. 85°
Loanhead, ...	N.?	W.?	Coal	South	50° to 60°	obt. 90° ac. 90°
Gilmerton Old Quarry, . .	NW.		–Lmei	SE.	40°	
Dryden, in Bilston Burn, .	NW. ½ N.		Lime	E.SE.	60°	
Cowbridge, ...	NE.		Sandstone	NW.	5°	
Burdiehouse, ...	W.NW.			SE.	30°	
Granton Quarry, ...	N. 55° W. N. 22° W.		Sandstone	SE.	15°	
Craigleith, ...	West		Sandstone	E. byN.	25°	
Fullarton, ...	NE.		Limestone	E.NE.	10°	
Arniston, ...	S. 37° W.	S. 60° E.	Splint Coal	W.	14°	obt. 97° ac. 83°
Ditto, ...	S. 80° W.	S. 13° E.	Parrot Coal	W. 21° N.	12°	obt. 93° ac. 87°
Ditto, ...			Great seam (cubical)	W.	14°	obt. 103° ac. 77°
Barleydean, ...	North	West	Coal	SE.	5°	obt. 90° ac. 90°
Rosewell Village, . . .	N. 33° W.	N. 57° E.	Coal	South	5°	Ditto
Rosewell Jewel Pit, . . .	N. 33° W.	N. 57° E.	Coal	South	5°	Ditto

Note.—The particulars afforded by the above Table are, I admit, far from satisfactory. They are not only incomplete, but, in several instances, exhibit results inconsistent or contradictory. That there must be many errors in the Table, I feel perfectly assured. One cause of error arises from this circumstance, that the colliers, and even the most intelligent overseers, often confound the *backs* with the *cutters*,—or rather the *terms*. This Table ought not therefore to be relied on, to any great extent; and I have put it into the Appendix, only because it is referred to in the Memoir, and because it may suggest to other observers, a convenient form for registering the results of their inquiries.

APPENDIX C.—P. 270.

REPORT ON THE WARDIE IRONSTONE. By WILLIAM GREGORY, Esq. M.D. Prof. of Chemistry.

EDINBURGH, 14*th May* 1836.

I have examined the three samples of Wardie Ironstone with the greatest care.

No. 1, from a depth of 20 fathoms 5 feet, contains in the calcined state, as given to me, in 100 parts—

Matter insoluble in Acid (Sand),	19.6
Peroxide of Iron,	72.5
Alumina (Clay),	3.5
Lime (a Trace),	0.0
Moisture and Loss,	4.4
	100.0

No. 2, from a depth of 26 fathoms 4 feet, contains, in a calcined state, in 100 parts—

Insoluble Matter,	37.8
Peroxide of Iron,	56.4
Alumina,	2.5
Lime (a Trace),	0.0
Moisture and Loss,	3.3
	100.0

No. 1, when calcined, contains therefore about *Fifty per cent.* of pure Iron, calculated in the *metallic state;* and No. 2 *Forty per cent.* nearly.

No. 3, in its natural state, contains in 100 parts—

Insoluble Matter,	19.3
Protoxide of Iron,	45.9
Alumina,	1.5
Water and Loss (a Trace of Lime), Carbon, &c.	33.3
	100.0

The Metallic Iron here is 32.2 per cent.; the reason of the difference is, that, by the calcination, a quantity of water was expelled, so that in Nos. *one* and *two*, the quantity of iron is increased in proportion to the weight of the mineral analyzed.

All the ores are remarkably good; and there can be no doubt, that, with the addition of lime, and other necessary fluxes, they will work admirably. I have scarcely seen any ores of the coal-field containing so much as *Forty-five* per cent. Protoxide; and it is probable that Nos. 1, and 2, contain a good deal more than this. All, I have no doubt, in the natural state, contain, as No. 3 does, some carbonaceous matter; but the quantity of this is not large.

I have much pleasure in making the foregoing Report, which is even more favourable than I had anticipated. The quality of your ironstone is not to be surpassed, scarcely equalled, in any ironwork in Scotland

WILLIAM GREGORY, M.D.

To Captain J. D. Boswall, R.N. of Wardie.

Note.—I understand from Captain BOSWALL that the iron-ore above reported on consisted of beds or bands about 17 inches in thickness, and situated at a depth of from 20 to 30 fathoms from the surface.

I may mention in connection with this subject, that there is a stratum of *yellow ochre*—(alumen and red oxide of iron),—situated between the North Greens Coal and its subjacent limestone. At Easter Duddingstone, this stratum is about 6 inches thick,—at Dryden it is from 16 to 18 inches. At the latter place, it is worked, and is sold in Edinburgh at the rate of a guinea per ton.

APPENDIX D.—P. 271.

I. Table of Slips in the Mid-Lothian and East-Lothian Coal-Fields.

No. on Map.	Direction by Compass.	Side, on which Strata are downcast.	Number of Fathoms, Strata are downcast.	Distance, to which Slip traced along surface.	End, at which Strata most dislocated.	Width of Slip, in feet.	Place, through or near which, Slip runs.	Source of information.	General Remarks.
1.	N. 60° W.	N. 30° E.	8 to 10				Walliford	Plans	
2.	S. 3° W.	N. 87° W.	6				Do.	Do.	
3.	W. by N.	S. by W.					Do.	Do.	
4.	W. by N.	S. by W.	15				Drumore	Grieve	
5.	W. 7° N.?	S. 7° W.					Wisp	Bald.	
6.	North						Rosewell		
7.	N. 70° W.	N. 20° E.	at Prest.gr. 30 —Tranent 16	2 miles	W. or trough		Prestongr. Bankton	Moore Cadell	Traced to Winton Loan
8.	S. by W.	W. by N.	At N. end 4 — S. — 1	140 yds.+	North		Myles	Moore	
9.	NW.	SW.	36+	$1\frac{1}{2}$ miles+			Ormiston		
10.	N. 41° W.	S 41° W.	8 or 9				Blindwalls		
11.	NW.	SW.	8				Pencaitland	Henderson	
12.	N.NW.	E.NE.	4				Do.	Do.	
13.	N.NW.	W.SW.	$3\frac{1}{2}$				Do.	Do.	
14.	N. 30° W.	W. 30° S.	16 to 20		N. or trough		Carberry	Grieve	
15.	NW.	NE.	8 or 9				Elphinston	Foster	
16.	NW.	NE.	6		S. or crop?		Fountainhall		
17.	North						Rosewell		
18.	N. 54° W.	N. 36° E.	80	$1\frac{1}{2}$ miles	E. or trough	9	Sheriffhall	Plans	
19.	S.SW.	E.SE.					Harelaw	Ross	
20.	NW.	NE?					Preston	Grieve	Quebec slip
21.	W. ?	North	40				Do.	Do.	North slip
22.	N. 60° W.	N. 30° E.	15				Brunstain	Plans	
23.	N. 35° W.	N. 55° E.	At S. end 13 at N. end $\frac{1}{2}$	1 mile +	S. or trough		Sheriffhall	Plans	
24.	W.NW.	E.NE.	$4\frac{1}{2}$	500 yds.	W. or trough		Morison's-haven	Moore	12 F. S. of Whin dyke
25.	W.NW.	E.NE.			E. or crop		Do.	Do.	16 F. S. of Whin dyke
26.	N. $13\frac{1}{2}$° W.	N. $76\frac{1}{2}$° E.	One slip 1 other $1\frac{1}{2}$				Morison's-haven	Moore	2 slips 4 or 5 yds. apart
27.	N. 68° W.	W. 68° S.	$2\frac{1}{2}$		E. or crop		Penstone	Do.	

I. Table of Slips in the Mid-Lothian and East-Lothian Coal-Fields—*continued.*

No on Map.	Direction by Compass.	Side, on which Strata are downcast.	Number of Fathoms, Strata are downcast.	Distance, to which Slip traced along surface.	End, at which Strata most dislocated.	Width of Slip, in Feet.	Place, through or near which, Slip runs.	Source of information.	General Remarks.
28.	NW.	SW.	5				Penstone	Do.	
29.	W. 6° N.	W. 84° S.	10?	½ mile +	W. or crop		Edmonstone	Weighan	
30.	N.NW.		0	1 Mile		54	Millerhill	Adams	5 slips running parallel
31.	N. 40° W.	S. at S. end N. at N. end	At S. end 5 at N. end 1½	1 mile+		9	Newton	Adams	
32.	N.NW.	Strata on N. side dip SE.; on S. side N.		1½ mile.			Cairnie	Adams	
33.	W. 5° S.	W. 85° N.	4 or 5 ?				Newbattle	Grieve	
34.	W.	N.	4½	2 miles	W. or trough		Bryants	Moffat	
35.	W.	N.	14	2 miles	W. or trough	0	Do.	Do.	70 F. fr. 34.
36.	W. 2½° N.	N. 2½° E.	4.0		W. or trough	0	Do.	Miller	Doubtful
37.	NW. by W.	NE. by N.	1.1				Catwell	Do.	
38.	S. 30° W.	S. 6° E.	15	½ mile			Blinkbonny	Gibson	
39.	N. 34° W.						Niddry	Bald	
40.	N. 34° W.						Do	Do.	
41.	N.NW.	E.NE.	4 feet				Blindwalls		
42.	NW. ½° N.	NE. ½ E.	4 feet		N. or trough		Do.		
43.	NW.	SW.	5 to 8				Cinderhall	Sherar	
44.	N.NW.	W.SW.	8				Do.	Do.	
45.	N. 5° W.	E. 5° N.	In lime qua. 1 50	1 mile+	S. or trough		Gilmerton		100 yds. NE. of engine
46.	North	East	10	Do.	Do.		Do.		
47.	North	East	35	Do.	Do.		Do.		50 F. W. of 46
48.	NW. by W.	N.E by N.	½				Sheriffhall	Plans	
49.	NW. by W.						Do.	Do	
50.	N. 70° W.						Dalkeith Pa.	Farey	Doubtful
51.	W.NW.	S.SW.	4		N. or crop		Bannockrig		
52.	W.NW.	S.SW.	5		Do.		Do.		50 F. fr. 51
53.	North	East	40				Gilmerton		
54.	North	East	60				Loanhead	Ross	
55.	North	East	16				Do.	Do.	
56.	North	East	30				Loanhead	Ross	
57.	N. W.?	W. ½ S.	15	1½ miles?			Edgehead	Cowan	

I. TABLE OF SLIPS IN MID-LOTHIAN AND EAST-LOTHIAN COAL-FIELDS—*continued.*

No. on Map.	Direction by Compass.	Side, on which Strata are downcast.	Number of Fathoms, Strata are downcast.	Distance, to which Slip traced along surface.	End, at which Strata most dislocated.	Width of Slip, in Feet.	Place, through or near which, Slip runs.	Source of information	GENERAL REMARKS.
58.	N. 67° W.	NN.E.	40		E. or crop		Arniston	Maxton	33 F. N. of Pit
59.	West	South	30		E. or crop		Do.	Do	83 F. N. of Pit
60.	S. 75° W.	NW. at NE. end. SE. at SW. end.	4 or 5 feet				Monteith houses	Do.	
61.	West	N.	7				Barleydean	Plans	
62.	W. 8° N.	S.	7				Do.	Do.	191 F. fr. 61
63.	NW.	SW.	5 feet				Cousland	Ainslie	
64.	W.SW.						Do.	Do.	
65.	SW.						Do.	Do.	
66.	N. 50° W.	NE.	2½				Monkton	Adams	
67.	N. 50° W.	NE.	2				Do.	Do.	
68.	N? or N. by E?	W. or W by N.	1⅓	400 yds.+	S. or trough	3 or 4	Myles	Moore	
69.	N. 18° W.? or NW?	W. 18° S. or SW.	33	400 yds.+			Huntlaw	Henderson	
70.	Do.	Do.	9	Do.	N. or trough		Do.	Do.	
71.	N. 3° W.	West	1⅓	200 yds.	N. or trough		Do.	Do.	
72.	N. 3° W	West	3	400 yds. +	S. or crop		Do.	Do.	
73.	N. ½ W. and S.SW.						Elphinstone	Sherar	This slip is crooked.
74.	SW.	NW.	1½		N. or trough		Preston	Grieve	
75.	NW.	SW.	1½		Do.		Do.	Do.	
76.	N. 70° W.	W. 70° S.			W. or crop		Joppa	Plans	
77.	North	West	4½				Birsley	Moore	16 F. E. of 109
78.	S.SW.	E.SE.					Harelaw	Ross	
79.	S.SW.	E.SE.					Do.	Do.	
80.	N.NW	E.SE.	2½				Do.	Do.	
81.	N. 38° W.	N. 38° E.	2½				Whim	Wright	
82.	N. 35° W.	N. 55° E.	At N. end 45 at S. 28	½ mile	N. or crop?		Gilmerton	Plans	
83.	N. 14° W.	N. 76° E.	At N. end 9 at S. end 27		S. or trough		Do.	Plans	
84.	N.NW.	E.NE.	2½				Cranston	Foster	
85.	N. 26° W.	N. 64° E.	At W. end 13 at Midfield 7		W. or trough		Midfield	Plans	
86.	N. 26° W.	N. 64° E.					Monkton	Plans	

I. TABLE OF SLIPS IN THE MID-LOTHIAN AND EAST-LOTHIAN COAL-FIELDS—*continued.*

No. on Map.	Direction by Compass.	Side, on which Strata are downcast.	Number of Fathoms, Strata are downcast.	Distance, to which Slip traced along surface.	End, at which Strata most dislocated.	Width of Slip, in feet.	Place, through or near which, Slip runs.	Source of information.	GENERAL REMARKS.
87.	N. 52° W.	N. 38° E.					Do.	Do.	
88.	NW. by N.					14	Jerusalem		
89.	NW.	NE.					Salton		
90.	NW	NE.					Do.		
91.	W. by S.	N. by W.	1		West		Wardie		
92.	N. 70° W.						Somerside	Plans	
93.	North	West					Cowden	Love	
94.	N. 16° W.	W. 16° S.	15				Do.	Do.	
95.	N.NW.	E.NE.	2 or 3				Dalkeith Pa.		
96.	SW.	SE.	10				Stobhill	Plans	
97.	S. 70° W.	W. 20° N.	A few feet				Do.	Do.	
98.	NW.?	NE.	80		N. or trough		Blinkbonny	Gibson	
99.	NW.?	SW.	30		Do.		Do.	Do.	
100.	NW.?	SW.	30		Do.		Do.	Do.	
101.	West ?	North	4				Bryants	Miller	
102.	N. 40° W. & N. 12° W.	N. 50° E. N. 78° E.	1				Arniston	Maxton	
103.	N. 10° W.?						Pathhead	Foster	
104.	N. 64° W.	N. 26° E.	At Sheriffhall Engine 10	$\frac{3}{4}$ mile	Scatters and ends at NW.		Sheriffhall	Plans	
105.	N. 75° W.						Monkton	Plans	
106.	North	East	$1\frac{1}{3}$				Cowden	Love	
107.	NW.	NE.	$\frac{1}{2}$				Do.	Do.	
108.	NW. by W.	NE.					Cowbridge		
109.	North	West	4 or 5 feet				Birsley	Moore	16 F. fr. 77
110.	WN.W.	SS.W.	5	100 yards		10 ?	Burdiehouse	Torrance.	
111.	N. by W.	E. by N.	50 + ?			50?	Do.	Do.	
112.	W. by N.	N. by E.	5 feet				Barleydean	Plans	130 F. fr. 62
113.	W. 5° N.	S. 5° W.	5 feet				Do.	Do.	21 F. fr. 112
114.	N. 25°. W.	N. 65° E.	10 ?				Carlops		Doubtful
115.	NW.	NE.	10 feet				Do.		
116.	NW.	NE.	5				Do.		
117.	W.NW.	S.SW.					Do.		Doubtful

II. Number of the above Slips running in different Points of the Compass.

[Each of the points stated in this and the following Table, is meant to include an arc of 11¼°.]

N.	N. by W. 11°.	N.NW. 23°.	NW. by N. 34°.	NW. 45°.	NW. by W. 56°.	W.NW. 67°.	W. by N. 79°.	W. 90°.	W. by S. 11°.	W.SW. 23°.	SW. by W. 34°.	SW. 45°.	SW. by S. 56°.	S.SW. 67°.	S. by W. 79°.
19	6	12	7	25	8	13	8	8	2	1	1	3	1	4	1

Slips running between N. and W. . . . . = 98
.... N. and E. . . . . = 21

III. Direction towards which the above Slips Dip, and amount of Dislocation of the intersected Strata, by these Slips, in Fathoms.

N.	N. by W. 11°.	N.NW. 23°.	NW. by N. 34°.	NW. 45°.	NW. by W. 56°.	W.NW. 67°.	W. by N. 79°.	W. 90°.	W. by S. 11°.	W.SW. 23°.	SW. by W. 34°.	SW. 45°.	SW by S. 56°.	S.SW. 67°.	S. by W. 79°.
40+5+ 40+4+ 4½+14+ 7=114½	1			1½			2+4=6	6+15+2 +3+2+ 1+4½+2 +1=36½	15	3+8+33 +9=53	20	36+9+8 +5+7+ 33+9+ 1½+30+ 30= 169½	2	2½+4+ +5+5= 16½	15+10 +1=46
S.	S. by E.	S.SE.	SE. by S.	SE.	SE. by E	E.SE.	E. by S.	E.	E. by N.	E.NE.	NE. by E.	NE.	NE. by N.	NN.E.	N. by E.
30+7 =37				10	15	2½		50+10+ 35+40+ 60+30+ 16+1= 242	2+27 +1+50 =80	4+5+1 +2½+13 +3+10 =38½	1+45= 46	9+1+6 +2+80 +1½+2½ +2+5+ 2½=111½	10+80+ +15+13 +1+½= 119½	30+10 +40= 80	1

Total sum of Fathoms thrown down towards points in quadrant between N. and W. . = 123 No. of Slips, . 11
.... S. and E. . = 64½ 5

More thrown down to North, ... 58½ Fathoms, . . 6 Slips.

Total sum of Fathoms thrown down towards points in quadrant between N. and E. . = 738½ No. of Slips, . 41
.... S. and W. . = 337½ 32

More thrown down to North, ... 401 Fathoms, . . . 9 Slips.
Add from above, ... 58½ 6

More thrown down in North half of Compass, 459½ Fathoms, ... 15 Slips.

Note.—It may excite surprise, that none of the "decided lines of fracture" described by Dr Hibbert in his paper on the Burdiehouse Limestone, as traversing the Mid-Lothian coal-field, are laid down on the Map accompanying this paper, or are specified in the above Table. The truth is, I have not been able to discover any one of these alleged lines of fracture. I regret this the more, on account of their being all described by Dr Hibbert as running in directions varying little from SW., and different therefore in this respect from the great proportion of the fractures existing in the district.

I may add, with reference to the mode in which the information regarding the Slips was obtained, that it was, in most instances, derived either from plans of the coal-workings, or from the under-ground overseers, as mentioned in the Table. Where there is no authority given in the Table, the statements rest on my own observation.

Since the first part of my Memoir was read and printed, I have acquired information regarding several slips of which I was not then aware. These have been added to the Table. This addition creates a slight discrepancy between the Table and the Memoir,—though, so far from invalidating the statements made in the text, it strongly confirms them.

APPENDIX E.

It is true, that there is not below the Burdiehouse limestone, any calcareous deposit nearly so thick as it. But it is a mistake to imagine, that there are no limestone strata whatever below it. In fact, there are carboniferous strata, several hundred fathoms in thickness, and forming part of the Esk coal-basin, which lie beneath the Burdiehouse limestone,—consisting of regular beds of sandstone, coal, shale, limestone, &c. Even at Straiton Mill, distant about three-fourths of a mile from Burdiehouse limestone, these calcareous and carboniferous strata may be seen,—and dipping in the same direction. As the strata are there nearly on edge, it is obvious, that the number and thickness of the deposits between the two places (which are situated from each other in nearly the direction of the dip and rise), must be very great. At Straiton Mill, there are two beds of coal;—one a cubical coal, about 2 feet thick,—the other a coarse parrot coal 4 to 5 feet thick. The former has been worked. It runs through the south end of the village of " Five-houses," situated on the Edinburgh and Loanhead road,—and is supposed to run near the south side of Gracemount House.

At Straiton Mill also may be seen two strata of limestone,—in composition and texture very similar to the Burdiehouse rock, filled with teeth, scales, and coprolites, similar to those discovered at Burdiehouse. For a farther account of these,—see p. 355 hereof. At the place now referred to, some of the sandstones are very coarse, and there is one bed of conglomerate, about 6 feet thick. This is an indication of its being near the bottom of the basin.

APPENDIX F.—P. 292.

Results of Experiments on a *small scale* of the Gas and Coke produced from various kinds of Parrot or Cannel Coal.

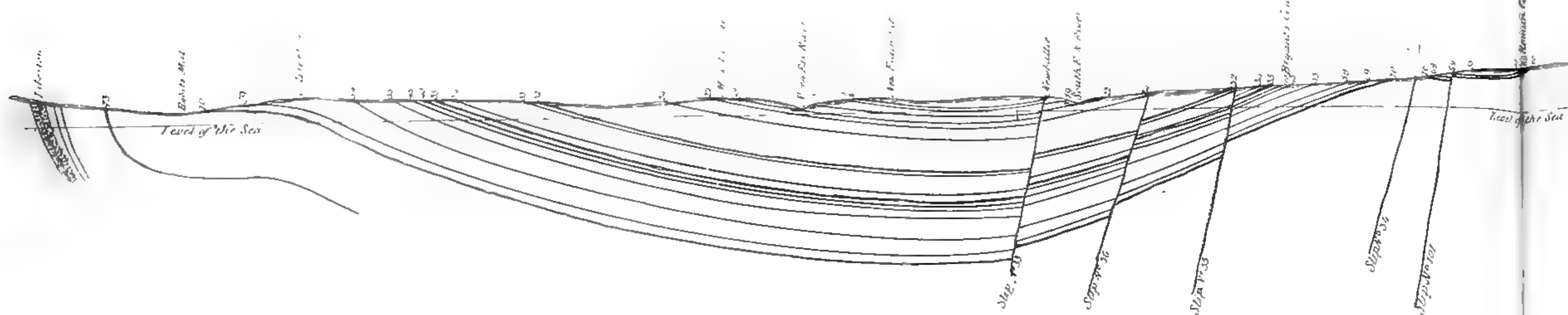
Description of Coal.	Weight of each charge of Coal.	Time required for Distillation.	Produce of Gas.	Proportion of Gas produced per Cwt.	Weight of Coke produced.	Weight of Ashes when Coke consumed.	Weight of Combustible Matter in Coke.	Consumption in one Hour of a Jet-flame 3 inches.
	Lb.	Hours.	Cubic Feet.	Cubic Feet.	Lb.	Lb.	Lb.	Cubic Feet.
Average of two Charges of J. and R. Aytoun's Coal (Fifeshire), ...	40	2½	204	599	19¼	4	15¼	·65
Average of two Charges of Mr Cuthbertson's Coal (East Lothian), ...	40	2½	192	537	20½	2¾	17¾	·78
Average of two Charges of Mr Marshall's Gilmerton Coal (Mid-Lothian), . . .	40	2½	219	613	20½	2½	18	·81
Average of two Charges of Halbeath Coal (Fifeshire),	40	2½	216	604	20	5¾	14¼	·72
Average of two Charges of Wemyss Methel Coal (Fifeshire), ...	40	2½	214	599	20¼	6¾	13½	·76
Average of two Charges of Sir Chas. Menteath's Coal from Mansfield (Dumfriesshire),	40	2½	240	672	20½	2½	18	·67
Average of three Charges of Mr Mercer's Coal, from Dryden (Mid Lothian),	40	2½	216	604	19	10¼	8¾	·96
Average of two Charges of Sir Chas. Menteath's *Rough* or *Cubical* Coal, from Mansfield, ...	40	2½	216	604	24½	3¼	21¼	·9

Note.—The above experiments were made during the year 1837–8 in the premises of the Edinburgh Gas Company, and I am indebted for the above statement of them to Mr Watson, the Manager.

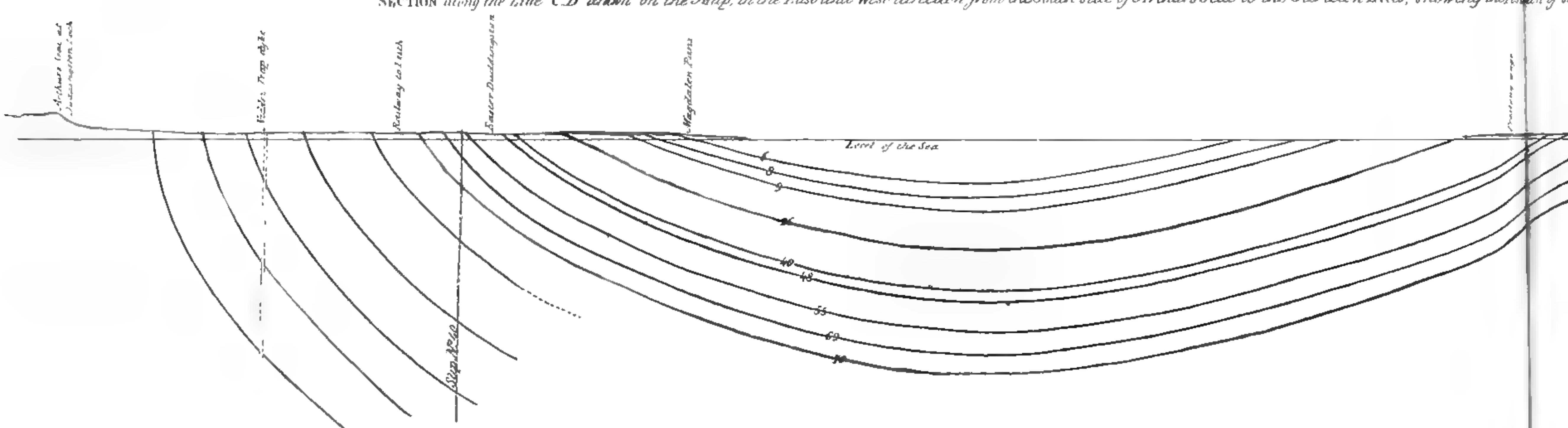
Mr Watson remarked, that the illuminating power of the gas is estimated inversely, according to the quantity indicated in the last column. But this is evidently a very uncertain test.

It appears that the results obtained in the manufacture of gas in large quantities, from any particular kind of coal, seldom agree with the results of experiments (such as the above), conducted on a small scale. This discrepancy arises chiefly from the great variation in the quality of the coals, though procured from the same seam, and out of the same pit. It is generally the best and purest specimens which are selected for experiments; so that there is necessarily less difference in the qualities of the coal *experimented* on, than there is in the immense supplies furnished to the *retorts*. But even among the specimens of coal so selected, it is remarkable how great a difference there is in the constitution of the coal,—arising from the different proportions in which its elements exist in the same seam.

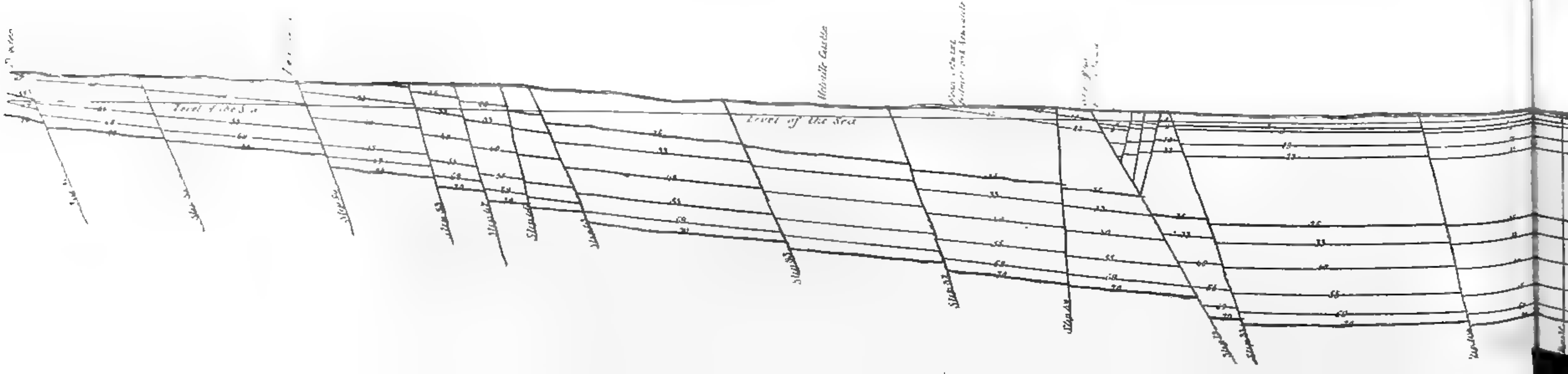
SECTION from Liberton to Seaton, drawn on a Scale of 2 Inches to a Mile along the line A.B. on the Map, showing the principal Coal seams, Lime strata, and Slips intersecting them —



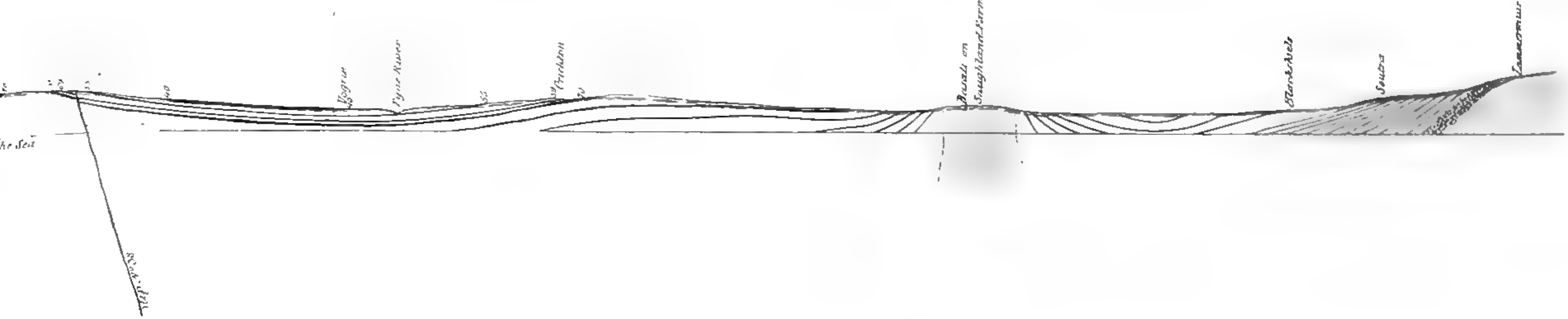
SECTION along the Line C.D drawn on the Map, in the East and West direction from the South side of Arthurs Seat to the Garleton Hills, showing the Position of s



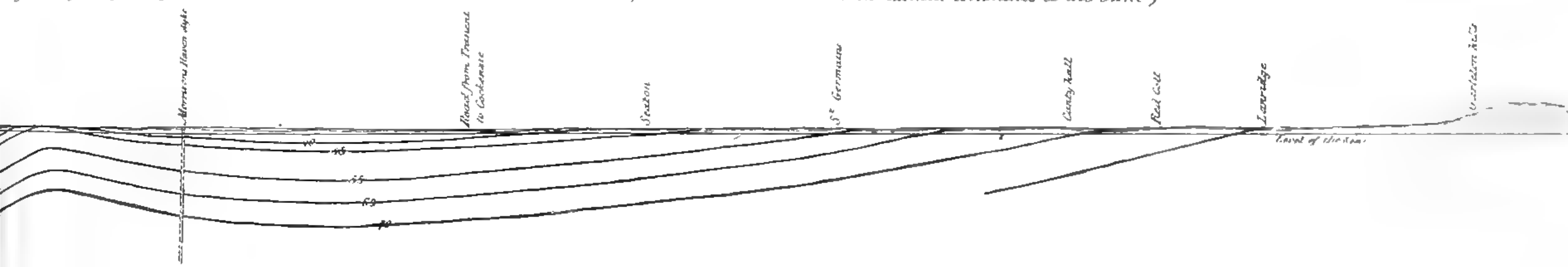
SECTION along the line E.F drawn on the Map from Dryden to Prestongrange, showing the manner in which the Basin of the Esk in tha



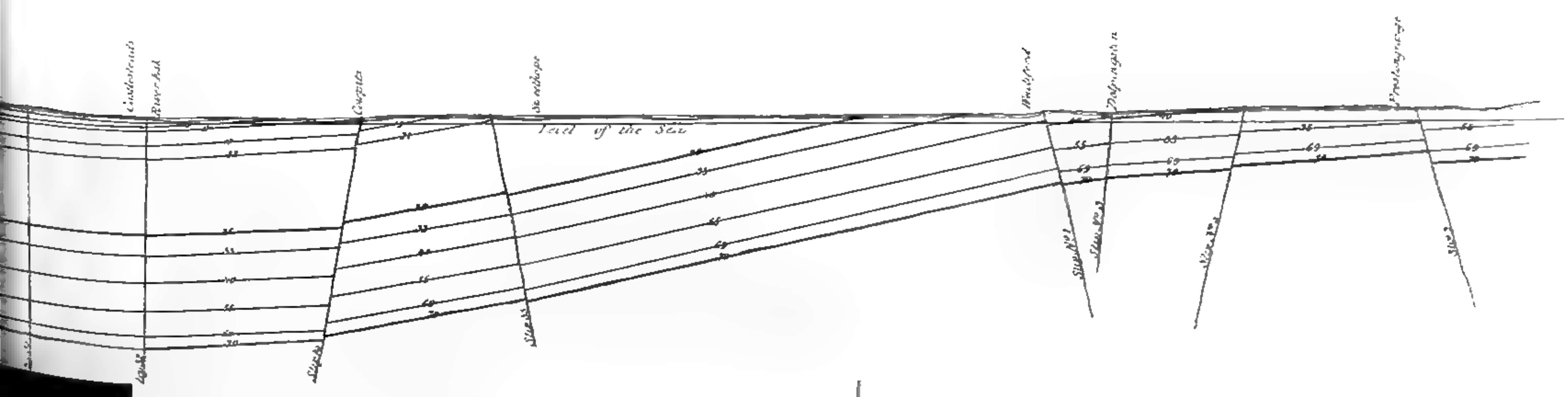
…es on the Section, refer to the accompanying Map of the Coal Field, the Table of Slips, and Table of Coal and Limestone Strata.)

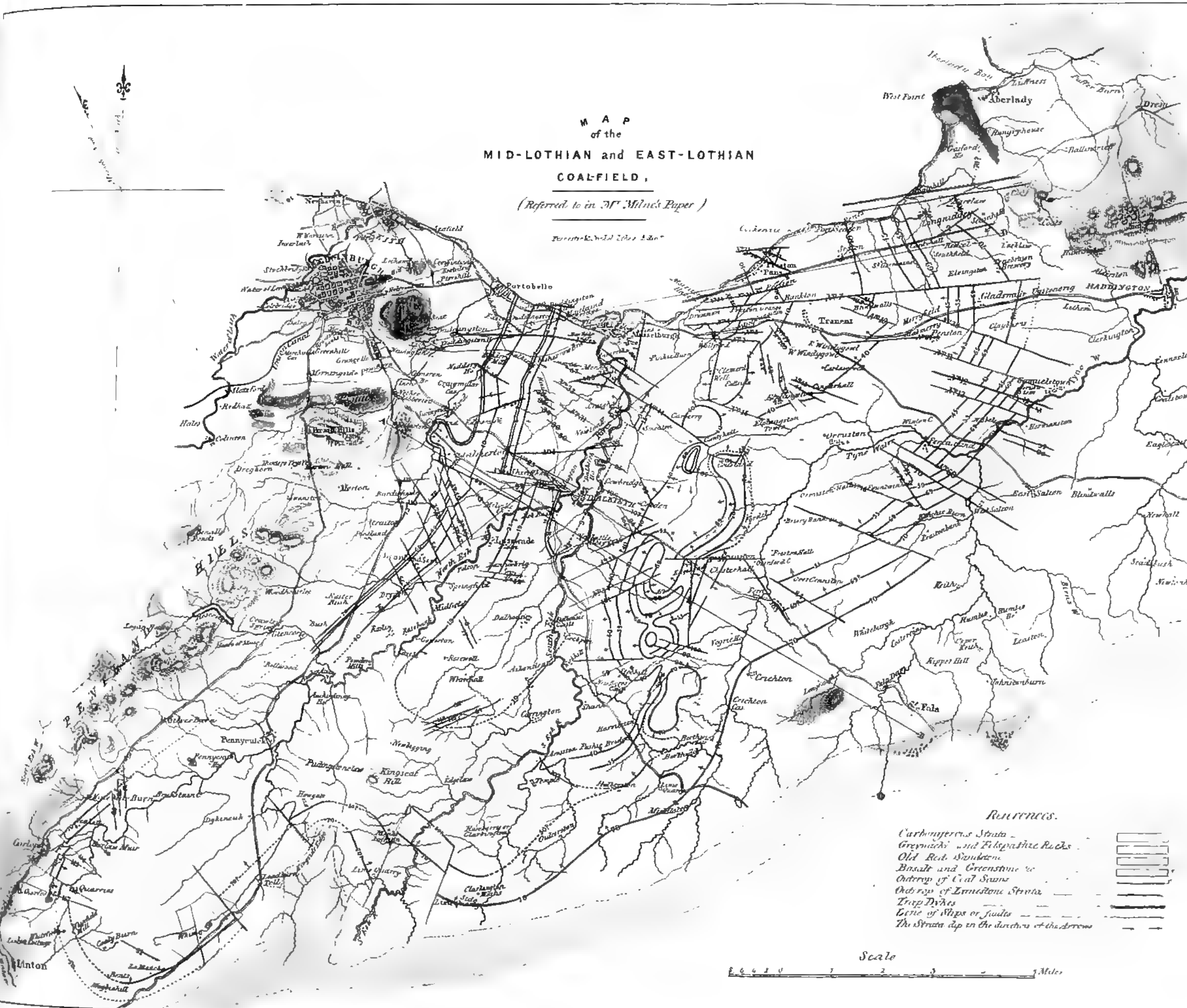


…n of some of the principal Coalseams and Limestone strata in that District — (The Scale both vertical & horizontal is 2 Inches to the Mile.)

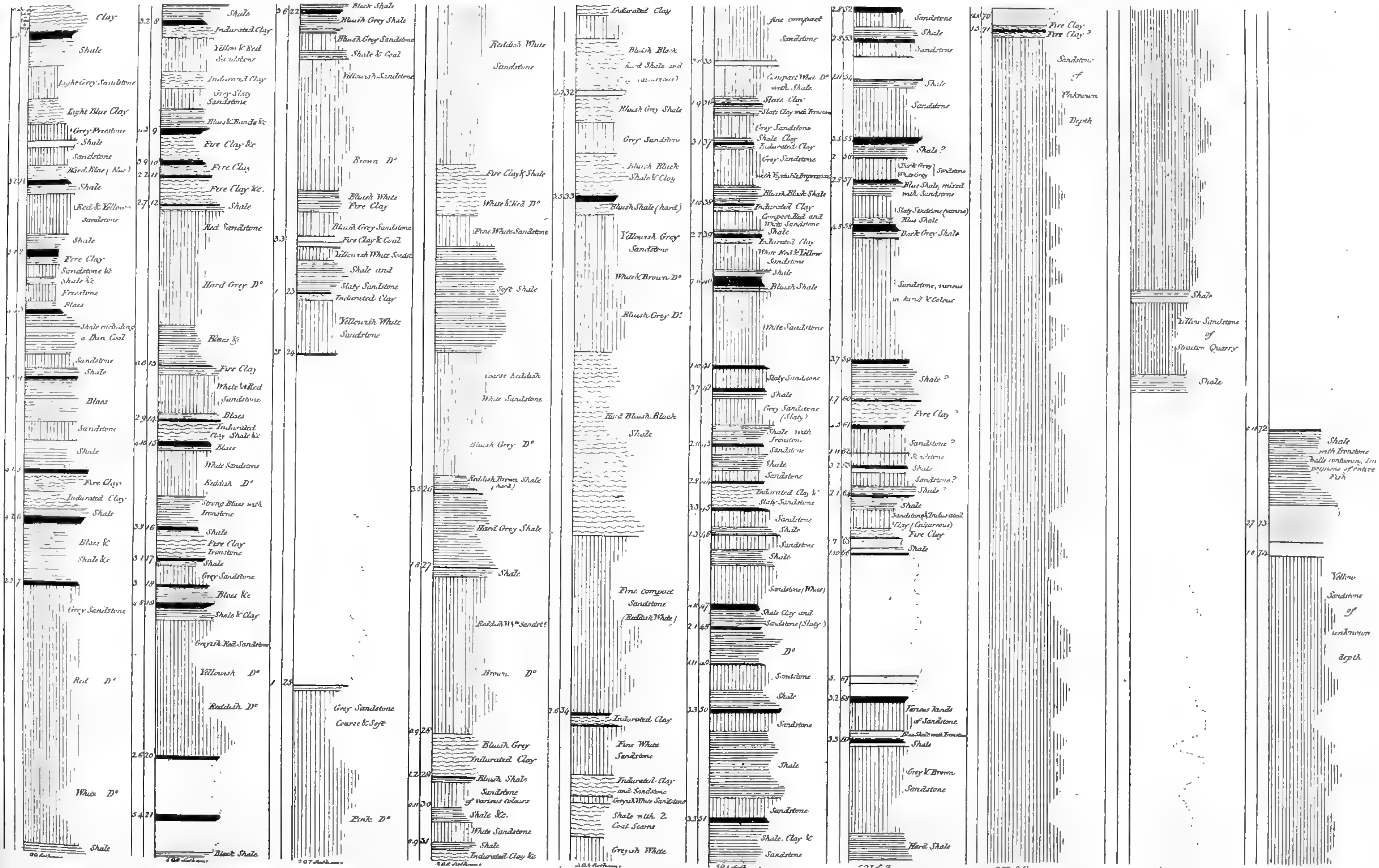


…in that direction intersected by Slips — (Scale, both vertical & horizontal 1½ Inches to a Mile.)




MAP
of the
MID-LOTHIAN and EAST-LOTHIAN
COAL-FIELD,
(Referred to in Mr. Milne's Paper.)
References.
Carboniferous Strata
Greywacke and Felspathic Rocks
Old Red Sandstone
Basalt and Greenstone &c.
Outcrop of Coal Seams
Outcrop of Limestone Strata
Trap Dykes
Line of Slips or faults
The Strata dip in the direction of the Arrows
Scale
Miles
Aberlady
West Point
Portobello
Musselburgh
Tranent
HADDINGTON
Dalkeith
Crichton
Fala
Linton
Pennycuik
Kingseat Hill

VERTICAL SECTION showing the Average Positions & Thicknesses of the Principal Coal Seams & Limestone Strata, in the Lothians, with the nature of the Intervening Strata. —— (Scale of One Inch to Ten Fathoms.)



NOTES EXPLANATORY OF THE PLATES ILLUSTRATING THE FOREGOING MEMOIR.

The Map on Plate XVII. is intended not merely to represent the extent of the *Coal-field* described in the preceding Memoir, but also to indicate the nature of the *formations* skirting the coal-field.

It should be borne in mind, however, that the strata of this coal-field reach beyond Haddington, and therefore beyond the limits of the Map. It is chiefly the lower part of the series of strata composing the coal-field, which exists so far to the east. The limestone that is worked or known there, certainly belongs to the lowest part, and the thin coal-seams which occur in Haddington, at Amisfield, Coalston, Morham, and other places mentioned in the Memoir, are also lower members of the basin. This extension of the coal-field reaches to the sea-coast between Dunbar and Dunglass.

It will be seen, that the part of the district occupied by *carboniferous strata*,—that is to say by strata alternating with coal-seams and shales (which I look upon as constituting the coal-field proper,) is coloured on the Map with a shade of indigo.

The *Old Red Sandstone* formation—consisting of red Clays, Sandstones, and Conglomerates, is indicated by a red colour. It must not, however, be supposed, that this formation exists only at the spots indicated on the Map. These are the places, where I have *ascertained* its existence,—and I was unwilling to extend the colour to other places which I had not examined, however strongly I might conjecture that the formation existed there. As the object of the Memoir was chiefly to describe the *coal-field*, and the manner in which its members—aqueous and igneous—were disposed, it seemed of less consequence to be very precise as to the older formations, and that little more was necessary in illustration of this Memoir than to indicate their existence and situation.

It will be seen from the Map, that the greywacke rocks, both in the Lammermuir and in the Pentland hills, have received the same colour as that which represents the felspathic rocks. If the description of these several rocks had formed any part of my Memoir, it would have been of course necessary, though difficult, to distinguish them by separate colours. But as my object in representing at all on the Map, any portion of the Lammermuir and Pentland range, was simply to point out the boundaries of the coal-field, and the position of the rocks, which, by their degradation, had afforded, in some degree at least, materials for some of its sedimentary strata, and which moreover had, in the opinions of some geologists, been the means of elevating and dislocating them,—I considered it might be useful to indicate the range of those hills, and the line of their nearest approximation to the coal-field. These objects were sufficiently attained by using one colour to represent the hills, coupled with an intimation, that it was by no means thereby intended to represent them as homogeneous, but as consisting of greywacke and of felspar in various states. In one point of view, there is even a propriety for using one and the same colour to represent these different rocks,—not merely because they form together one and the same range of hills, but because they have, in my opinion, been raised at one and the same period. I consider that the greywacke strata were burst through and elevated by great eruptions of felspathic rocks. These eruptions must have taken place, probably about the same period, and apparently along certain lines which run in a direction nearly east and west,—in many parts of the south of Scotland. At most places, they brought up with them the deep-seated strata of the greywacke and silurian epochs.

At other places, these strata have not been brought up, at least to the very surface. In proof of this last remark, reference might be made to the *Garlton Hills* (marked on the Map), which I conceive to be contemporaneous with the Pentland and Lammermuir hills, and among which no transition rocks are to be seen.

No one can traverse the Lammermuir hills, without perceiving that at least one-third of the rocks composing them are felspathic. The latter occur not merely in the form of dykes and veins, of great width and extent,—but also of amorphous masses or hills. St Abb's Head consists of a coarse flesh-coloured felspar. At Fassney the felspar is associated with quartz and large scales of mica, forming a regular granite. In the burn which flows over the northern brow of Soutra Hill from Laurie's Den to Woodcot, enormous masses of sienite and granite may be seen mixed with the greywacke strata. It is hardly necessary to say, that these strata in the Lammermuir, as in the Pentland range, are generally vertical, and that the direction of the strata is always east and west, or very nearly so,—circumstances which strongly support the inference suggested by the general direction of the hilly chain, that the force, by whatever agent produced, which was the immediate cause of those felspathic eruptions, acted in lines running nearly east and west.

The Basalt, Greenstone, and other augitic trap-rocks, are denoted on the Map by a green colour. But it is right to mention, that, besides those so indicated, there are rocks of the same species among the Pentland hills which it was unnecessary, and would have been very difficult, to have separately represented. Among the Pentland hills, as among the Lammermuir range, the greywacke strata occur, and are most frequently nearly vertical.

Having thus spoken generally of the different formations or sets of rocks represented on the Map, I proceed to offer a few remarks, in detail, with regard to each of these formations.

(1.) The *Carboniferous* formation.

The object I had in view, was to trace *individual strata* belonging to this formation, through the district as far as they reached,—to lay down on the Map the outcrop of the most important of these strata,—and to mark their variations of thickness. The strata whose outcrop I have laid down on the Map, are merely coal-seams and limestone strata,—and on account of the very reduced scale of the Map, I have been obliged to leave out many of these. The perfect accuracy of all the lines, I will not vouch for ;—though it is proper to observe, that, in the original map, they were laid down from information obtained on the spot, or from plans, with an examination of which I was occasionally favoured. But, however accurately these lines may be *mapped*, every one must be aware that they *must* represent very incorrectly the actual facts. Lines drawn on a horizontal plane, never can represent with truth the circumstances which occur on an uneven and undulating surface. From this cause it happens that the distances between the outcrops of the strata, as marked on the Map, particularly along the Roman Camp ridge, cannot be in exact conformity with fact. Still, I believe, that the outcrop of the different seams, as laid down, is in no case very distant from the truth :—where I was uncertain of the line of outcrop, dots have been used instead of a thick line.

I ought to mention, that there is some uncertainty as to the identity of some of the coal-seams which crop out on the outskirts of the coal-fields, comprehended between the east and south-west points. These belong to the lower members of the basin, which, from being in those parts much reduced in thickness, and consequently little worked and sought after, are not accurately known. It is possible, therefore, that, on future examination, some of those lowest members of the series may have been found to be wrong numbered.

I may extend these remarks to embrace the limestone strata which lie at the bottom of the basin. There are two or three such strata, and from their proximity to each other, it is very difficult to recognise them. That there are two separate strata of limestone along the south part of the district, must be evident to any one who has gone over. I consider the one worked at Crichton Dean to correspond with the Gilmerton limestone lying immediately below the *North Greens* coal. But I do not know whether the limestone that lies below the Crichton Dean limestone, and which was formerly worked near Woodcot and Fala, corresponds with any known seam beneath the Gilmerton limestone. This lower limestone at Fala, and all the other places where it occurs, is a marine limestone, containing ***Producta, Terebratula, Orthocera,*** and all the other shells common in marine limestones. Now the next lowest limestone known on the north side of the district, is the Burdiehouse limestone, which is supposed to be of fresh-water origin, and which certainly has many characters extremely different from the other limestones of the district. Notwithstanding these differences, however, it is possible that the Fala limestone and the Burdiehouse limestone may have been deposited at the same period, and may actually form parts of one general deposit, which varies in character at different places on account of local peculiarities. That the Burdiehouse limestone runs for at least half a mile, in a regular stratum of uniform thickness, is certain,—for the old workings, to the south of Straiton village, were pointed out to me. It therefore hardly deserves the character given to it by Dr HIBBERT, of being a mere local deposit of calcareous matter. But I do not go so far as to say, that this limestone forms part of a stratum which reaches to the southern limits of the coal-field. I only maintain, that there is as yet no evidence to the contrary, and that there are some circumstances which render this idea probable. The mere fact, that at Burdiehouse there are in this stratum impressions of terrestrial vegetables, and remains of fish and shells which, for anything yet known, may have lived in fresh or in salt water,—is no reason why in a part of the sea, much more distant from the land, similar remains should not occur.

I may add, that Dr HIBBERT was mistaken in imagining that he had discovered a marine limestone at Moredun Mill older and lower than the Burdiehouse limestone. This Moredun Mill limestone forms, in fact, part of the Gilmerton bed, which lies a long way above the Burdiehouse limestone. It will be seen from the Map, that it forms there an extraordinary loop, arising from its taking a counter dip.

Neither is it true that the Burdiehouse limestone is the lowest limestone in the district, or the only one possessing vegetable exuviæ and animal exuviæ of ambiguous character. There are below it, and at a great distance,—two other strata, in which I have found these remains in the greatest abundance. These two strata are each from $1\frac{1}{2}$ to 3 feet thick, and they are about 50 yards apart from each other. They crop out at Straiton Mill,—about one-half of a mile from the Burdiehouse quarry. These two strata, there is some reason to think, run all the way to Carlops, for I have seen two strata there, of much the same thickness, containing the same fossils, and associated with coal-seams, similar in thickness and character with those occurring at Straiton Mill.

I have mentioned that the carboniferous strata extend through East-Lothian, as far as the German Ocean. I may add, that though the coal-field apparently terminates on the SW. limits of the district at Coaley Burn, Whitfield, and the Bents,—carboniferous strata re-appear a few miles to the SW., in Peeblesshire, dipping the opposite way, and forming, as it were, the commencement of another basin.

2. *The Old Red Sandstone Formation* is the next in order marked on the Map. But I have scarcely any observations of detail to make on it. This formation in many respects resembles the New Red Sandstone; and judging from its mineralogical appearance, and the entire absence of organic remains, it might be easily confounded. In some places, too, especially along the flanks of the Lammermuirs, it is very horizontal,—a circumstance countenancing the above opinion; and indeed, in several places along that part of the district, there is no reason to suppose that the old red sandstone strata have been elevated since their deposition. On the NW. side of the district, where this formation is in contact with the east parts of the Pentland Hills, the case is different. There they dip at considerable angles. All difficulty and doubt, however, is obviated by the fact, that there red sandstone dips *under*

the coal-measures, and that there are not in its conglomerates any rocks newer than the greywacke and felspar of the hills, along the sides of which they are deposited.

I have often been perplexed with the question, from what source the great abundance of iron could have been derived, which colours the sandstones and conglomerates that repose on the Lammermuir Hills. Near Soutra, I discovered that the greywacke rocks with which these strata are in contact, are reticulated with veins of red iron-ore or hematite; and it is possible that this circumstance may suggest a solution of the question. If the waters, shortly after these hills were raised by the igneous rocks, were saturated near them with sedimentary matter derived from the greywacke rocks, the deposit would have a red colour. In another place, on the south side of the Lammermuirs (near Cowdenknows in Berwickshire), I observed the older rock,—a felspar rock,—also intersected with multitudes of red iron veins,—in close proximity to the red sandstone formation.

This theory receives some countenance from the fact, that in Lyndale (near Carlops), the old red sandstone strata are not red, but of a brown or dark yellow colour. The trappean claystone on which they there rest, and from the disintegration of which they are derived,—contains no iron. The only metal I have found in it is lead. The lead was worked formerly there to a considerable extent.

3. *The Greywacke and Felspathic Rocks* are next to be noticed.

It is a matter of doubt, whether the Lammermuir range has been elevated since the deposition of the coal-strata. Generally speaking, I think they have not;—though, at particular localities, as near Dunglass in East-Lothian, and near Leadburn Toll, they probably have been elevated. Except at these points, and there may be a few others,—I am inclined to hold, that the stratified rocks along the southern parts of the district possess the slope which they received on their original deposition.

It is likewise a question attended with difficulty, to determine whether the Pentland Hills were elevated before or after the deposition of the carboniferous strata. The latter opinion is countenanced by three facts, which are at first sight very decisive. One of them is, that these strata, along the south as well as the north flanks of the Pentland Hills, dip rapidly from the hills, at an angle which is never less than 60°, and generally greater. Another fact is, that in many places near and among these hills, the old red sandstone formation is found broken and interrupted, patches of it being found only here and there. Besides all this, there are places where the stratified rocks, near the Pentland Hills, are traversed by dykes of trap, which are undoubtedly offsets from the great mass of igneous matter belonging to these hills. An example of one of the *coulées* occurs at Heartside, near Carlops, where the new road was lately cut. These are circumstances, clearly indicating elevation subsequent to the deposition of the carboniferous strata.

On the other hand, we find that the old red sandstone conglomerate along the flanks of the Pentland Hills, contains fragments of some of the rocks composing these hills,—shewing, therefore, that certain rocks at least forming part of these hills, were in existence, before the old red sandstone conglomerate was formed;—and of course before the carboniferous strata were deposited. It is important to observe, what the rocks are which exist in the conglomerate, and what are the rocks not found in the conglomerate. I have found abundance of greywacke, and of many varieties of felspar,—but never any specimens of basalt or greenstone, though undoubtedly basalt and greenstone exist among the Pentland Hills. The inference from this is, that the conglomerate had been formed after the elevation of the greywacke and felspathic rocks, but previous to the elevation of the augitic rocks. This inference is confirmed by an examination of several localities near West Linton, where the stratified are seen in contact with the felspathic rocks. Along the banks of the Tyne, the stratified rocks are resting on a yellow-coloured claystone, of igneous origin. At Linton Bridge, they are horizontal; and, though they acquire a dip approaching to 30° about a mile farther up, they do not present any of the marks of having been elevated since their deposition. The nature and appearance of these slaty rocks, affords additional evidence of the same fact. They have evidently been formed from the decomposition and detritus of the igneous rocks. There is a quarry about a mile to the NE. of West Linton, where the nature of both kinds of rocks can be very closely examined, and their junction distinctly seen. The igneous rocks consist of a brownish-yellow claystone or felspar, which easily disintegrates. Upon it, lies a series of slaty strata, consisting of claystones and sandstones, which slope away from the hill top (where the quarry is) at an angle of about 8°, and they are in no degree indurated by the igneous rock on which they rest. It is impossible then to doubt, that the greywacke and felspar in the Pentland Hills,—or at least in a considerable part of them,—existed previous to the deposition of the carboniferous strata.

The only way of solving the difficulty, is by supposing, that there were two eruptions of trap in that part of the district,—one previous to the deposition of the old red sandstone, and the other subsequent to the deposition of the carboniferous strata. The first eruption would be of the felspathic rocks, which brought up with it the greywacke in *this* part of the district, as it did also along the Lammermuir range. During this first eruption, it is probable that there was neither greenstone nor basalt evolved, as we do not find any pebbles of these rocks in the old red sandstone. We know, indeed, *aliunde*, that the great proportion of these rocks in the district were not, in fact, erupted till after the deposition of the carboniferous strata;—so that it is extremely likely, that the whole of that class of igneous rocks existing among the Pentlands, appeared only at this later epoch. It is to a certain extent a confirmation of this opinion, that the part of the Pentland Hills where the stratified rocks are most highly inclined, is towards the east, where Greenstone most abounds among them, and where they are in close vicinity to Arthur's Seat, Craiglockhart, &c. If this inference be correct, then the two eruptions might be designated the Felspathic and the Augitic, being, as they are, characterized and distinguished by a difference in the nature of the volcanic matter indicated by these names. In many parts of the Pentland Hills, Greenstone and Basalt occur, in the form both of hills and of dykes. These dykes are seen traversing the felspar and greywacke rocks,—as well as the old red sandstone and carboniferous formations;—so that there can be no doubt, I think, that they must have given the Pentland Hills, especially in their eastern parts, an additional upheave.

Whilst such is my own opinion, I am aware that it is not concurred in by several geologists who have examined the Pentland Hills with great care and assiduity. Mr Maclaren is about to publish a geological account of these hills, which will probably contain ample materials for a satisfactory decision of the above question.

The basalt and greenstone rocks are not nearly so abundant in the district as in other coal-fields. The direction towards which the trap flowed seems to have been from Arthur Seat or its neighbourhood;—for the dykes on the east side of it, all thin away in that direction,—whilst those on the opposite side thin away to the west. The Niddry dyke terminates near Brunstain House. The Cockenzie dyke terminates in the field NE. of Lanridge. Between Red Coll and Lanridge, its thickness is only 50 feet.

The *Sections* on plates XV. and XVI. hardly require observation,—as the explanations already given, in regard to the colours on the Map, are applicable equally to them. These sections are not entirely imaginary. The position and direction of the slips,—the dip of the strata and their relative distances from each other, with the points where they crop out to the surface, have all been laid down, according to the fact. There may be some error in the relative distances of the strata from each other,—and also in the identification of the lowest bed of limestone. But as the object was to give a general idea of the form of the two basins, as they now exist, and of the manner in which the strata have been dislocated, these possible errors seem immaterial.

The *Sections* on Plate XVII. are intended to exhibit a vertical section of the *whole* strata,—in the district from top to bottom. They have been necessarily divided on the Plate, as it would have been inconvenient to have represented them all in one column. It will be seen, that in some parts these sections are blank, and have no strata indicated on them. In these, the nature of the strata has not been ascertained by me, and they are left to be filled up by future observers. Even as to the parts which *are* filled up, it will not be supposed, that the strata marked are meant to afford more than an indication of what *generally* exists in the district. This column has been constructed almost entirely from information supplied by the tacksmen or the managers of the coal, in different parts of the district. It may be proper to mention, that several of the terms employed by them, I have ventured to translate into geological language:—as, for example, "*blaes*," I have translated "*shale*,—"*freestone*," I have translated "*sandstone*,"—"*faikes*" has been described as "*slaty sandstone*."

Since the foregoing Memoir and most of this appendix was thrown off, it has been resolved to append hereto the *coal and lime table*, referred to on page 260 of the Memoir. It has been found possible to reduce it to a smaller scale than was at first anticipated, and it was thought, that the Memoir would be more complete by having it attached. It is accordingly now given on Plate XVIII. After what has been said of this table both in the Memoir itself and in Appendix A, it is unnecessary to offer here any explanations as to its objects, or the mode of its construction. I would only observe, in regard to the names of places and persons at the top of the table, that the names of the former represent *localities*, where the strata are of the thickness and at the distance from each other, stated in the same column below,—the one being stated in feet and inches, the other in fathoms and feet. The names of persons indicate the *individuals* from whom the subjacent details were obtained, or by whose permission and assistance they were procured. Where there are no persons' names attached to the names of places, the information was obtained from my own observation, or from individuals whose names were unknown to me.

Perhaps I may be here permitted to mention, how readily every person connected with the district, whom I applied to for information, not only communicated to me what they themselves knew, but gave me access to any plans or reports in their possession likely to contain information. It is of consequence, even in a scientific point of view, to mention this,—as it affords much encouragement to such as are disposed to prosecute geological researches, to learn that those who are most able to assist them in their difficulties and labours, are also the most willing to do so.

Average of Whole

Total of COAL	289	937.4
D° of LIMESTONE	64 !	

XVII *Results of Observations made with Whewell's Anemometer.* *By* Mr John Rankine. *Communicated by* Professor Forbes.

Read 24th December 1838.

The following observations of the direction and force of the wind, were made with an anemometer lately invented by Professor Whewell of Cambridge.

In this instrument the only part that is fixed is a japanned cylinder, on which the points of the compass are marked by black lines dividing it in its whole length into compartments, corresponding to the spaces intervening between any two double points. On this the other parts are supported by means of a strong rod which runs down the centre of the cylinder, and, terminating in a sharp point, turns easily round as the wind changes. A single broad vane, having the rod running down the centre of the cylinder for its axis, presents to the wind a fly resembling the sails of a wind-mill, and causes the moveable part of the instrument to revolve round the fixed cylinder as the wind changes; so that the aërial current, come from what quarter it may, blows against the circular disk of the fly, and turns it with a velocity proportional to the force of the wind at the time. The motion thus produced is diminished by two endless screws working in the circumference of toothed wheels. The axis of the second wheel is continued downwards nearly to the foot of the cylinder, where it is supported and turns in a collar connected by a graduated rod with the upper part of the instrument. A pencil, attached to a nut, descends on the axis of this wheel, and presses against the surface of the cylinder, tracing in its progress, and as the vane wavers, a thick irregular line like the shadings on the coast of a map: the middle of this line is easily ascertained, and, from the compartment of the cylinder on which the marks are made, shews to the eye at one view the average direction of the wind, or, in other words, the point of the compass on either side of which the wind continually oscillates. The length of the line is measured by means of two indexes, which slide along the graduated rod connecting the upper part of the instrument with the collar near the bottom of the cylinder. The descent of the pencil, thus ascertained, is proportional to the velocity of the wind, and the time during which it blows in one direction jointly. This gives what Mr Whewell calls the *Integral Effect* of the wind, or the total amount of the aërial current that passes over the instrument in any direction, during the interval that elapses between the recordings of its indications. The space through which the pencil descends is proportional to the joint effect of the velocity of the wind, and the time during which it blows from any one point of the compass; but the anemometer does not record what portion of the joint result each element has performed. An increase of time or velocity would be indicated in the same manner, but it is impossible to discover from the tracings of the pencil what share either has had in the joint result; the velocity may vary from interval to interval, and the same current may continue for a longer or shorter time, but the instrument will always give the sum of all the elements of the current, or, in other words, will "integrate the velocity multiplied into the differential of the time."

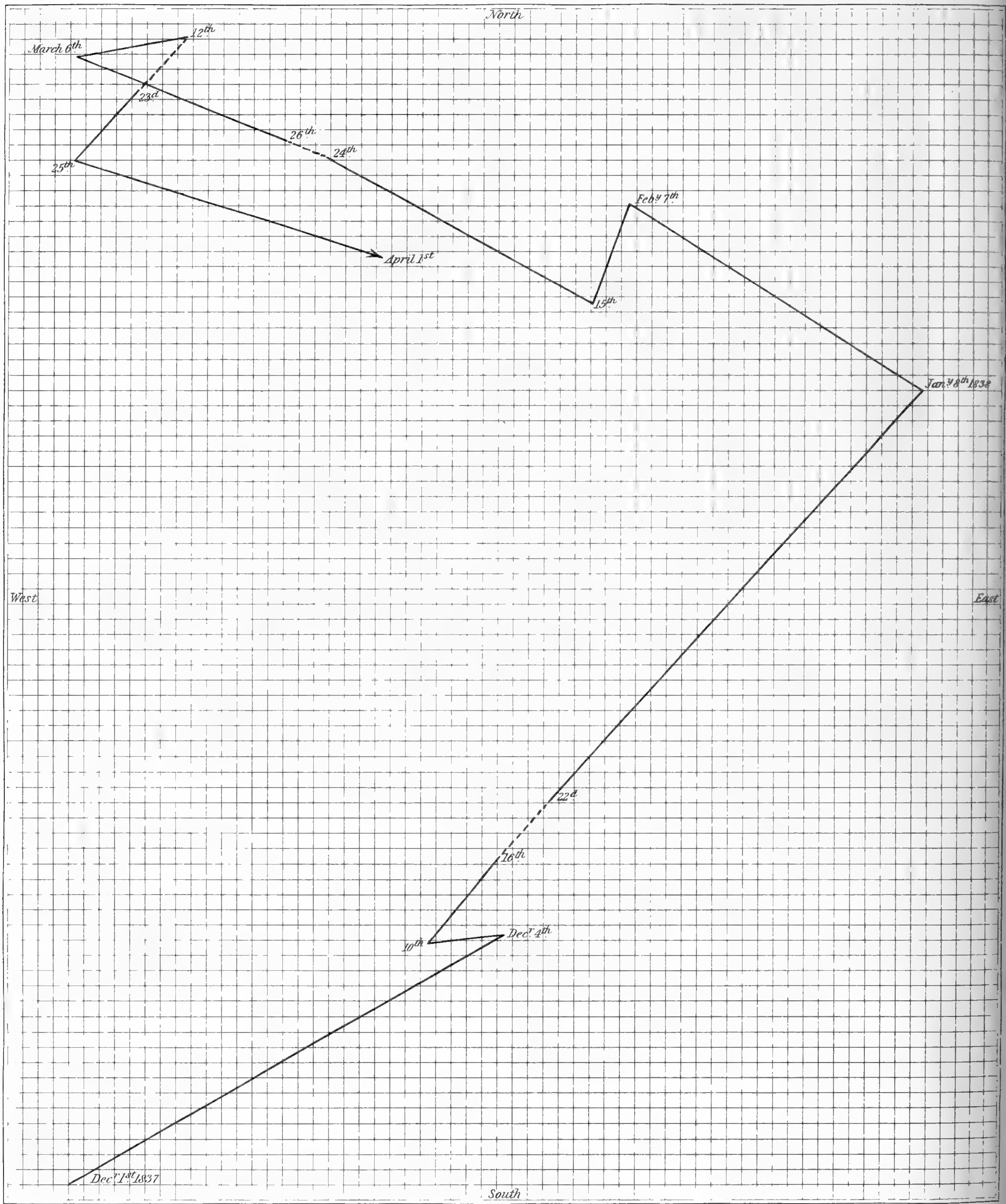
The accuracy of this result, however, rests on the assumption, that the velocity of revolution of the fly is proportional to the velocity of the wind; this has not as yet been ascertained, but it seems exceedingly probable that a very near approximation to such a result is given.

The importance of the kind of results given by this anemometer may be estimated, if it is considered how imperfectly the phenomena of atmospheric currents are observed, if the direction only be recorded, and that too by merely reckoning the number of days that the wind blows from each point of the compass. Such a method is quite fallacious, for, in point of effect, the gentle breeze of one day is placed on a par with the storm of another. The general relations which some suppose to exist between the mean annual directions at different stations, must depend very materially on the *quantity* of fluid transferred; it is plain, therefore, that unless instruments registering the *force* as well as the *direction* of the wind be employed, we may hope in vain to acquire sound data for meteorological speculation.

This anemometer was erected on the roof of the University, about the middle of November 1837, and its indications recorded up to the 1st of April 1838, except for a few days whilst it was undergoing repairs. The First Table, at the end of this paper, shews the register as kept during the months of December, January, February, and March. The readings are in inches, tenths, and hundredths, on the scale.

The method of obtaining the mean direction of the wind for any given time, is to reduce the partial winds into their component parts N, E, S, W. The sum of all the east winds taken, and subtracted from the west, gives the effective west wind, and the sum of all the north winds taken, and subtracted from the south, gives the effective south wind. By compounding the magnitude and proportion of the two effective winds, you find the magnitude and direction of the effective wind between west and south which belongs to the whole time. With the view of obtaining the mean direction and magnitude for these four months, the Second Table has been constructed: it presents the same observations reduced from thirty-two, to the four cardinal, points, by means of certain multipliers, found by considering each intermediate Point as the hypotenuse of a right-angled triangle,

Integral Effect of the Wind _ 1st Decr. 1837 _ 1st April 1838.



Scale. 1 Inch to 9. of the Reduced Results in Table II.

Lizars sc.

whose sides will necessarily represent the portion referrible to the two adjacent cardinal points. Thus for reducing the intercardinal NE, SE, SW, NW, to the cardinal points, the fraction $\frac{7}{10}$ is used as the multiplier; for reducing in a similar manner the subordinate winds NNE, ENE, SSE, &c. $\frac{12}{13}$ and $\frac{4}{10}$ are used; for the oblique winds N*e*, S*e*, E*s*, &c. $\frac{\cdot 2}{10}$ and $\frac{98}{100}$; and for NE*n*, SE*s*, SW*s*, &c. $\frac{55}{100}$ and $\frac{83}{100}$ are used as multipliers.* In this reduction the days are resolved into periods during which a certain group of neighbouring winds were prevalent.

The mode of registration employed renders it impossible to present in a graphical form every successive change in the direction of the wind. At each reading the amount was put down in the column corresponding to the direction in which such a quantity of wind had passed over the anemometer, leaving out of account altogether the order of succession: for instance, within the space of twelve hours the wind may have blown from two or three points of the compass, but we are unable to find, by a mere reference to the register, what direction was first in the *order of time*. It must be recollected, however, that this is not an imperfection in the instrument, but merely in this particular mode of registering its indications.

But though from the register employed the minute changes cannot be presented in a graphical form, yet we may take the mean direction and magnitude of certain groups of neighbouring winds, as found in the Second Table, and compounding them, give in a continuous line the average direction and magnitude that belongs to the whole time of observation. This accordingly has been done, and shews very remarkably the general features of the wind during last winter. (See Plate XIX.) It will be remembered that the month of December was very mild, with south and west winds, and though frost set in about the beginning of January, SW winds still continued to prevail till the 8th; a change then took place and east winds prevailed, accompanied by severe frost, with little intermission, up to the 6th of March. All these changes are very distinctly shewn in the line drawn in the way just stated; in fact, the *general* character of the atmospheric currents, and the severity or mildness of the season, is much more remarkably shewn than if every minute change in the direction had been traced out.

It would be interesting to compare the foregoing results with those of similar instruments at other stations, but, so far as I am aware, the observations made at Plymouth and Cambridge during the last winter have not been published.

* The notation N*e*, S*e*, &c. denotes what is commonly represented by NbE, SbE, &c.; and NE*n*, SE*s*, &c. what is commonly represented by NEbN, SEbS, &c.

TABLE I.

Date of each Observation.	N	Ne	NNE	NEn	NE	NEe	ENE	En	E	Es	ESE	ESe	SE	SEs	SSE	Se	S	Sw	SSW	SWs	SW	SWw	WSW	Ws	W	Wn	WNW	WNw	NW	NWn	NNW	Nw	Total.
1837, Nov. 14. Hours 24									.3			.05	.25	.2														*					.8
15. 24			1.45																				.33										1.78
	Not	in op	erati	on.																													
21. 24																								2.7									2.7
22. 24																			.65		.4		1·05	.7									2.8
23. 24																					5.4												5.4
24. 24																					1.3		.9		1.6								3.8
25. 24																									.4								.4
27. 48																							2.45	.9		2.4	.4						6.15
28. 24																									.9								.9
29. 24																									1.4								1.4
30. 24																			1.35		.37												1.72
Dec. 1. 24																							3.6										3.6
2. 24																				1.8			.6										2.4
4. 48																			4.4														4.4
5. 24														.25																			.25
6. 24											1.				.55																		1.55
7. 24										1.8																							1.8
8. 24										1.6																							1.6
9. 24			.6					2.3	.35																								3.25
10. 24			.6																														.6
11. 24																					.4												.4
12. 24																								2.									2.
13. 24																						.45											.45
14. 24																			1.45														1.45
15. 24																			1.63														1.63
16. 24																			2.2														2.2
	Un	der	repa	ir.																													
22. 24																	1.3		1.05		2.	.9											5.25
23. 24																					.7	.75	1.55	1.1									[illegible]

Date of each Observation.	Hours	N	Ne	NNE	NEn	NE	NEe	ENE	En	E	Es	ESE	ESe	SE	SEs	SSE	Se	S	Sw	SsW	SWs	SW	SWw	WSW	Ws	W	Wn	WNW	WNw	NW	NWn	NNW	Nw	Total.
1837, Dec. 25.	48																					3.65	.6	2.4	2.65									9.3
26.	24																			.6						1.1								1.7
27.	24																				1.4													1.4
28.	24																							.43										.43
29.	24												.38			.3		.45																1.13
30.	24													.4		.35	.27				1.95													2.97
31.	24																				1.1													1.1
1838, Jan. 1.	24																				1.4													1.4
2.	24																			.2				.4										.6
3.	24																.65		1.1															1.75
4.	24																					4.4												4.4
5.	24																			2.82			.35		.25									3.42
6.	24																			3.3														3.3
7.	24																			.15														.15
8.	24																	.3																.3
9.	24									.2		.5																						.7
10.	24											.6																						.6
11.	24										1.3																							1.3
12.	24																								.24									.24
13.	24																								.2									.20
15.	48															.23																		.23
16.	24	none																																0.00
17.	24																								.35	.3								.65
18.	24										.4																							.4
19.	24												.5																					.5
20.	24	none																																0.00
21.	24											.42		.5	.5																			1.42
22.	24													.25	1.7																			1.95
23.	24													2.1		2·4																		4.5
24.	24											2.15		2.3																				4.45

TABLE II.—ANEMOMETER (1837-8) REDUCTION.

	N	Ne	NNE	NEn	NE	NEe	ENE	En	E	Es	ESE	ESe	SE	SEs	SSE	Se	S	Sw	SSW	SWs	SW	SWw	WSW	W	W	Wn	WNW	WNw	NW	NWn	NNW	Nw
Nov. 21. to Dec. 4. Ws SSW. 2.7 4.4																			.65 1.35 4.4	1.8	.4 5.4 1.3 .37		.33 1.05 .9 2.45 3.6 .6	2.7 .7 .9	1.6 .4 .9 1.4	2.4	.4					
S = 16.22 W = 28.24	.48 .1 .58	=N														N= S=	5.9 1.49 5.22 3.57 .86 17.04 .58 16.46		6.4	1.8	7.47		8.93	4.3 W=	4.3 2.56 .99 5.22 8.24 4.21 2.35 .37 28.24	2.4	.4					
Dec. 5. to 10.			.6 .6					2.3	.35	1.8 1.6	1.			.25	.55																	
SEs NNE .25 .6 S = 0.22 E = 7.68	1.1 .46 1.56	=N	1.2					2.3 E=	.35 .48 2.25 3.33 0.92 .13 .22 7.68	3.4	1.			.25	.55	N= S=	.68 .4 .2 .5 1.78 1.56 0.22															
Dec. 11. to 16. SW SSW .4 2.2																			1.45 1.63 2.2		.4	.45		2.								
S = 5.79 W = 4.72																S=	4.87 .28 .24 .4 5.79		5.28		.4	.45		2. W=	2.11 .28 .37 1.96 4.72							
Dec. 22. to Jan. 8. S S 1.3 .3												.38	.4		.3 .35	.27 .65	1.3 .45 .3	1.1	1.05 .6 .2 2.82 3.3 .15	1.4 1.95 1.1 1.4	2.7 3.65 4.4	.9 .75 .6 .35	1.55 2.4 .43 .4	1.1 2.65 .25	1.1							
S = 29.10 W = 24.74								E=	.31 .28 .26 .18 1.03			.38	.4		.65	.92	2.05 .20 .28 .6 .9 1.07 7.49 4.85 7.52 1.43 1.91 .8	1.1	8.12	5.85	10.75	2.6	4.78	4.0 E= W=	1.1 .22 3.24 3.21 7.52 2.15 4.41 3.92 25.77 1.03 [illegible]							

TABLE II.—ANEMOMETER (1837–8) REDUCTION.

	N	Ne	NNE	NEn	NE	NNe	ENE	En	E	Es	ESE	ESe	SE	SEe	SSE	Se	S	Sw	SSW	SWs	SW	SWw	WSW	Ws	W	Wn	WNW	WNw	NW	NWn	NNW	Nw
Jan. 9. to Feb. 7. E SE*s* .2 1.5		.45	.15		.3	.65 .2	.5		.2 1.9	1.3 .4 2.2 .6 .1	.5 .6 .42 2.15 .15 .3 1.6	.5 .15	.5 .25 2.1 2.3	.5 1.7 .4 1.5	.23 2.4										1.3							
S = 11.59 E = 19.53	.44 .13 .21 .46 .2 1.44	.45 =N	.15		.3	.85	.5	W= E=	2.1 .09 .06 .21 .7 .46 4.5 5.28 .53 3.6 2.25 1.05 20.83 1.3 19.53	4.6	5.72	.65	5.15	4.1	2.63	S = N= S =	.92 2.28 .35 3.6 3.4 2.42 12.97 1.44 11.53								1.3							
Feb. 8. to 15.			1.3		4.2	1.8 .2	.3 .3	.2																	.8 2.3 .15 .16 3.41 .16 3.57				.23			
NE*e* 1.8 0 N = 5.68 E = 2.45	1.2 2.94 1.1 .24 .04 .16 5.68	=N	1.3		4.2	2.	.6	.2 W= E=	.52 2.94 1.66 .55 .19 5.86 3.57 2.29																							
Feb. 16. to 24. E*s* E*s* .2 4.6					.1					.2 .5 1.4 4.6	3.3 .4 .4 .3	4.6 2.65 .2	.7	.5			.5 .35	.5	.2													
S = 9.54 E = 17.45	.07	=N			.1			W= E=	.07 6.56 4.06 6.18 .49 .27 17.63 .18 17.45	6.7	4.4	7.45	.7	.5		N= S =	.85 1.34 1.76 4.09 .49 .41 .49 .18 9.61 .07 9.54	.5	.2					W=	.1 .08 .18							

	N	N*e*	NNE	NE*n*	NE	NE*e*	ENE	E*n*	E	E*s*	ESE	ES*e*	SE	SE*s*	SSE	S*e*	S	S*w*	SSW	SW*s*	SW	SW*w*	WSW	W*s*	W	W*n*	WNW	WN*w*	NW	NW*n*	NNW	N*w*
Feb. 26. to Mar. 6. E*s* SW. 4.3 3.35						.7	.15	.85	3.4 .1 .6	4.3 2.6 1.3 .6 .3	2.3		1.1								3.35											
S = 5.24 E = 15.10	.38 .06 .17 .61	=N				.7	.15	.85 W= E=	4.1 .58 .13 .83 8.91 2.12 .77 17.44 2.34 15.10	9.1	2.3		1.1			N= S=	1.82 .92 .77 2.34 5.85 .61 5.24								2.34	=W						
March 7. to 12. W W 3.4 .3															.15 .3		.05 .35		.05	.2 .3			.2		3.4 2.3		.8	.4				
S = 0.80 W = 7.06	.32 .22 .54	=N						E=	.18						.45	N= S=	.4 .41 .04 .41 .08 1.34 .54 .80		.05	.5			.2	E= W=	5.7 .02 .27 .18 .74 .33 7.24 .18 7.06		.8	.4				
March 23. to 25. W E 1.1 .3	1.1	3.2	.1	.1		.4 .15	.6	.4	.25 .3	1.9	.05																					
N = 4.62 E = 4.58	1.1 3.13 .09 .08 .30 .24 .08 5.02 .40 4.62	3.2 =S =N	.1	.1		.55	.6	.4 E=	.55 .64 .04 .05 .45 .55 .39 1.87 .04 4.58	1.9	.05					S=	.38 .02 .40															
Mar. 25. to April 1. S*w* WNW .6 .2				2.4	1.2					1.15								.1 .6	.2 .4		.7		.3 2.1		2.7 2.2 1.5	2.2 3.1 1.2 .8	.6 2. .2		.6	1.5 1.3	.4	
W = 16.71 N = 5.02	1.46 1.12 .42 2.32 .37 1.99 .84 8.52 2.91 5.61	=S =N		2.4	1.2			E=	1.32 .84 1.12 3.28	1.15						S=	.23 .68 .55 .49 .96 2.91	.7	.6		.7		2.4	E=	6.4 .14 .24 .49 2.22 7.15 2.58 .42 1.54 .16 21.34 3.28	7.3	2.8		.6	2.8	.4	

SUMMARY.

Periods during which groups of neighbouring Winds prevailed.	N.	E.	S.	W.
November 21. to December 4.	....	....	16.46	28.24
December 5. to ...... 10.	....	7.68	0.22	....
December 11. to ...... 16.	....	....	5.79	4.72
December 22. to January 8.	....	....	29.10	24.74
January 9. to February 7.	....	19.53	11.59	....
February 8. to ...... 15.	5.68	2.29	....	....
February 16. to ...... 24.	....	17.45	9.54	....
February 26. to March 6.	....	15.10	5.24	....
March 7. to ...... 12.	....	....	0.80	7.06
March 23. to .. ... 25.	4.62	4.58	....	....
March 25. to April 1.	5.61	....	....	18.06
	15.91	66.63	78.74	82.82
			15.91	66.63
			S = 62.83	W = 16.19

XVIII.—*On the Colour of Steam under certain circumstances.* By JAMES D. FORBES, *Esq., F.R.SS.L. & Ed., Professor of Natural Philosophy in the University of Edinburgh.*

Read 21st January 1839.

IN the end of May, or beginning of June last, I happened to stand near a locomotive engine on the Greenwich Railway, which was discharging a vast quantity of high-pressure steam by its safety-valve. I chanced to look at the sun through the ascending column of vapour, and was struck by seeing it of a very deep orange-red colour, exactly similar to that of dense smoke, or the colour imparted to the sun when viewed through a common smoked glass.

I did not pay much attention to the fact at the moment, nor attempt to vary the experiment; but, reflecting on it afterwards, it seemed to me not only as in itself very singular, but as still more extraordinary, because I had never heard of a property of steam which must have been witnessed by thousands of persons. Some months after (in the end of October), being on the Newcastle and Carlisle Railway, I resolved to verify the fact, which I had no difficulty in doing, and I farther discovered a very important modification of it. For some feet or yards from the safety-valve at which the steam blows off, its colour for transmitted light is the deep orange-red I have described.* At a greater distance, however, the steam being more fully condensed, the effect entirely ceases. Even at moderate thicknesses, the steam-cloud is absolutely opake to the direct solar rays; the shadow it throws being as black as that of a dense body, and when the thickness is very small, it is translucent, but *absolutely* colourless, just like thin clouds passing over the sun, which have, indeed, a perfect analogy of structure. When the steam is in this state, no indication of colour is perceptible in passing from the thickness corresponding to translucency, to that which is absolutely opake.

Having made these observations, which were all that the circumstances enabled me to accomplish, I was very anxious to verify them under steam of various pressures, and to determine the following, amongst other points. (1.) Whether steam, in its purely gaseous form, is really, as commonly supposed, colourless; (2.) Whether the colour depends on a stage in the process of condensation, and on that alone; (3.) What effect the tension of the steam has upon the phenomena.

But there was another inquiry which interested me much more than all these, which was, to examine how the spectrum was affected by the absorbent action of the steam, which appeared to leave the red and orange rays predominant. Judging

* The same may be observed, during the ordinary progress of the engine, in the steam thrown into the chimney, but the presence of smoke renders the experiment less satisfactory.

from the phenomena of absorption of light by gaseous bodies, and especially the singular action of nitrous acid gas in dividing the spectrum into a vast number of bands, discovered by Sir D. Brewster, I thought it by no means improbable that steam, acting in a similar manner, might exercise its specific action upon the prismatic colours at many points. Should this conjecture be confirmed, I also foresaw an application to the phenomena of the atmosphere and the production of the atmospheric lines of the solar spectrum, also remarked by Sir D. Brewster.

After various ineffectual attempts to obtain the requisite facilities, Mr Edington, of the Phœnix Iron Works at Glasgow, most kindly put at my disposition an excellent high pressure boiler, and farther afforded me every facility for prosecuting my experiments on the optical properties of steam. I first examined the simple phenomena of colour as seen by the naked eye. A lantern * was held behind a jet of steam, issuing from a stopcock in the top of the boiler, having a bore of $\frac{1}{4}$ inch. When the safety-valve (which acted with great promptness) was loaded with 50 lb. on the inch, the steam issued nearly invisible, and, at the small thickness of the jet in that part, perfectly colourless. As the light was raised, the orange colour appeared at a height of a few inches above the cock, and rapidly deepened up to a height of about 20 inches; after which it appeared that the rapid condensation of the steam only rendered it more opake, without deepening its hue. At that height, therefore, I resolved to transmit the light, and to analyze it by a prism. A theodolite, and good prism in front of the telescope, were placed at a distance of about 25 feet in front of the boiler. Beyond the steam-cock a lantern, with a lens refracting parallel rays, was adjusted, and between the steam-cock and the prism a slit of variable width. The light, reaching the prism through the slit, must first pass through the column of steam at a height of about 20 inches from the orifice. To test the adjustment of the apparatus, and also for the purpose of contrast, I had provided a bottle, about five inches diameter, full of remarkably dense nitrous acid gas, which Mr Kemp was so good as to prepare. When this was placed where the steam was to issue, the appearance of the nitrous acid spectrum was magnificently displayed. I then removed the bottle, and opened the steam-cock gradually (the pressure on the safety-valve being 55 lb. above the atmosphere, or the tension of the steam $4\frac{2}{3}$ atmospheres.) The violet end of the spectrum was almost instantly absorbed, then the whole blue, and part of the green, just as in the nitrous acid spectrum, but *no lines were visible in the remaining part.* When the cock was fully opened, the spectrum exhibited a singular appearance. The bright red was the only part which seemed natural. The extreme red was slightly invaded by the opacity of the steam. Most of the orange, the yellow, and as much of the green as was not absorbed, had a dirty and disagreeable hue, which I described in a memorandum at the time

* The experiments were performed at night.

as " dingy, alternating between yellow and purple, with shades of green. When the steam had its highest pressure, there was a decidedly purple tinge." The appearance to the naked eye of the slit was now identically the colour of the nitrous acid gas, through which I from time to time viewed a distant gas flame, and compared it with the colour of the slit. The experiment was performed under 50 and 55 lb. many times over. The light examined was then caused to pass through the steam only 10 inches above the orifice of the stop-cock, under the idea that though the colour there was fainter, possibly there might be a tendency to develope lines in the spectrum. But the experiment being made under the same pressure as before, the effect was similar, only much less intense: the slit had now but a faint tawny colour, and prismatic analysis shewed the violet alone absorbed.

Steam blowing off at 25 lb. The lantern and slit 20 inches above orifice, as at first. To the eye the light appears as red as under 55lb. Mr Edington observed, that the colour is deeper than that of the nitrous acid gas bottle. Neither he nor his assistants ever observed the colour of steam before. Prismatic phenomena as before, only the obscuration not quite so great.

Steam blowing off at 15 lb. " Evidently redder than the gas bottle. Same phenomena of spectrum, but green remains pure throughout, and verges on (bounds immediately with) orange. During the absorption of violet before vanishing (the steam-cock being gradually opened), it assumes a dirty white colour, verging on yellow and purple." A common lamp was viewed through different parts of the column of steam of this pressure, from the orifice up to a height of five or six feet, and wherever it was not entirely obscured, it appeared of different shades of smoke colour, up to an intense tawny orange.

With 7 lb. on the inch, still visibly red to the eye: prismatic phenomena similar, but slighter.

With 4 lb., no longer visibly red to the eye, when arranged as above; and even with the prism the violet appears but little affected. When let off in large quantity from the safety-valve, and a lamp viewed through it, there is a faint redness close to the orifice, but every where above, the transition is from colourless translucency to complete opacity. At about 2 and 1 lb., no colour can be detected.

From these experiments I would deduce the following conclusions:

(1.) Steam in its purely gaseous form, is, as commonly supposed, colourless, at least at small thicknesses.

(2.) The orange-red colour of steam, by transmitted light, appears to be due to a particular stage of the condensing process. In the incipient state of condensation, steam is colourless and transparent; it is next transparent and smoke-coloured; finally it becomes colourless at small thicknesses, absolutely opake at greater.

(3.) The state of tension of the steam seems only to affect the phenomenon in so far as it renders the critical colorific stage of condensation more or less completely observable.

(4.) The absorptive action of steam on the spectrum is not exerted in the same way as that of other gaseous coloured bodies, such as nitric acid gas and iodine vapour. It cuts off, however, totally the same part of the spectrum as nitrous acid does. Its phenomena perhaps have a greater analogy to those of opalescence than any other.

The effect of mere change of mechanical structure in altering the optical properties of bodies, is a phenomenon likely to give important information, both as to the constitution of matter, and the constitution of light; and the present observation may perhaps be one day received as a contribution towards a mechanical theory of vapour, including that most singular stage which intervenes between the gaseous and completely liquid form, and which is probably connected with the mechanical suspension of clouds. It is at all events very important to know that a portion of watery vapour confined in a close vessel, and subjected to change of temperature alone, without chemical change, is capable of undergoing the alterations of colour and transparency which have been adverted to. The singular fact noticed by Sir D. Brewster in the case of nitrous acid gas, whose colour deepens to an intense orange-red by the simple application of heat, seems to be a fact of the same kind.

I cannot doubt that the colour of watery vapour under certain conditions, is the principal or only cause of the red colour observed in clouds. The very fact that that colour only appears in the presence of clouds, is a sufficient refutation of the only explanation of the phenomena of sunset and sunrise having the least plausibility, given by optical writers. If the red light of the horizontal sky were simply complementary to the blue of a pure atmosphere, the sun ought to set red in the clearest weather, and then most of all. But experience shews that a lurid sunrise or sunset is *always* accompanied by clouds, and in a great majority of cases, when the changing state of previously transparent and colourless vapour may be inferred from the succeeding rain. In like manner terrestrial lights seen at a distance grow red and dim, when the atmosphere is filled with vapour soon to be precipitated. Analogy applied to the preceding observations would certainly conduct to a solution of such appearances; for I have remarked that the existence of vapour of high tension is by no means essential to the production of colour, though of course a proportionably greater thickness of the medium must be employed to produce a similar effect when the elasticity is small.

Glasgow, *29th December* 1838.

XIX.—*The Colours of the Atmosphere considered with reference to a previous Paper "On the Colour of Steam under certain circumstances."* *By* JAMES D. FORBES, *Esq. F. R. SS. L. & Ed., Professor of Natural Philosophy in the University of Edinburgh.*

Read 4th February 1839.

IN the following Paper, it is proposed to illustrate more fully the hint explanatory of certain atmospheric colours, given in a notice of the remarkable red hue of condensing steam, communicated on the 21st January. Since that time, I have examined with care the principal authors who have adverted to the subject of the colour of the sky generally, and of the redness of sunset in particular; and since, in the course of that research, I have found much to confirm, and little to modify, the view which I have already taken of the subject, I hope that the present Paper may be considered as a fit appendix to my former experimental notice. It will be recollected that in it I stated the singular fact, that steam does not pass at once from the state of invisible pellucid vapour to that of a misty white cloud, such as issues from the spout of a tea-kettle; but that an intermediate stage occurs, in which it is coloured, even very highly, giving to transmitted light a hue varying from tawny yellow up to intense smoke-red. I then observed, that, since this phenomenon does not require steam of high tension for its production, it is very probable that the tints of sunset and of artificial lights seen through certain fogs, may be owing to the absorptive action of watery vapour in this critical condition.

EBERHARD, a writer of more than sixty years ago, states that the multitude of opinions of authors on the colour of the sky alarmed him when he came to analyze them; and as, since his time, these have perhaps been doubled, some idea may be formed of the labour required to collect and classify the scattered notices which are to be found in special treatises, academical collections, and periodical works, respecting it. The most copious references I have found amongst German authors, but these I have, in almost every case, been able to verify by a reference to the original authorities. The result has been a reduction to a few of those authors who have added any thing of consequence to a subject which has rather been one of opinion than of science, until lately; and to still fewer of those who, by any one original observation or experiment, have added a single mite to the data for reasoning. The mass of copyists I may pass over in silence, or with little notice, and thus I hope to be able to reduce into moderate compass the results of a considerably tedious investigation.

It is impossible to advance any consistent theory about the colours of dawn, sunset, and clouds generally, without including the fact of the blue colour of the sky. The first notice I find quoted on the subject by way of explanation, is LEONARDO DA VINCI'S,* who attributed it to the mixture of the white solar light reflected from the matter of the atmosphere, with the intense darkness of the celestial spaces beyond. This doctrine was also maintained by FROMOND, and later by DE LA HIRE, FUNK, WOLFF, and MUSSCHENBROEK, after the Newtonian theory of colours should have banished such reasonings from science. It was still later revived, to the disgrace of modern physics, amongst the chromatic fancies of Göthe.† OTTO GUERICKE had nearly similar views.

The first trace of a more reasonable doctrine I find quoted from the writings of HONORATUS FABRI,‡ probably from his Optical Essays, published at Lyons in 1667, and which must, therefore, have been independent of NEWTON'S observations.§ In opposition to the doctrines of FROMOND, FABRI attributes the colour of the sky to the reflection of light, by corpuscular particles floating in the atmosphere; and MARIOTTE, about the same time, seems boldly to have maintained that the colour of air is blue.||

NEWTON'S thoughts on this subject are given, with his customary modesty, rather in the form of suggestions than assertions; and as many writers of the last century have only reproduced his ideas with slight alterations, it is important to observe his own exact statement of them. NEWTON'S opinion respecting the colours of natural bodies, whatever judgment we may form as to its universal application, was singularly ingenious, and well worked out. He had discovered, in the course of his memorable investigation on the colours of thin plates, that every transparent body begins to reflect colours at a certain thickness; that these vary according to definite laws, as the thickness diminishes, passing through an immense variety of compound tints, until at length it becomes so thin (as in the case of the soap-bubble) as to be incapable of reflecting any colour at all: the last colour it reflects being orange, yellowish-white, and finally blue, before they vanish; these are called colours of the first order. Now, on this subject, NEWTON says, "The blue of the first order, though very faint and little, may possibly be the colour of some substances; and particularly the azure colour of the sky seems to be of this order. For all vapours, when they begin to condense and coagulate into small parcels, become first of that bigness whereby such an azure must be reflected before they can constitute clouds of other colours. And so this being the

* Traité de la Peinture, quoted in GEHLER'S Wörterbuch, art. *Atmosphäre.*

† Farbenlehre, i. 59, quoted by HUMBOLDT. ‡ EBERHARD in Rozier, i. 620.

§ FABRI'S Dialogues (1669), of which I have found a copy in the Advocates' Library, contain many allusions to the imperfect transparency of the air, and the foreign particles mixed with it; but I do not find his theory of the blue colour clearly stated.

|| "On peut croire qu'il y a des couleurs primitives dans quelques corps, comme du bleu dans l'air. . . . . . Il semble qu'il y ait du verd dans l'eau."—MARIOTTE, Œuvres, i. 299. Leide 1717.

first colour which vapours begin to reflect, it ought to be the colour of the finest and most transparent skies in which vapours are not arrived to that grossness requisite to reflect other colours, as we find it by experience."* In another proposition, he says: "If we consider the various phenomena of the atmosphere, we may observe, that when vapours are first raised, they hinder not the transparency of the air, being divided into parts too small to cause any reflection in their superficies. But when, in order to compose drops of rain, they begin to coalesce and constitute globules of all intermediate sizes, those globules, when they become of a convenient size to reflect some colours, and transmit others, may constitute clouds of various colours, according to their sizes; and I see not what can be rationally conceived in so transparent a substance as water for the production of these colours, besides the various sizes of its fluid and globular parcels."†

The theory of NEWTON, therefore, embraces the colour of clouds, whether by reflected or transmitted light, as well as that of the blue sky. He applied a modification of the same theory to explain the *coronæ* round the sun and moon.‡ The air he seems to have believed to be devoid of colour, and the reflective particles to consist of vapour foreign to it.

The idea of MARIOTTE of the inherent quality of the sky to reflect blue light, was next prominently stated by BOUGUER, who farther put it in so palpable a form as to have been generally quoted since as a complete explanation of aërial colours.§ He observes, that as red light penetrates farther than blue (the reason is not mentioned), the latter is wholly reflected, whilst the former reaches the eye; and this theory was farther improved by later writers, by ascribing superior *momentum* to the red rays, and inferior to the more refrangible ones. SMITH, the author of the System of Optics, states the same view, but with greater clearness. "The blue colour of a clear sky," he says, "shews manifestly that the blue-making rays are more copiously reflected from *pure air* than those of any other colour; consequently they are less copiously transmitted through it among the rest that come from the sun, and so much the less as the tract of air through which they pass is the longer. Hence the common colour of the sun and moon is whitest in the meridian, and grows gradually more inclined to diluted yellow, orange, and red, as they descend lower; that is, as the rays are transmitted through a longer tract of air;"‖ and so he explains the colour of the moon in eclipses by the altered light refracted by the earth's atmosphere.

Next, EULER (1762) maintained the same opinion as to the blueness of the sky. "It is more probable," he says, "that all the particles of the air should have a faintly bluish cast, but so very faint as to be imperceptible, until presented

* Optics, Book ii. Part iii. Prop. vii. † Ibid. Prop. v. end. ‡ Book ii. Part iv. Obs. 13.

§ Traité d'Optique, p. 365-368. He likewise explains the coloured shadows noticed by Buffon.

‖ SMITH's Optics, vol. ii. Remarks, 378.

in a prodigious mass, such as the whole extent of the atmosphere, than that this colour is to be ascribed to vapours floating in the air, which do not pertain to it. In fact, the purer the air is, and the more purged from exhalation, the brighter is the lustre of heaven's azure, which is a sufficient proof that we must look for the reason of it *in the nature of the proper particles of the air*."*

The Abbé NOLLET (1764) attributes the blue colour of the sky to its reflecting those rays; but, strangely enough, he supposes, that, in order to convey that tint to the eye, they must previously have come to the earth, been reflected by it, and stopped in their second transit through the atmosphere. The colour of the sun in a fog he attributes to the fog stopping the blue rays, at which time, he says, the atmosphere must appear blue externally to an observer in the moon.†

A very clever but little known writer, Mr THOMAS MELVILL, who died in 1753, aged twenty-seven, has left some interesting observations exactly to our purpose, in a paper published in the second volume of the Edinburgh Physical and Literary Essays.‡ Amongst other acute remarks on optical subjects, after approving of NEWTON'S theory of the blue colour of the sky, he objects to his explanation of the tints of sunset, justly inquiring, "Why the particles of the clouds become just at that particular time, and never at any other, of such magnitude as to separate these colours; and why they are rarely, if ever, seen tinctured with blue and green, as well as red, orange, and yellow?" "Much rather," he adds, "since the atmosphere reflects a greater quantity of the blue and violet rays than of the rest, the sun's light transmitted through it ought to draw towards orange-yellow or red, especially when it passes through the greatest tract of air; accordingly, every one must have remarked that the sun's horizontal light is sometimes so deeply tinctured, that objects directly illuminated by it appear of a high orange or even red; at that instant, is it any wonder that the colourless clouds reflect the same rays in a more bright and lively manner." This he more fully illustrates, and then adds,—"Does it not greatly confirm this explication, that these coloured clouds immediately resume that dark leaden hue which they receive from the sky as soon as the sun's direct rays cease to strike upon them? For if their gaudy colours arose *like those of the soap-bubble*, from the particular size of their parts, they would preserve nearly the same colours, though much fainter when illuminated *only* by the atmosphere. About the time of sunset, or a little after, the lower part of the sky to some distance on each side from the place of his setting seems to incline to a faint sea-green, by the mixture of his transmitted beams, which are then yellowish, with ethereal blue; at greater distances, this faint green gradually changes into a reddish-brown, because the sun's rays, by passing through more air, begin to incline to orange; and on the opposite side of

* EULER'S Letters (translation), ii. 507. † NOLLET, Leçons de Physique, vi. 17. 1765.

‡ Page 81-89, &c. Edin. 1770.

the hemisphere, the colour of the horizontal sky inclines sensibly to purple, because his transmitted light, which mixes with the azure, by passing through a still greater length of air, becomes reddish." I have quoted this passage because, so far as it goes, it explains with remarkable elegance the actually observed phenomena, and because it exposes the insufficiency of the theory of iridescent colours to explain the hues of sunset. The theory of vesicular vapour, or floating bubbles of water as constituting clouds, was prevalent even at a far earlier period than this. LEIBNITZ had supported it in the seventeenth century,* and had calculated the rarity of the ethereal fluid with which they were supposed to be filled. KRATZENSTEIN (1740) had, by actual experiment on the colours which they reflected, attempted to estimate their thickness by direct measurement, to find their diameter.† SAUSSURE demonstrated the existence of bodies apparently so constituted, in clouds themselves; but I nowhere find that he has applied it to explain their coloration on the principle which MELVILL justly condemns in this passage. SAUSSURE's opinion of the blue colour of the sky was, so far as I can judge, that of MARIOTTE and BOUGUER,‡ although he alludes very particularly to bluish vapours as foreign matters floating in the upper regions of the sky, which he says were decidedly not aqueous, since they did not affect the hygrometer.§ He thinks this may illustrate the obscure phenomena of dry fogs.‖

The memoir of EBERHARD of Berlin on this subject,¶ contains nothing to detain us. The author seems to coincide in the theory of MARIOTTE, and spends much labour in refuting that of DA VINCI.

DELAVAL's elaborate Theory of the Colour of Bodies, we may also rapidly dispose of. He adopts the idea of FABRI, that the foreign matters suspended in the air become the means of reflecting blue light, and transmitting red, on the same principle as arsenic dispersed through glass. This comparison to the acknowledged phenomena of opalescence, is not unimportant.**

The greater part of the optical writers of the present century have closely followed one or other of those already quoted. The writer of the article Optics in the 4th edition of the Encyclopædia Britannica, which was revised by Professor ROBISON, gives, as an opinion which he considers new, that of BOUGUER and MELVILL, with very little modification or addition. He assumes the greater mo-

* Opera Omnia, ii. p. ii. 82. Edit. 1768. " Cur vapores eleventur non spernenda quæstio est, atque inter alia non malè concipiuntur in illis bullæ insensibiles ex pellicula aquæ et aëre incluso constantes, quales sensus in liquoribus spumescentibus ostendit."

† Theorie de l'Elevation des Vapeurs et des Exhalaisons, &c. Bordeaux, 1740. Quoted in SAUSSURE's Hygrometrie, § 202, and in KÄMTZ, Lehrbuch der Meteorologie, iii. 48. The diameter he made $\frac{1}{3600}$, and the thickness $\frac{1}{500000}$ inch.

‡ Voyages dans les Alpes, iv. § 2083.

§ Hygrometrie, § 355.

‖ Hygrometrie, § 372.

¶ ROZIER, Introduction, i. 618.

** Manchester Memoirs, 1st Series, ii. 214, &c.

mentum of the red ray (deduced, I presume, from the Newtonian theory of refraction), as the explanation of its greater transmissibility, and the reflection of the blue, attributing the colours of sunset to the former, those of a pure atmosphere to the latter. It would have been more correct, however, simply to assume the blueness of the atmosphere for reflected, and its redness for transmitted light, since we see in differently coloured media, that the assumed prerogative of the red ray does not hold, being absorbed by a green or blue glass, whilst the other rays persevere.

HUMBOLDT gives no positive opinion upon the colours of the atmosphere, or of water.*

It is singular that I have been unable to discover in Dr YOUNG's various writings very positive notices of his opinion on this subject, though it is probable that he coincided in general with the view last stated.† He seems to have leaned strongly to NEWTON's theory of the colour of bodies, though he was not insensible to its difficulties.

Sir JOHN LESLIE very explicitly adopts the theory of air reflecting blue light, and transmitting orange, as a full and adequate solution of the colour of a pure sky, and also of the tints of yellow, orange, red, and crimson, which characterize the sun's light when near the horizon.‡ The important observation of Sir D. BREWSTER,‖ that the blue light of the sky is polarized, and therefore has undergone reflection, is conclusive on that point, although the cause of the peculiarities of the plane of polarization in different regions of the sky is not easily explained.§

Sir JOHN HERSCHEL coincides with NEWTON in considering the colour of the sky as the blue of the first order, and as one of the most satisfactory applications of the Newtonian theory.¶

But the author who, of all others I have met with, supports BOUGUER's theory of the colour of the sky with greatest fulness and ingenuity, is BRANDES, in the article *Abendröthe* (evening redness), in GEHLER's Physikalisches Wörterbuch.** He maintains the colour of the sun, and surrounding clouds, at sunset and sunrise, to be due *solely* to the colour of pure air,—a doctrine which he supports by many striking arguments. The presence of vapours, he observes, is always indicated by a dull white, mixed with the azure of the

* See his Relation Historique, 8vo, ii. 116, &c.

† See his Nat. Phil. ii. 321. Compare pages 637, 638, 646, on NEWTON's Theory of the Colour of Bodies.

‡ Encyclopædia Britannica, art. *Meteorology*. The same theory is maintained in the article *Physical Geography* by Dr TRAILL, just published.

‖ On New Philosophical Instruments, p. 349.

§ PECLET, Traité de Physique, ii. 307. Brussels edit.; HERSCHEL on Light, art. 858, and QUETELET's Supplement to the French translation.

¶ Essay on Light, art. 1143.

** Vol. i. p. 4. &c. 1825.

sky, and the complementary colour of that white which should belong to the transmitted ray can never be red. On the contrary, he says, the colour of the sun seen directly through clouds, when on the meridian, is always white, and the effect even of so strong a mist as to render his disc easily viewed by the naked eye, is to give it the appearance of a silver plate.* The beauty of the sunset, he further observes, is in exact proportion to the purity of the atmospheric blue during the day; and the only reason, he asserts, why the sun appears to set red through vapours, is because his light is by them so much diluted that the colour can be more distinctly perceived. The colour of elevated clouds, at some distance from the horizon, he imputes (as MELVILL had done) to the great space of air which the light must traverse before it reaches them, and, after doing so, before it falls on the eye. The green colours of the sky he attributes, as LESLIE and most other writers have done, to the reflected blue light mixing with the transmitted orange. This theory was never so ably handled.

A totally different hypothesis from any of the preceding, as regards the blue of the sky, was about the same time started by MUNCKE. He asserts that this hue is, what the German writers call purely *subjective*, that is, an ocular deception, received by the eye on looking into vacant space.† This theory has been well discussed by BRANDES, but I think he has not succeeded in explaining MUNCKE's fundamental experiment, which is this:—If the sky be viewed by one eye directly, and by the other through a long blackened tube, the colour in the latter case gradually seems to vanish. Now, the explanation of this optical difficulty is to be found, I conceive, in the general fact first observed by Mr SMITH, ‡, and which I have verified in a great variety of cases, that when a white object is viewed at once by both eyes, one shaded, and the other powerfully illuminated, though its natural colour is undoubtedly white, it appears *red* to the shaded eye, and *green* to the other. The shaded eye in MUNCKE's experiment, therefore, superimposes a red impression (by the effect of contrast with the exposed eye) on the blue which it sees, and being its complementary colour, or nearly so, it must tend to diminish the blueness, and finally to produce white.

BERZELIUS adopts the view which considers the air itself coloured.‖

In the older writings of Sir DAVID BREWSTER, we find the theory of BOUGUER maintained §; but since he has been led to what we must consider, for a majority of cases, a refutation of the Newtonian doctrine of the

* GEHLER's Physikalisches Wörterbuch, vol. i. p. 6, Note.

† SCHWEIGGER's Journal, xxx. 81; and article *Atmosphäre* in GEHLER.

‡ Edin. Journal of Science, v. 52.

‖ Lehrbuch der Chemie, WÖHLER's edit. 1825, i. 346.

§ Edin. Encyclopædia, art. *Optics*, p. 620. Compare articles *Atmosphere* and *Cyanometer*.

colours of bodies, he was naturally induced to view with doubt the composition of the celestial blue, and especially of the colours of clouds. That the reflected and transmitted tints should be complementary, as NEWTON'S theory assigns, is well known to be rather the exception than rule in coloured bodies generally; and a very simple prismatic analysis, which it seems difficult to misconstrue, proves that the composition of colours—the green of leaves, for instance,—is widely different from that which the doctrine of thin plates would infer.* "I have analyzed too," he says, "the blue light of the sky, to which the Newtonian theory has been thought peculiarly applicable, but, instead of finding it a blue of the first order, in which the extreme red and extreme violet rays are deficient, while the rest of the spectrum was untouched, I found that it was defective in rays adjacent to some of the fixed lines of FRAUNHOFER, and that the absorptive action of our atmosphere widened, as it were, these lines. Hence, it is obvious, that there are elements in our atmosphere which exercise a specific action upon rays of definite refrangibility. . . . . . I have obtained," he adds, "analogous results in analyzing the *yellow, orange, red,* and *purple* light which is reflected from the clouds at sunset."† Such a prismatic analysis as is here referred to, is even more satisfactory than in the case of the juices of plants, because here the very reflected light itself is examined in the state it reaches the eye. I need hardly add, that this experiment is not less conclusive against the *subjective* theory of MUNCKE, than against the theory of thin plates of water of NEWTON and his followers.

FORSTER, in his treatise on Atmospheric Phenomena, maintains the doctrines of MELVILL respecting the colour of clouds. "We observe," he says, "that clouds of the same variety, having the same local or angular position with respect to the sun, sometimes appear richly coloured, and at other times scarcely coloured at all,—a circumstance which renders it questionable whether the colour is from the cloud itself, or whether the cloud only reflects the light which is coloured by refraction in passing through the haze of the atmosphere in the evening. The former is, however, probably the case; for different clouds, in nearly the same angular position with respect to the sun, shew different colours at the same time."‡

I must quote myself as having formerly adopted the theory of BOUGUER, with regard at least to the celestial blue. In one of a series of papers on the Bay of Naples, published about ten years ago, I noticed the occurrence of a strictly purple tinge (the poetic *lumen purpureum*), in a perfectly clear sky, which I attributed to a part of the violet rays, mixed with the blue, finding their way to

* Life of NEWTON, p. 78. 1831. Ed. Trans. xii. 538.

† Ed. Trans. xii. 544. Compare Encyc. Brit. new Edition, art. *Optics*, p. 510.

‡ Researches about Atmospheric Phenomena, 3d edit., 1823, p. 86. The continuation of the passage will be quoted further on.

the eye. There is no question (notwithstanding the authority of EUSTACE*), that VIRGIL's epithet was founded on the accurate observation of Nature. The fact has also been observed by HUMBOLDT and by LESLIE.†

We now come to the theory of M. LEOPOLD NOBILI of Reggio, and which, after what has been stated, may be very briefly expounded. In quoting M. NOBILI's speculations on this subject as new to me, I must observe, that they are contained in a memoir ‡ on a certain uniform scale of colours, for the use of artists, produced by the elegant method of depositing thin layers of transparent substances on metallic surfaces, by precipitation from solutions by means of galvanic decomposition. This beautiful art of forming what NOBILI calls his "Apparences Electro-chimiques," was first pointed out to me, as well as the papers describing it, by Professor NECKER of Geneva, as far back as the winter 1831–2, when some members of the Society may recollect that I exhibited in this room specimens of NOBILI's chromatic scale, prepared by myself.‖ From an attentive comparison of the beautiful series of tints, identical with those of thin plates, so produced, NOBILI endeavours to assign *empirically*, as NEWTON had done, the orders to which the colours of Nature belong; only, instead of cautiously proposing them as guesses, like his illustrious predecessor, he assigns them, with a degree of confidence but ill sustained by the now almost untenable character of NEWTON's theory of the colour of bodies. Many of the remarks are very ingenious, but whenever he contradicts NEWTON, he seems, I think, to fall into evident inaccuracy. The general question is one with which we have now nothing to do, and therefore I confine myself only to the statements which concern the present subject. Because he has banished the *blue of the first order*, as having no existence,§ he is forced to assign to the blue of a clear sky

* "In the splendour of a Neapolitan firmament, we may seek in vain for that *purple light* so delightful to our boyish fancy."—*Tour in Italy*.

† Encyclopædia Britannica, art. *Meteorology*.

‡ Bibliotheque Universelle (1830), tom. xliv. p. 337.—Translated in TAYLOR's Scientific Memoirs, vol. i.

‖ It is a curious circumstance, which I have never heard remarked, that Dr PRIESTLEY in a great measure anticipated the experiment of NOBILI; for, by successive electric discharges on the surface of *many* kinds of metal, he produced rings identical with those of NEWTON.—PRIESTLEY, Phil. Trans. 1778. These colours were no doubt produced by the heat developed in the same way as those mentioned in one part of NOBILI's paper. The explanation of these colours, by supposing with the philosopher of Reggio (if I understand him aright), that they are produced by thin plates of *adhering oxygen gas*, is too evidently founded in error to require any notice.

§ NOBILI quotes AMICI's authority in confirmation of this novel assertion, and also for the alleged absence of green in the second order of colours. I think I can speak with much confidence as to the existence of blue of the first order in the depolarized tints of mica plates: but the attempt to shew (Bibl. Univ. xliv. p. 343 and 344, *note*), that there *ought* to be no blue, and that the first colour of NEWTON's scale should be *white*, seems to me a failure, arising from a degree of misconception of first principles which it is difficult to admit.

the character of the second order; whilst he attributes the tints of flocculent clouds, partially illuminated by the sun or moon, to the first order; in other words, he supposes the vesicular vapour of which he speaks, to have double the thickness in an azure sky, than in the midst of a fog, whilst NEWTON expressly assigns the blue of the first order to the air, because "it ought to be the colour of the finest and most transparent skies in which vapours are not arrived at that grossness requisite to reflect other colours, as we find it is by experience." This is only one of the various contradictions into which the artist-like view of matching colours by external resemblances, and assuming a common origin, has led the ingenious author. The application of the colours reflected from vapours to measure the thickness of the vesicles* was, we have seen, completely anticipated by KRATZENSTEIN, and the generality of the application disproved by MELVILL half a century ago, when he speaks of the theory of the "gaudy colours" of the clouds arising, "like those of the soap bubble, from the particular size of their parts."

I have perused NOBILI'S Memoir with a most anxious wish to arrive at his true meaning, disembarrassed of the somewhat poetical vagueness of his own expressions, and the serious mistakes of his translator; and I believe his view to be this:—There are both transmitted and reflected tints in the sky. The transmitted ones are complementary to the blue of the sky, and therefore, acccording to NOBILI, of the *second* order, whilst all the fiery tints which particularly characterize sunset as contrasted with the dawn, are colours of the *first* order reflected from the vesicular vapours of clouds.

An ingenious paper by Count XAVIER DE MAISTRE on the colour of air and water, appeared in the *Bibliotheque Universelle* for November 1832.† With regard to the atmosphere, the author's theory is so far similar to that of DELAVAL, that its colour is to be ascribed to the peculiar state of the particles of water contained in it acting on the principle of opalescence, the reflected light being blue and the transmitted orange. He thence refers to the colours of sunset, and adds,—"But it often happens that the colours are not observed, and the sun sets without producing them. It is not, therefore, to the pure air alone that we must attribute the opaline property of the atmosphere, but to the mixture of air and vapour in a particular state, which produces an effect analogous to that of the powder of calcined bones in opaline glass. Neither is it the quantity of water which the air contains that occasions these colours, for when it is very humid, it is more trans-

* In the translation of the paper in TAYLOR'S Scientific Memoirs, i. 99, by an oversight, the maximum thickness of the cloudy vesicles is stated at *the ten-millionth* of an inch, instead of *ten millionths* of an inch, or a hundred times greater, as in the original. There is even a slight mistake in the latter; the tint he describes corresponding to plates of water, not of air, would require a thickness of seven millionths.

† Translated in the Edin. New Phil. Journal, vol. xv.

parent than it is in an opposite state, the distant mountains then appearing more distinct,—a well known prognostic of rain, and the sun then sets without producing colours; in the fogs and vapours of the morning, the light of the sun is white, but the red colour of the clouds at sunset is generally regarded as the forerunner of a fine day, because these colours are a proof of the dryness of the air, which then contains nothing more than the particular disseminated vapours to which it owes its opaline property." In this interesting passage we have, I am persuaded, all that is known of the cause of atmospheric colours, with the single want of the link which shall shew that watery vapour is sometimes capable of absorbing all but red rays, and sometimes not.*

The late Mr HARVEY of Plymouth, gives a minute analysis of the colours of the clouds,† which he considers only explicable on the theory of absorption, which office he assigns to the particles of the clouds themselves, though he admits that these often transmit pure white light. He is even ready to believe that the sun has sometimes been observed blue or green, an observation which I think M. ARAGO has rightly considered as an optical deception arising from the contrasted colour of an intensely red sky, such as that which occurred in many parts of the world on the occasion of the dry fog of 1831.‡

BRANDES's theory of the evening red, is especially applicable to the rich purple hue thrown over Mont Blanc and the higher Alps|| after the sun has set to the plains, and that kind of redness is usually observed in cloudless skies, not like the gorgeous colouring of our northern sunsets to which I particularly referred in my former paper. In a communication read to the British Association in 1837, M. DE LA RIVE accounts ingeniously for a repetition of this phenomenon which is sometimes observed 10 or 15 minutes after the first disappeared. This he plausibly attributes to a total reflection undergone by the rays of light in the rarer regions of the atmosphere when in a state of great humidity and transparency.§

* Count MAISTRE explains the colour of the water by similar reasoning. He considers it blue for reflected, and yellowish-orange for transmitted light, and the green colour of the sea and some lakes he attributes to diffused particles which reflect a portion of the transmitted tint, and mingle with the blue. This is well confirmed by DAVY's Observations, (Salmonia, 3d edit. p. 317). ARAGO has very ingeniously applied the same reasoning to the ocean, shewing that when calm it must be blue, but when ruffled, the waves acting the part of prisms, refract to the eye some of the transmitted light from the interior, and it then appears green, (Comptes Rendus, 23d July 1838.) Most authors have admitted the intrinsically blue or green colour of pure water, as NEWTON (Optics, b. i., part ii., prop. x.), MARIOTTE (already quoted), and EULER: HUMBOLDT seems doubtful, (Voyage, 8vo, ii. 133).

† Encyc. Metropolitana, art. Meteorology, p. 163, &c.

‡ Annuaire 1832, p. 248. Whilst this Paper is passing through the press, I have seen a notice by M. BABINET (Comptes Rendus, 25th Feb. 1839), on the subject of the blue colour of the sun, which he considers as real, and endeavours to explain by the theory of mixed plates.

|| Germ. "Glühen der Alpen."

§ Seventh Report of British Association. Transactions of Sections, p. 10.

Probably upon the principle of multiplied reflections, the cases of preternaturally protracted twilights may be explained, such as those recorded by KÄMTZ*.

It is now time that we endeavour to sum up briefly the evidence we have collected.

If we exclude the theory of LEONARDO DA VINCI and GÖTHE, attributing the colour of the sky to a mixture of light and shade; and that of MUNCKE, which would make it a mere optical deception, we shall find the chief principles which have been maintained, reduced to three.

(1.) That the colour of the sky is that reflected by pure air, and that all the tints it displays are modifications of the reflected and transmitted light. This is more or less completely the opinion of MARIOTTE, BOUGUER, EULER, LESLIE, and BRANDES.

(2.) That the colours of the sky are explicable by floating vapours acting as thin plates do in reflecting and transmitting complementary colours. This was NEWTON'S theory which has been adopted in whole or in part by many later writers, and especially by NOBILI.

(3.) On the principle of opalescence and of specific absorption depending on the nature and unknown constitution of floating particles. To this theory in its various stages, we find FABRI, MELVILL, DELAVAL, Count MAISTRE, and Sir D. BREWSTER, attached.

These different views are so easily blended, and have often been so far misunderstood even by their supporters, that it is impossible to draw any definite line between them. I will notice a few of the leading points of difficulty which present themselves to some of these opinions, and tend to restrict the field of inquiry.

1. The azure of the sky cannot, I think, with any probability, be referred to the existence of those vesicular vapours which are supposed to act so important a part in the mechanism of clouds. We have no evidence direct or indirect of their existence, whenever the hygrometer is not affected, nor indeed where it does not indicate absolute dampness. The atmosphere we know to be pre-eminently transparent when loaded with uncondensed vapour. That vapour may be colourless, or it may not; the presumption is, I think, that it has no colour, since the blue of heaven is always most fully developed when the dryness of the air is intense; and *that* even at heights which render it in the last degree improbable that any condensed vapour should exist at heights still greater. We are as ignorant of the constitution of the parts of pure vapour, as we are of the parts of pure air: vesicles are *water*, not *vapour*;—to speak of films capable of reflecting definite colours when no *water* exists in the air, or the hygrometer does not indicate absolute dampness, is to speak (as BERKELEY said of Fluxions) of the ghosts of departed quantities.

* Lehrbuch der Meteorologie, iii. 58.

2. Admitting that the blueness of the reflected light of the sky is an inherent quality, of which we can give no account, we must next say that it is running too fast to a solution to admit with Brandes that the red of evening is solely caused by the colour of the air being complementary to its reflected tint. His explanation of the variable redness of sunset, owing to the variable opacity of white vapours allowing the redness to be more or less distinctly perceived, though ingenious, is palpably wrong. The simplest experiments prove that the redness is not merely apparent, but depends upon the admixture of the variable ingredients of the atmosphere. The proof is the Prismatic Analysis of the sun's light, and we may add, the observation of artificial lights in different states of the atmosphere, which at some times are seen in their natural condition, at others lose all their rays but the red, and finally vanish in fogs with an intense red glare.

3. If fogs and clouds modify the solar light on the principle of reflecting the rays they do not transmit, why do not such fogs and clouds appear vividly blue by reflected light, as Nollet supposed a foggy atmosphere must do to a spectator placed beyond it?

4. If the vesicles constituting the clouds give to the colourless light falling upon them the various hues of sunset, why, in the first place, do we not perceive bows of various hues, as Kratzenstein did in operating on the small scale; and how comes it that clouds, identical in structure, nay the very same clouds, do not exhibit sunset tints at any other time of day? But the most convincing proof of any, is simply to watch the progress of the solar rays tinging a cloud successively with different hues, just as it would a lock of wool similarly placed; or as it does the snowy Alpine summits. Forster mentions an instance of detached cirro-cumuli being of a fine golden-yellow, but in a *single minute* becoming deep red.

5. To these unanswerable difficulties the prismatic analysis of the blue and sunset tints of the sky superadds one conclusive against the theory of Newton as it at present stands. The reflected blue and transmitted red-orange are *not* colours of thin plates. They are derived from all parts of the spectrum by the mysterious process of transmission, which has preserved them and absorbed the rest. It is hopeless at present to inquire what is the mechanical constitution of the medium which has effected this alchemy.

One question, however, which is quite within our reach, remains to be answered. The colours of the sky cannot indeed be explained, if by explanation we mean an ultimate analysis of the mechanism producing them; but the theory of absorption is incomplete until we can shew in what part of the course of the rays of light, and under what varying circumstances, the different phenomena of colour may be produced. Hassenfratz observed, that the light of the horizontal sun was deficient, when analyzed by the prism, in all the violet and blue rays.* Sir D. Brewster, making a similar observation with more care, has detected a *speci-*

* Kamtz, Lehrbuch iii. 40.

fic action of the earth's atmosphere affecting every part of the spectrum by absorbing, or annihilating certain luminous rays of every colour. The analogy which he has observed to exist between the deficient lines of the atmospheric spectrum, and those of the common solar spectrum, (which Sir DAVID supposes to have been produced in the transit of light through the sun's atmosphere), and those developed in artificial light by the absorptive action of nitrous acid gas, is truly remarkable, and has led him farther to conclude, "that the same absorptive elements exist" in all those media.* Now, since it is the strata of air nearest to the earth whose effect is chiefly conspicuous in producing the tints of evening, it is to be presumed that the elements which produce this action, are within reach of chemical analysis. The air, containing as it does the constituents of nitrous acid gas, is naturally first looked to for their origin. But this supposition, even if it be true, for the atmospheric *lines* of the spectrum, cannot explain the extraordinary variety of absorptive action observed in hazy weather, when, as we have said, the atmosphere at a thickness of but a few miles suffers only the red rays to pass; a fact familiar to those who have attended to the subject of light-house illumination, and in consequence of which crimson signal-lights were proposed a few years ago for adoption in hazy weather by Sir JOHN ROBISON,† on account of the persistence of such rays in a foggy atmosphere. The absorptive elements are clearly within our reach; can they be nitrous gas, or what are they? The experiment detailed in my last paper comes in to answer the question. Vapour has hitherto been known (to philosophers at least) under but two characters,—a colourless gaseous body, and a translucent pure white mass of particles generally called vesicular.‡ I have shewn that it passes through a third or intermediate state, in which it is very transparent, but having a more or less intense colour graduating through the very shades which nitrous acid gas assumes,—that is, tawny yellow, orange, deep orange-red, intense smoke-red, verging on blackness. I say that this discovery, to a great extent, supplies the gap which was wanting to make the absorption theory intelligible. It is the "mixture of air and vapour in a particular state," which Count MAISTRE supposed (see the passage quoted above), but could not prove to exist. The threefold condition of vapour in the sky we can now exhibit in a room;—the pure elastic fluid devoid of colour, which gives even to pure air its greatest transparency,—next, the transition state, when, still invisible in form, and almost certainly not vesicular, it transmits a steady orange glare, not the play of colour which is often seen in clouds and fogs forming a glory round a radiant body;—and lastly, the vesicular steam, such as we every day see issuing from the spout of a tea-kettle reflecting iridescent colours, just as the semi-opake clouds do which seem to float across the disk of the sun or moon.

* Ed. Trans. xii. 530.

† Phil. Mag. 1833.

‡ See ROBISON's Works, ii. 2, &c.

These coronæ, notwithstanding their apparent analogy to the colours of thin plates, seem rather to be due to the effect of diffraction.*

The non-appearance of the lines of the spectrum in my experiment, may be plausibly explained in the following manner, which, however, I offer merely as a conjecture. When steam of high pressure issues from an orifice, a horizontal section of the expelled column will include vapour in every stage of condensation. Its centre, up to a certain height, will be pure invisible steam; at the exterior of all, in contact with the cold air, there will manifestly be vesicular steam, and a cylindrical space between the two will contain red steam. Now it is extremely probable, that when the experiment is performed on the small scale, as I have described it, by suffering light to pass through such a compound column, and then analyzing it by the prism, enough of unabsorbed rays are reflected from the highly luminous surface of the vesicular steam to prevent the fine lines from being seen if they exist. And I am strongly confirmed in this conjecture by the fact, that when the rush of steam is very violent, and always when much vesicular vapour is present, the unabsorbed part of the spectrum presents a washy and impure tint (particularly mentioned in my former paper), which probably arises from a blending of the colours, produced by this cause.

In conclusion, I have only a word or two to say respecting the application of these facts to atmospheric appearances regarded as prognostics of weather. The modified hues of the sky, and of the sun and moon near the horizon, have, for so many ages, and in so many countries, been regarded as the surest indications of atmospheric changes, that we cannot doubt that it is to the variety of conditions in which vapour exists in the air, more or less nearly condensed, that these phenomena are due. HUMBOLDT describes the colour and form of the sun's disc at setting in tropical regions, as the most infallible prognostic,† and elsewhere ascribes these variations "to a particular state of the vesicular vapour."‡ Since the red steam occurs only during the critical stage of its partial condensation (and perhaps conversely during evaporation), it is evident that it must correspond to a critical state of diffused vapour of the atmosphere. The applications might be very extended; I will only advert to one, the surest, most consistent, and probably the most ancient of such prognostics. The red evening and grey morning as the signs of fine weather, are recorded in the verses of ARATUS,|| in the New Testament,§ and in one of our most familiar proverbs. It is wholly inexplicable on the theory of BRANDES, which considers the redness as due solely to the purity of the atmosphere, since that is usually greater in the morning than the evening. According to my view it occurs thus: Soon after the maximum

* See YOUNG's article *Chromatics*, in Encyc. Brit., and FRAUNHOFER in SCHUMACHER's Astronomische Abhandlungen. Drittes Heft. 1825.

† Relation Historique, 8vo, ii. 128.

‡ New Spain (translation), ii. 326.

|| Diosemeia, 93. Quoted by KÄMTZ.

§ Matt. xvi. 2, 3.

temperature of the day and before sunset, the surface of the ground, and likewise the strata at different heights in the atmosphere, begin to lose heat by radiation. This is the cause of the deposition of dew, and consequently in severe weather we have vast tracts of air containing moisture in that critical state which precedes condensation, and yet it may be exceedingly doubted whether any vapour properly called vesicular is necessarily formed in this process. Be that as it may, every accurate observer of nature in alpine countries will confirm me in stating, that fine weather is almost invariably accompanied by the formation of dew on exposed surfaces, and by the progressive depression of the moister strata, until at length visible fogs are formed in the bottom of the valleys, and especially over water.* This is the surest sign of a following fine day in mountainous regions. Now SAUSSURE in his ascent of Mont Blanc, "observed that the evening vapour which tempered the sun's brightness, and half concealed the immense space he had below him, formed the finest purple belt, encircling all the western horizon, and as the vapour descended and became more dense, became narrower and of a deeper colour, and *at last of a blood-red.*"† Now this phenomenon corresponds, I imagine, precisely to the development of colour which I have remarked in vapour in the act of being condensed, and DE LA RIVE's remark, that the nocturnal illumination of Mont Blanc takes place in serene evenings, *when the air is highly charged with moisture,* is to the same purpose. But a remark of Mr FORSTER, in his "Researches about Atmospheric Phenomena,"‡ is even more pointed, and is valuable, because his work is pre-eminently descriptive, rather than theoretical. "Sometimes the tints in the twilight haze come on so suddenly and are so circumscribed, as to induce a belief that very sudden and partial changes take place in the atmosphere at eventide; *which may perhaps be somehow connected with the formation of dew.*" He then records an observation made 2d November 1822. "Being about four o'clock in the evening, near Croydon in Surrey, I observed a very beautiful western sky, caused by the bright edge and dependent fringes of a light bed of cloud being finely gilded by the setting sun. Some detached cirrocumuli also, which formed the exterior boundaries of the aforesaid cloud, were likewise of a fine golden-yellow, and the same colour appeared in different clouds in other parts of the sky, while the scud-like remains of the nimbus floated along in the west wind below. In the course of about a quarter of an hour, the lofty gilded clouds all assumed a deep red appearance, and the change was effected so suddenly, that while looking at them, I only took my eyes

* For the reason why over water, see DAVY's Paper, Phil. Trans. 1819.

† Quoted by HARVEY in Encyc. Metrop. *Meteorology*, p. *166. The cause of the purple light mentioned here, probably arises from a mixture of the reflected blue of the pure sky (*which is always present when purple is seen*) with the yellow-orange, which condensing vapour first transmits. I do not think it at all necessary to affirm, however, that pure air has no transmitted colour of its own.

‡ Third edit. p. 87.

off them for a minute to stop down the tobacco in a pipe that I was smoking, and when I looked up at them again, the colour was totally changed. Now, what renders the phenomenon remarkable is, that it happened just about the period of the vapour point. The descending sun had scarcely had time to make any great difference in the angle of reflection, and it seemed therefore, that some sudden change, produced by the first falling dew, was the cause of this simultaneous change of colour in all the clouds then visible." I confess it seems to me that this passage is nothing short of a demonstration of the truth of my theory of Atmospheric Colour, the more interesting, because I was unacquainted with it until after writing nearly the whole preceding part of this paper.

With regard to the Morning the case is very different. In fine weather the strata near the surface of the earth alone, and in the lowest and most sheltered spots, are in a state of absolute dampness. The vapours, which, during the reversion of the process, might probably produce colour, are not elevated until the action of the sun upon the earth's surface has continued long enough to impart a sensible warmth, by which time the moment of sunrise is past, and the sun's disc has risen above the horizontal vapours. It would be easy, by a more lengthened discussion, to shew, that the slowly progressive transition of vast masses of air through the temperature of the dew-point, can only occur in serene weather at sunset and not at sunrise. The inflamed appearance of the morning sky, considered indicative of foul weather, is, I have no doubt, owing to such an excess of humidity being present, that clouds are actually being formed by condensation in the upper regions, contrary to the direct tendency of the rising sun to dissipate them, which must therefore be considered as indicating a speedy precipitation of rain.

Edinburgh, *4th February* 1839.

XX.—*On Fresnel's Formulæ for the Intensity of Reflected and Refracted Light.*

By PHILIP KELLAND, *M. A., late Fellow of Queen's College, Cambridge, Professor of Mathematics, &c., in the University of Edinburgh.*

Read February 18. 1839.

INTRODUCTION.

IT is well known, that when light is incident on a refracting surface, a portion of it is reflected, whilst both the transmitted and the reflected light undergo polarization. The obvious mode of accounting for this, is to attribute to the particles on whose motion light is supposed to depend, the property of transmitting one class of vibrations more freely than another, limited, however, by the direction and mode of action of the adjacent particles. M. FRESNEL, in order to determine the intensity of light reflected and refracted under different circumstances, assumed that the density of the particles of ether is greater in refracting media than *in vacuo*. By means of this assumption, and other subsidiary ones, he deduced formulæ for the intensity of the reflected and refracted light, by means of which the amount of polarization, as well as the change which the plane of polarization undergoes, can be readily deduced. The obvious interpretation of the formulæ coincided precisely with discoveries which had been long known, and the more difficult deductions from them have been tested by numerous experiments of Sir DAVID BREWSTER and others. It appears that, although for highly refractive media, they may be only approximations, yet, in most cases, they are so close as to deserve the most careful attention of those who endeavour to establish a correct mechanical theory.

M. CAUCHY, in different memoirs, has laboured to deduce M. FRESNEL's formulæ from the equations of motion, and, in one instance, from assumed conditions of a nature not widely different from M. FRESNEL's own. The fact that these expressions had been deduced from the assumption of a greater density within refracting media than without, appeared to throw a doubt over the truth either of the molecular hypothesis, which seemed to require the reverse, or of the formulæ themselves.

Whilst M. CAUCHY is tossed about with various and conflicting conclusions, Mr M'CULLAGH is led, by totally different considerations, to one of the most important of them, viz. that the vibrations which constitute light polarized in the plane of incidence are vibrations effected in that plane, a result which is directly opposed to that of M. FRESNEL. How these philosophers have succeeded in the more complex case of crystalline reflexion, it concerns us not to inquire, until the principles which guide their hypotheses shall be shewn to be sound and mechanical.

It appears, however, that M. CAUCHY has actually inferred, from mechanical principles, that the vibrations of polarized light are the opposite to those assumed by M. FRESNEL. Of the amount of evidence which M. CAUCHY adduces I am altogether ignorant; but it ought to be overpowering indeed to shake our faith in an hypothesis which has so successfully overcome all difficulties, and brought the apparently complex phenomena of double refraction to the level of common optics.

However this be, the matter is not yet set at rest, for a paper has just been printed for the next part of the CAMBRIDGE TRANSACTIONS, in which M. FRESNEL'S hypothesis as to the direction of vibration is assumed to hold, and his formulæ corresponding to light polarized in the plane of incidence are established, whilst an approximate demonstration is offered for those corresponding to the perpendicular plane.

My primary object in drawing up the present memoir has been to remove from the molecular theory some difficulties in which Mr GREEN'S researches seem to involve it. As a preliminary step, I will therefore point out the most important of these, and endeavour to shew that the arguments which naturally arise out of them are such as can be answered without compromising any of the principles on which the molecular hypothesis is based. Having done this, I shall apply the equations of motion deduced from molecular forces, to shew that the formulæ result in the most satisfactory manner from the state which such forces induce.

To effect my purpose of explaining the difficulties which Mr GREEN'S memoir opposes to the molecular theory, it will be requisite that I point out in few words the nature and results of that theory.

Almost all mathematicians have admitted the idea of discrete molecules to be philosophical; but very few have attached any weight to the results to which this hypothesis leads. LAPLACE, in his *Mécanique Céleste*, supposes the atoms of matter to be permeated by the molecules of caloric; but he assigns forces to the molecules, which are conceived to diminish with great rapidity as the distance from the molecule is increased, and actually to vanish at all appreciable distances. By a similar hypothesis, in the same great work, he solves the problem of Capillary Attraction.

POISSON also, in his Memoir on the Equilibrium and Motion of Fluids, as well as in his Capillary Attraction and Theory of Heat, conceives the particles to be separated by finite intervals, and makes use of a force which results from this circumstance; but neither he nor LAPLACE appears to have investigated the complex arrangement of actions and their counteracting opposites, to which this force is due. This last investigation was reserved for M. CAUCHY, who managed it with great skill, in his *Exercices de Mathématiques*, vol. iii. p. 188, and vol. iv. p. 129. In the fifth volume, M. CAUCHY applied his results to the theory of light; but his success was not complete at first, owing to the circumstance that he had recourse to the method of expansion so universally adopted in physical investigations. In subsequent publications, however, M. CAUCHY has solved the difficult problem of obtaining a relation between the velocity of transmission and the length of the wave. This very important result, which removed from the undulatory theory almost the only obstacle to its being entitled to the designation of a true *physical theory*, appeared in 1830. Since that time M. CAUCHY has published various memoirs on the reflexion of light, and on other points of the theory, in one of which he has determined the law of force by which the particles act on one another to be that of the inverse fourth power of the distance.

In a memoir of my own (*Transactions of the Cambridge Philosophical Society*, vol. vi. p. 153), another law is arrived at, viz. that of the inverse square of the distance. This conclusion, agreeing as it does with the great law of gravitation, and necessary, moreover, as it appears to be, from the very condition of attraction, I have retained in all my subsequent investigations. One important corollary from it will be found in page 180 of the same memoir, viz. that the vibrations are altogether transversal to the direction of a *wave*. This conclusion Professor LLOYD has also obtained from CAUCHY'S law of the inverse fourth power. His paper was read to the Royal Irish Academy. It would be too wide a field to enter on the discoveries of Sir WILLIAM HAMILTON. Copious information on the subject, together with a translation of M. CAUCHY'S most important memoir, will be found in the pages of the *Philosophical Magazine*.

Nor is the arrangement and action of force thus assumed less in consistence with statical than with dynamical truths. The great problem of cohesion, as connected with expansion, &c., appeared to defy a law of force such as that of the inverse square, until M. MOSSOTTI, by a most skilful application of analysis, removed the most glaring difficulties. The same subject has been commenced by myself in the *Transactions of the Cambridge Philosophical Society*, vol. vii., in which I have deduced results which demonstrate the possibility, or at least afford argument for the probability, of the universality of the law of universal gravitation. This, therefore, is the present state of the molecular theory: it coincides with the great law of attraction, and is the extreme limit to that law; it accounts for the complicated phenomena of light, which defy more simple investigation, at the same time that it requires the introduction of no modification into those processes which are adequate to effect their purposes without its aid; it demonstrates the necessity of a circumstance which had previously been only suspected to exist, the perfect transversality of vibration; and, lastly, it promises an insight into the perplexing phenomena of absorption. Having thus pointed out the nature and results of the molecular hypothesis, I return to the examination of the memoir, which appears in some points to argue against it.

Mr GREEN states, that two waves will result from giving a motion to a fluid, such as that commonly supposed to be the medium the vibrations of which constitute light, the one transversal and the other normal. On a careful examination of his memoir, I cannot discover this normal vibration; the nearest approach to it appears to result from the circumstance, that two waves, the incident and the reflected, may be transmitted at the same time, and therefore cross each other. If, then, in this case, the angle of incidence be an angle of 45°, one vibration may be at right angles to the other; but this circumstance does not in the slightest degree militate against any conclusions which have been arrived at by the molecular hypothesis. The coexistence of vibrations travelling in different directions, is distinctly recognised in that theory. It may be well to state clearly, that the point, and I think a most important one, which has been proved from the molecular hypothesis, is this; that *one* wave cannot consist partly of normal, partly of transversal vibrations. Of course, the definition of the wave restricts it to a state of motion transmitted in *one* direction with *one* velocity.

There can be little doubt, however, that the normal vibration to which Mr GREEN refers, is supposed to be contained in that function which he introduces in the body of his memoir, as the result of the change of motion from an incident and reflected to a refracted one. This vibration is, however, merely a vibratory motion, *not* transmitted in the same direction as the incident; and in the sequel of the present memoir, it will appear that it is really and *bonâ fide* a transverse vibration. Thus a statement, which at the first sight appears to argue powerfully against the molecular theory, does, when attentively examined, afford strong presumptive evidence in its favour.

I have deemed it right to be explicit on this subject, as the admission of Mr GREEN's statement, if it left hypotheses such as LAPLACE's as to the constitution of media uninjured, would absolutely crush the more probable hypothesis of the Newtonian law of gravitation applied to the ultimate atoms.

There is another point in Mr GREEN's paper which, although not so important as the one just noticed, will require an answer of a very different nature, and ought consequently to be attended to. It is this: in order to obtain the law which FRESNEL has deduced for the intensity of light polarized in the plane of incidence, it is found requisite to assume that the velocity of transmission varies inversely as the square root of the density.

This overthrows, apparently, all the previous conclusions of the molecular hypothesis; for all its advocates, as far as I recollect, have come to the conclusion that the density of the caloric within refracting media is less than it is *in vacuo*. But it is desirable that great caution should be exercised in judging of this and like apparent oppositions. We have no very precise notion of the proper signification of density within a medium, nor, if we had, is it quite obvious that the aggregate attraction of combined molecules should of necessity vary as the density. It must be recollected that, in estimating the effect of forces resulting from molecules, the whole result consists of the sum of a large number of terms, not diminishing in magnitude with the same rapidity in all cases. It does not then follow, that the attraction varies as the density, nor even according to any simple function of it.

But I do not stop here. Allowing the assumptions of FRESNEL to be correct, —and from the coincidence of Mr GREEN'S conclusions with his, most persons will be inclined to think them substantially so,—all the discrepancy between the molecular hypothesis as viewed by M. CAUCHY, and that deviation from it adopted by M. POISSON and Mr GREEN, amounts to this, that one party (suppose the former) have misinterpreted the formulæ relative to the density of the particles. I shall shew presently that the formulæ themselves are not at all affected by the apparent contradiction of conclusion, since the results of M. FRESNEL may be deduced as easily, and I think with as little assumption, by the molecular hypothesis as by the other.

ANALYTICAL INVESTIGATION.

My object in the investigation which follows, is to deduce M. FRESNEL'S formulæ for the intensity of rays reflected at the common surface of two media, air and glass, the incident rays being polarized. It will not be requisite in this place to enter into a discussion of the results obtained by grouping particles. Suffice it to say, that, by strict mathematical investigation, it can be shewn that the assumption of NEWTON'S law of force for the particles of the media surrounding the material particles, gives rise to an expression of the following form, for the aggregate attraction or repulsion of those particles which surround one particle of matter

$$f = m \,.\, e^{-\alpha a} \frac{1 + \alpha a}{a^2}$$

a being the distance between two material particles. This expression is insensible at sensible distances, and consequently we may limit our summation in the subsequent process to such distances. We will adopt the following notation.

The media being both perfectly symmetrical, and bounded by a plane surface; let that plane be called the plane of $y\,z$, the axis of z being parallel to the line at which the front of the wave cuts the plane, and that of x' the direction of transmission. When the incident vibrations are polarized in the plane of incidence, all the motion will be in a direction parallel to the axis of z.

Take x, y, z as the co-ordinates of any particle in a state of rest; $x, y, z + \gamma$

those of the same particle at the time t,

$$x+\delta x,\ y+\delta y,\ z+\delta z,\ \text{and}\ z+\delta z+\delta\gamma$$

the corresponding quantities for another particle in the upper medium;

$$x+\delta x_{,},\ y+\delta y_{,},\ z+\delta z_{,},\ z+\delta z_{,}+\delta\gamma_{,}$$

the co-ordinates of another particle in the lower medium at the same time.

When discussing the lower medium separately, we will adopt $x_{,}, y_{,}, z_{,}, \gamma_{,}$ etc. in all cases for which we use x, y, z, γ, etc. in the upper.

r is the distance between the two particles in a state of rest.

$r+\varrho$ their distance in a state of motion.

Let $r\phi r$ be the force on the particle under consideration, arising from another particle at the distance r. If, however, it be thought requisite, we may consider $r\phi r$ as the aggregate attraction of a group of particles about a material particle. The law of force, whenever a law is wanted, will be assumed to be that of NEWTON.

Let r', $\delta x'$ etc. denote the distance, etc. of particles in one medium from those in the other.

The notation $\gamma_{,x_{,}+\delta x_{,},\, y_{,}+\delta y_{,}}$ denotes the value which $\gamma_{,}$ assumes when $x_{,}+\delta x_{,}$, $y_{,}+\delta y_{,}$ are written for $x_{,}$ and $y_{,}$.

Slight deviations from these arrangements will occasionally be made, which will be pointed out when they occur.

SECTION I.

ON LIGHT, CONSISTING OF VIBRATIONS PERPENDICULAR TO THE PLANE OF INCIDENCE.

We shall adopt the following process: first, deduce the equations of motion on the supposition that the force is insensible except at very small distances from its origin, and then take the law of force, varying inversely as the square of the distance. The object of the first process is the discovery of the *form* which the results assume, to serve as a guide to the more complex calculations of the second.

The two media will be supposed to be arranged in a perfectly symmetrical manner, so that all terms which involve the odd powers of the distances of the particles, will vanish when the sums of such terms are taken, extending throughout the whole mass.

1. The expression for the force on the particle P, resolved parallel to the axis of z, is $\Sigma\phi\ (r+\varrho)\ (\delta z+\delta\gamma)$.

$$=\Sigma(\phi r+\phi' r\,.\,\varrho)\ (\delta z+\delta\gamma)$$

$$=\Sigma(\phi r+\frac{\phi' r}{r}\,\delta z\,\delta\gamma)\ (\delta z+\delta\gamma)$$

$$=\Sigma\{\phi r\,.\,\delta z+\frac{\phi' r}{r}\,\delta z^2\,\delta\gamma+\phi r\,.\,\delta\gamma\}$$

Now $$\delta\gamma = \frac{d\gamma}{dx}\delta x + \frac{d^2\gamma}{dx^2}\frac{\delta x^2}{2} + \frac{d^2\gamma}{dy^2}\frac{\delta y^2}{2} + \ldots$$

δx being supposed small for those limits to which the force produces a sensible effect. We conclude that

$$\frac{d^2\gamma}{dt^2} = \Sigma\left\{\phi r\delta z + \left(\phi r + \frac{\phi' r}{r}\delta z^2\right)\cdot\left(\frac{d\gamma}{dx}\delta x + \frac{1}{2}\frac{d^2\gamma}{dx^2}\delta x^2\right)\right\} + \text{etc.}$$

$$= \Sigma\left\{\phi r + \frac{\phi' r}{r}\delta z^2\right\}\left\{\frac{d\gamma}{dx}\delta x + \frac{d^2\gamma}{dx^2}\frac{\delta x^2}{2} + \frac{d^2\gamma}{dy^2}\frac{\delta y^2}{2}\right\} \text{........(1)}$$

$$\frac{d^2\gamma_{,}}{dt^2} = \Sigma\left\{\phi r_{,} + \frac{\phi' r_{,}}{r_{,}}\delta z_{,}^2\right\}\left\{\frac{d\gamma_{,}}{dx_{,}}\delta x_{,} + \frac{d^2\gamma_{,}}{dx_{,}^2}\frac{\delta x_{,}^2}{2} + \frac{d^2\gamma_{,}}{dy_{,}^2}\frac{\delta y^2}{2}\right\} \text{...(2)}$$

the symbol Σ being supposed to extend to the sensible limits of the force; but the particle in each case not near the confines of the medium.

It is clear that the term involving $\frac{d\gamma}{dx}\delta x$ vanishes, and the equations are thereby simplified.

By the same process, we may obtain the equation of motion of a particle in the upper medium, near the confines, to be the following:

$$\frac{d^2\gamma}{dt^2} = \Sigma\left(\phi r + \frac{\phi' r}{r}\delta z^2\right)\left(\frac{d\gamma}{dx}\delta x + \frac{d^2\gamma}{dx^2}\cdot\frac{\delta x^2}{2} + \frac{d^2\gamma}{dy^2}\frac{\delta y^2}{2}\right)$$

$$+ \Sigma\left(\phi r' + \frac{\phi' r'}{r'}\delta z'^2\right)\left(\frac{d\gamma_{,}}{dx_{,}}\delta x' + \frac{d^2\gamma_{,}}{dx_{,}^2}\frac{\delta x'^2}{2} + \right) \text{............(3)}$$

the symbol now extending indefinitely on one side, but being bounded by the common surface on the other.

Also, $$\frac{d^2\gamma_{,}}{dt^2} = \Sigma\left(\phi r_{,} + \frac{\phi' r_{,}}{r_{,}}\delta z_{,}^2\right)\left(\frac{d\gamma_{,}}{dx_{,}}\delta x_{,} + \frac{d^2\gamma_{,}}{dx_{,}^2}\frac{\delta x_{,}^2}{2} + \frac{d^2\gamma_{,}}{dy_{,}^2}\frac{\delta y_{,}^2}{2}\right)$$

$$+ \Sigma\left(\phi r' + \frac{\phi' r'}{r'}\delta z'^2\right)\left(\frac{d\gamma}{dx}\delta x' + \frac{d^2\gamma}{dx^2}\frac{\delta x'^2}{2} + \right) \text{............(4)}$$

If the particles last considered coincide in the bounding surface, the results will be simplified, as we shall see presently.

Let $\Sigma\left(\phi r + \frac{\phi' r}{r}\delta z^2\right)\frac{\delta x^2}{2}$ in equation (1) be designated by n^2, it is clear that

$$\Sigma\left(\phi r + \frac{\phi' r}{r}\delta z^2\right)\frac{\delta y^2}{2} \text{ is also equal to } n^2.$$

Denote $\Sigma\left(\phi r_{,} + \frac{\phi' r_{,}}{r_{,}}\delta z_{,}^2\right)\frac{\delta x_{,}^2}{2}$ the expression in equation (2) by $n_{,}^2$

$\Sigma\left(\phi r + \frac{\phi' r}{r}\delta z^2\right)\delta x$ in (3) by P

$\Sigma\left(\phi r' + \frac{\phi' r'}{r'}\delta z'\right)\delta x' = \Sigma\left(\phi r_{,} + \frac{\phi' r_{,}}{r_{,}}\delta z_{,}^2\right)\delta x_{,}$ in equation (3) by $P_{,}$

then the four equations in order become

$$\frac{d^2\gamma}{dt^2}=n^2\left(\frac{d^2\gamma}{dx^2}+\frac{d^2\gamma}{dy^2}\right)$$

$$\frac{d^2\gamma_{,}}{dt^2}=n_{,}^2\left(\frac{d^2\gamma}{dx_{,}^2}+\frac{d^2\gamma_{,}}{dy_{,}^2}\right)$$

$$\frac{d^2\gamma}{dt^2}=\frac{n^2}{2}\left(\frac{d^2\gamma}{dx^2}+\frac{d^2\gamma}{dy^2}\right)+\frac{n_{,}^2}{2}\left(\frac{d^2\gamma_{,}}{dx_{,}^2}+\frac{d^2\gamma_{,}}{dy_{,}^2}\right)+\mathrm{P}\frac{d\gamma}{dx}+\mathrm{P}_{,}\frac{d\gamma_{,}}{dx_{,}}$$

$$\frac{d^2\gamma_{,}}{dt^2}=\frac{n^2}{2}\left(\frac{d^2\gamma}{dx^2}+\frac{d^2\gamma}{dy^2}\right)+\frac{n_{,}^2}{2}\left(\frac{d^2\gamma_{,}}{dx_{,}^2}+\frac{d^2\gamma_{,}}{dy_{,}^2}\right)+\mathrm{P}\frac{d\gamma}{dx}+\mathrm{P}_{,}\frac{d\gamma_{,}}{dx_{,}}$$

that last two equations requiring that x have the value o written for it.

2. Since a particle at the confines of the medium must be so acted on that it is in equilibrium when $\gamma=0$, it is easy to perceive that

$$\Sigma\phi r\,\delta x=-\Sigma\phi r_{,}\,\delta x_{,}\text{ taken as in equation (3).}$$

This must of course arise from the variation of density near the common surface. On examining the expression, it will appear that, when expressed in language, it is equivalent to the equalization of the sum of a series of terms of different values, but of given dimensions. Now $\frac{\phi' r}{r}\,\delta z^2\,\delta x$ is of the same dimensions as the above term; hence we should expect that

$$\Sigma\frac{\phi' r}{r}\,\delta z^2\,\delta x=-\Sigma\frac{\phi' r_{,}}{r_{,}}\,\delta z_{,}^2\,\delta x_{,}$$

and . . $\mathrm{P}=-\mathrm{P}_{,}$

3. The solution of equation (1) is

$$\gamma=f(ax+by+ct)+\mathrm{F}(-ax+by+ct)$$

the function f corresponding to the incident, and F to the reflected wave; that of (2) is

$$\gamma_{,}=f_{,}(a_{,}x_{,}+by+ct)$$
$$=f_{,}(a_{,}x+by+ct)$$

by writing x as the general symbol. Now we suppose the wave motion to continue unbroken, so that the equations (3) and (4) give the same results respectively as (1) and (2).

If, then, we substitute the results already obtained, we shall satisfy the two equations (3) and (4).

$$c^2\{f''(by+ct)+\mathrm{F}''(by+ct)\}=$$

$$\frac{n^2}{2}(a^2+b^2)\{f''(by+ct)+\mathrm{F}''(by+ct)\}+\frac{n_{,}^2}{2}(a_{,}^2+b^2)\{f_{,}''(by+ct)\}$$

$$+\mathrm{P}\{af'(by+ct)-a\mathrm{F}'(by+ct)-a_{,}f_{,}'(by+ct)\}$$

And the right hand side of equation (4) is the same as this. Let us now write f for $f(by+ct)$ and so on, then taking notice that by (1) and (2)

$$c^2 = n^2 (a^2 + b^2)$$
$$= n_{,}^2 (a_{,}^2 + b^2)$$

we reduce the two equations to these

$$f'' + F'' = f_{,}''$$
$$= \frac{1}{2}(f'' + F'' + f_{,}'') + \frac{P}{c^2}\{af' - aF' - a_{,}f_{,}'\}$$

or
$$f'' + F'' - f_{,}'' = \frac{2P}{c^2}(af' - aF' - a_{,}f_{,}')$$

$$f_{,}'' - F'' - f'' = \frac{2P}{c^2}(af' - aF' - a_{,}f_{,}')$$

whence
$$f'' + F'' - f_{,}'' = 0$$
$$af' - aF' - a_{,}f_{,}' = 0$$

which equations give

$$F'' = \frac{a - a_{,}}{a + a_{,}} \cdot f''$$

$$f_{,}'' = \frac{2a}{a + a_{,}} \cdot f''$$

by differentiating the second and eliminating successively $f_{,}''$ and F''.

4. Now if ϕ be the angle of incidence,

$\phi_{,}$ that of refraction,

$r = x\cos\phi + y\sin\phi$ is the space described in a given time without the medium,

$r_{,} = x_{,}\cos\phi_{,} + y_{,}\sin\phi_{,}$ within;

and if $\lambda, \frac{\lambda}{\mu}\left(=\frac{\lambda}{\sin\phi}\sin\phi_{,}\right)$ be the lengths of the waves respectively,

$$a = l \cdot \frac{\cos\phi}{\lambda} \qquad b = l \cdot \frac{\sin\phi}{\lambda}$$

$$a_{,} = l \cdot \frac{\cos\phi_{,}}{\lambda} \frac{\sin\phi}{\sin\phi_{,}} = \frac{l}{\lambda} \cdot \frac{\sin\phi\cos\phi_{,}}{\sin\phi_{,}}$$

$$b = l \cdot \frac{\sin\phi_{,}}{\lambda\sin\phi_{,}}\sin\phi = \frac{l\sin\phi}{\lambda}$$

$$F'' = \frac{\sin\phi\cos\phi_{,} - \sin\phi_{,}\cos\phi}{\sin\phi\cos\phi_{,} + \sin\phi_{,}\cos\phi} \cdot f''$$

$$= \frac{\sin(\phi - \phi_{,})}{\sin(\phi + \phi_{,})} f''$$

$$f_{,}'' = \frac{2\cos\phi\sin\phi_{,}}{\sin(\phi + \phi_{,})} \cdot f''$$

The notation F, &c. is the same as that used by Mr Green, and the present page is added merely to make the subject complete.

5. These are the results deduced, in a manner apparently widely different, by M. Fresnel. That the results should coincide is not by any means a matter

of surprise, even supposing in both cases the argument fallacious; for in all the ways of establishing them the same grand assumption extends throughout the whole, viz. that the particles at the common surface of the media have motions resulting from, and conversely affecting, the motion without the latter medium, and that these motions are regulated by the usual laws of the result of forces. Perhaps I shall be better understood if I illustrate my meaning by giving the following demonstration of the results in question.

A particle at the surface is acted on by three sets of forces, in the directions respectively of the directions of incidence, reflection, and refraction: not that the particle is urged in these directions, but is acted by a force which gives it a motion as much depending on the direction as though it were. We have then three forces acting on the particle, and any one may be considered as the resultant of the other two. If this be allowed, we know by the laws of mechanics, that each force is in the proportion of the sine of the angle contained by the other two.

Let then I R and T denote the incident reflected and transmitted vibration;

then
$$\frac{\mathrm{R}}{\mathrm{I}} = \frac{\sin \overline{\phi - \phi_{,}}}{\sin \overline{\phi + \phi_{,}}}$$

$$\frac{\mathrm{T}}{\mathrm{I}} = \frac{\sin 2\,\phi}{\sin \overline{\phi + \phi_{,}}}$$

$$= \frac{2 \sin \phi \cos \phi}{\sin \overline{\phi + \phi_{,}}}$$

$$= \frac{2\,\mu \sin \phi_{,} \cos \phi}{\sin \overline{\phi + \phi_{,}}}$$

μ being the refractive index.

But when the motion actually takes place within the medium, the length of the wave has to be diminished in the ratio $\frac{1}{\mu} : 1$; if, then, we conceive the new wave to remain similar to the old one, as we doubtless ought, we must diminish the vibration in the same ratio: hence the value of the vibration within the medium is

$$\frac{1}{\mu}\mathrm{T} = \frac{1}{\mu}\mathrm{I} \,.\, \frac{2 \sin \phi \cos \phi}{\sin \overline{\phi + \phi_{,}}}$$

$$= \mathrm{I} \,.\, \frac{2 \sin \phi_{,} \cos \phi}{\sin \overline{\phi + \phi_{,}}}$$

the same result as before.

This consideration, then, leads us to M. Fresnel's formulæ.

6. Next let us adopt the molecular hypothesis, without having recourse to the approximations mentioned in the introduction: then

$$\frac{d^2 \gamma}{d t^2} = \Sigma \left(\phi r + \frac{\phi' r}{r} \delta z^2\right) \delta \gamma$$

$$\frac{d^2 \gamma_{,}}{d t^2} = \Sigma \left(\phi r_{,} + \frac{\phi' r_{,}}{r_{,}} \delta z_{,}^2\right) \delta \gamma_{,}$$

and if we assume

$$\gamma = a\cos(ex+fy+ct) + b\cos(-ex+fy+ct+g)$$

$$\gamma_{,} = a_{,}\cos(e_{,}x_{,}+fy+ct+h)$$

then $$\delta\gamma = a\cos(ex+e\,\delta x+fy+ct+f\,\delta y) - a\cos\overline{ex+fy+ct}$$

$$+ b\cos(-ex-e\,\delta x+fy+ct+g+f\,\delta y) - b\cos(-ex+fy+ct+g)$$

$$= -a\cos(ex+fy+ct)\,(1-\cos\overline{e\,\delta x+f\,\delta y})$$

$$-a\sin(ex+fy+ct)\sin(e\,\delta x+f\,\delta y) + \text{etc.}$$

Let $ex+fy$ be abbreviated by ϱ

$ex-fy$ ...... R

$e_{,}x+fy$ ...... $\varrho_{,}$

and $$\delta\gamma = -a\cos\overline{\varrho+ct}\,(1-\cos\delta\varrho) - a\sin\overline{\varrho+ct}\sin\delta\varrho$$

$$-b\cos(-\mathrm{R}+ct+g)(1-\cos\delta\mathrm{R}) + b\sin(-\mathrm{R}+ct+g)\sin\delta\mathrm{R}$$

$$= -\mathrm{I}\,.\,2\sin^2\frac{\delta\varrho}{2} - \mathrm{R}\,2\sin^2\frac{\delta\mathrm{R}}{2}$$

$$+\frac{1}{e}\frac{d\mathrm{I}}{dx}\sin\delta\varrho + \frac{1}{e}\frac{d\mathrm{R}}{dx}\sin\delta\mathrm{R}$$

denoting γ by $\mathrm{I}+\mathrm{R}$.

Now the wave is similarly situated with respect to the line along which R is measured, and that along which ϱ is measured; hence

$$\Sigma\left(\phi r + \frac{\phi' r}{r}\delta z^2\right)\sin^2\frac{\delta\varrho}{2} = \Sigma\left(\phi r + \frac{\phi' r}{r}\,\delta z^2\right)\sin^2\frac{\delta\mathrm{R}}{2}$$

which gives each of them $=\dfrac{c^2}{2}$.

For $$\frac{d^2\gamma}{dt^2} = -\Sigma\left(\phi r + \frac{\phi' r}{r}\delta z^2\right)2\sin^2\frac{\delta\varrho}{2}\,.\,\gamma$$

&c. = &c.

$$\therefore c^2 = \Sigma\left(\phi r + \frac{\phi' r}{r}\delta z^2\right)2\sin^2\frac{e\,\delta x \mp f\,\delta y}{2}\,.$$

$$= \Sigma\left(\phi r_{,} + \frac{\phi' r_{,}}{r_{,}}\,\delta z_{,}^2\right)2\sin^2\frac{e_{,}\delta x_{,} + f\,\delta y_{,}}{2}$$

The equation corresponding to (3) is

$$\frac{d^2\gamma}{dt^2} = \Sigma\left(\phi r + \frac{\phi' r}{r}\,\delta z^2\right)\delta\gamma + \Sigma\left(\phi r' + \frac{\phi' r'}{r'}\,\delta z^2\right)\left(\gamma_{,\,x_{,}+\delta x_{,}\,,\,y_{,}+\delta y_{,}} - \gamma\right)$$

and $$\gamma_{,\,x_{,}+\delta x_{,}\,,\,y_{,}+\delta y_{,}} - \gamma = a_{,}\cos(e_{,}x + e_{,}\delta x + fy + f\,\delta y + ct + h)$$

$$-a\cos(ex+fy+ct) - b\cos(-ex+fy+ct+g)$$

$$= \gamma_{,}\cos\delta\varrho_{,} + \frac{1}{e_{,}}\frac{d\gamma_{,}}{dx}\sin\delta\varrho_{,} - \gamma$$

$$\frac{d^2\gamma}{dt^2} = -\Sigma\left(\phi r + \frac{\phi' r}{r}\delta z^2\right) 2\sin^2\frac{\delta\varrho}{2}\,.\,\gamma + \Sigma\left(\phi r + \frac{\phi' r}{r}\delta z^2\right)\left(\sin\delta\varrho\,\frac{1}{e}\frac{d\mathrm{I}}{dx} + \sin\delta\mathrm{R}\,\frac{1}{e}\frac{d\mathrm{R}}{dx}\right)$$

$$+\Sigma\left(\phi r' + \frac{\phi' r'}{r'}\delta z'^2\right)\left(1 - 2\sin^2\frac{\delta\varrho_{,}}{2}\right)\gamma_{,} + \Sigma\left(\phi r' + \frac{\phi' r'}{r'}\delta z'^2\right)\left(\sin\delta\varrho_{,}\frac{1}{e_{,}}\frac{d\gamma_{,}}{dx}\right) - \Sigma\left(\phi r' + \frac{\phi' r'}{r'}\delta z'^2\right).\gamma$$

$$= -\frac{c^2}{2}\gamma + \Sigma\left(\phi r + \frac{\phi' r}{r}\delta z^2\right)\sin\delta\varrho\,\frac{1}{e}\frac{d\gamma}{dx} - \frac{c^2}{2}\gamma_{,} + \Sigma\left(\phi r' + \frac{\phi' r'}{r'}\delta z'^2\right)\sin\delta\varrho_{,}\frac{1}{e_{,}}\frac{d\gamma_{,}}{dx} + \Sigma\left(\phi r' + \frac{\phi' r'}{r'}\delta z'^2\right)(\gamma_{,} -$$

Similarly for the lower medium

$$\frac{d^2\gamma_{,}}{dt^2} = -\frac{c^2}{2}\overline{\gamma + \gamma_{,}} + \Sigma\left(\phi r_{,} + \frac{\phi' r_{,}}{r_{,}}\delta z_{,}^2\right)\sin\delta\varrho_{,}\frac{1}{e_{,}}\frac{d\gamma_{,}}{dx}$$

$$+\Sigma\left(\phi r' + \frac{\phi' r'}{r'}\delta z'^2\right)\sin\delta\varrho\,\frac{1}{e}\frac{d\gamma}{dx} + \Sigma\left(\phi r + \frac{\phi' r}{r}\delta z^2\right)(\gamma - \gamma_{,})$$

7. By substituting for $\frac{d^2\gamma}{dt^2}$ and $\frac{d^2\gamma_{,}}{dt^2}$ their values, and calling

$$\Sigma\left(\phi r + \frac{\phi' r}{r}\delta z^2\right)\sin\delta\varrho = \mathrm{P}$$

$$\Sigma\left(\phi r_{,} + \frac{\phi' r_{,}}{r_{,}}\delta z_{,}^2\right)\sin\delta\varrho_{,} = \mathrm{P}_{,}$$

$$\Sigma\left(\phi r + \frac{\phi' r}{r}\delta z^2\right)\ldots\ldots = \mathrm{Q}$$

$$\Sigma\left(\phi r_{,} + \frac{\phi' r_{,}}{r_{,}}\delta z_{,}^2\right)\ldots\ldots = \mathrm{Q}_{,}$$

we get

$$-c^2\gamma = -\frac{c^2}{2}(\gamma + \gamma_{,}) + \frac{\mathrm{P}}{e}\frac{d\gamma}{dx} + \frac{\mathrm{P}_{,}}{e_{,}}\frac{d\gamma_{,}}{dx} + \mathrm{Q}_{,}(\gamma_{,} - \gamma)$$

$$-c^2\gamma_{,} = -\frac{c^2}{2}(\gamma + \gamma_{,}) + \frac{\mathrm{P}}{e}\frac{d\gamma}{dx} + \frac{\mathrm{P}_{,}}{e_{,}}\frac{d\gamma_{,}}{dx} + \mathrm{Q}(\gamma - \gamma_{,})$$

Or

$$0 = \frac{c^2}{2}(\gamma - \gamma_{,}) + \frac{\mathrm{P}}{e}\frac{d\gamma}{dx} + \frac{\mathrm{P}_{,}}{e_{,}}\frac{d\gamma_{,}}{dx} + \mathrm{Q}_{,}(\gamma_{,} - \gamma)$$

$$0 = \frac{c^2}{2}(\gamma_{,} - \gamma) + \frac{\mathrm{P}}{e}\frac{d\gamma}{dx} + \frac{\mathrm{P}_{,}}{e_{,}}\frac{d\gamma_{,}}{dx} + \mathrm{Q}(\gamma - \gamma_{,})$$

By means of these equations we obtain

$$c^2(\gamma - \gamma_{,}) - \overline{\mathrm{Q} + \mathrm{Q}_{,}}\,\overline{\gamma - \gamma_{,}} = 0$$

or

$$(c^2 - \mathrm{Q} - \mathrm{Q}_{,})(\gamma - \gamma_{,}) = 0\,;$$

And

$$\frac{\mathrm{P}}{e}\frac{d\gamma}{dx} + \frac{\mathrm{P}_{,}}{e_{,}}\frac{d\gamma_{,}}{dx} + \overline{\mathrm{Q} - \mathrm{Q}_{,}}\,(\gamma - \gamma_{,}) = 0$$

From the nature of the functions we cannot have $c^2 = \mathrm{Q} + \mathrm{Q}_{,}$,

$$\therefore\ \gamma - \gamma_{,} = 0$$

is the only mode of satisfying the first equation, and thus the second equation becomes

$$\frac{\mathrm{P}}{e}\frac{d\gamma}{dx}+\frac{\mathrm{P}_{,}}{e_{,}}\frac{d\gamma_{,}}{dx}=0 \;\ldots\ldots\; (\mathrm{A})$$

We need hardly repeat that the latter hypothesis, by which equations (3) and (4) are deduced and combined, is true only for the particular value $x = 0$.

Now, even without retaining the restrictions imposed on the functions in art. 2, we may shew, by the reasoning used there, that

$$\frac{\mathrm{P}}{e}+\frac{\mathrm{P}_{,}}{e_{,}}=0$$

$$\therefore \frac{d\gamma}{dx}=\frac{d\gamma_{,}}{dx}$$

two conditions which completely satisfy the equation (A)

Hence the final result is, that when $x = 0$

$$\gamma=\gamma_{,}$$

$$\frac{d\gamma}{dx}=\frac{d\gamma_{,}}{dx}$$

and the general values of γ are already determined.

Thus the results above obtained, approximately in the case of particles whose action is insensible at sensible distances, is proved true, *without any approximations,* by the reasoning we have employed.

SECTION II.

ON LIGHT CONSISTING OF VIBRATIONS IN THE PLANE OF INCIDENCE.

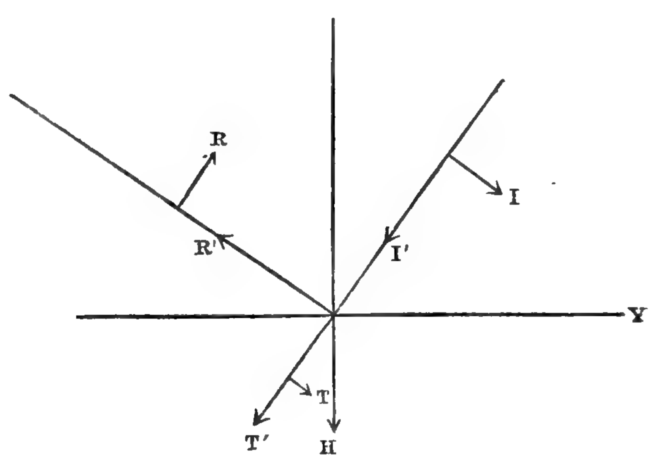
8. We shall assume that it has been demonstrated that light cannot consist of vibrations partly transversal, partly normal, and shall consequently distinguish strictly between a *motion* in the direction of transmission, and a *vibration* in that direction.

At a distance from any break in the state of the molecules, one function will be sufficient to represent the motion of a particle, since any motion not belonging to the type of that function will be transmitted independently of it, and unaffected by it, on the principle of the coexistence of vibrations. When, on the other hand, the state of the particles in the immediate neighbourhood of that under consideration is discontinuous, we cannot assume that a state of motion represented by one type will have no influence on that which is represented by another. On the contrary, we should expect, from the ordinary laws of fluids, that the particular type of the wave itself should undergo a considerable change, and possibly anticipate a reversion of some of the previous axioms by which our calculations were guided. It will be necessary then to retain every term which enters into our expressions, except those only which disappear of themselves by the conditions of symmetry.

Now, before we proceed to apply analogous reasoning to the case of waves whose vibrations take place in the plane of $x\,y$, it must be remarked, that, when the motion arrives at the surface, a sudden change takes place. But this sudden change, which occurs necessarily at the first instant the light falls on the surface, will in all the future part of the motion materially affect, not only the vibrations beyond the surface, but those also above it; and the change which takes place is nothing else than that of twisting a vibration which previously had been perpendicular to the direction of motion, so as to cause it no longer to be so. Now, I have shewn in the Transactions of the Cambridge Philosophical Society, vol. vi. p. 180, that a motion perpendicular to the front of the wave cannot be transmitted as a vibration along with the wave. The assumption that it can be so transmitted gives rise to the result that the velocity is an impossible quantity. In other words, some part of the expression which we assumed to be a function of sines and cosines depends on *possible*, as sines and cosines do on impossible, exponentials.

To apply this conclusion to the case in question, we must observe, that, if we reckon along the axis of y, whatever be the motion in question, its value must be a reciprocating one; and further, it is necessary that, whatever value it has for one value of y at a particular time, the same will it have for another value of y at some other time: hence the function which expresses the motion must be a circular function of y and t, but a possible exponential function of x. The motion thus introduced will consequently be a vibration transmitted along the axis of y, and consequently the direction of motion is parallel to the axis of x.

We proceed then to deduce the equations of motion of a particle situated near the common surface of the media, on the hypothesis that the light consists of vibrations in the plane of incidence. As a preliminary step, partly for the purpose of exhibiting the correctness of the method employed, I have deduced the equations of motion of a particle situated at such a distance from the surface that the vibrations transmitted along the axis of x do not affect the forces. Afterwards I have deduced the general equations corresponding to a particle situated at the common surface.



9. We adopt the following notation in addition to that already used:

α, β are the motions parallel to x and y of a particle in the upper medium.
$\alpha_{,} \beta_{,}$ do. do. in the lower.
I, R, T are the incident reflected and refracted vibrations.
$I_{,}$ and $T_{,}$ the corresponding normal motions.

Occasionally δx and δy will be replaced by

$$\delta x' \cos \phi + \delta y' \sin \phi$$
$$\delta y' \cos \phi - \delta x' \sin \phi$$

respectively, when combined with a function depending on the incident wave, and by

$$\delta x'' \cos \phi + \delta y'' \sin \phi$$
$$\delta x'' \sin \phi - \delta y'' \cos \phi$$

when combined with one which depends on the reflected wave.

From the values of δx and δy, it is clear that the axis of x' is the line of transmission at incidence, and that of x'' at reflexion. The *values* of I, R, are in general not required, but for the purpose of fixing the ideas, they may be conceived to be as follows:

$$\begin{aligned}
\mathrm{I} &= a \cos (e x + f y + c t) \\
\mathrm{R} &= b \cos (-e x + f y + c t + g) \\
\mathrm{T} &= c \cos (e_{,} x + f y + c t + h) \\
\mathrm{I}_{,} &= \mathrm{A}\, e^{-m x} \cos (f y + c t + \eta) \\
\mathrm{T}_{,} &= \mathrm{C}\, e^{-m_{,} x} \cos (f y + c t + h + \eta_{,})
\end{aligned}$$

If it should be thought that these values belong only to a particular case, I would remark that, from the linearity of our equations, the results which we deduce for one circular function, are equally true, *mutatis mutandis*, of a series of such functions.

10. The values of $\alpha, \beta, \alpha_{,}, \beta_{,}$, deduced from the figure, are:

$$\begin{aligned}
\alpha &= \overline{\mathrm{I} - \mathrm{R}} \sin \phi + \mathrm{I}_{,} \\
\beta &= (\mathrm{I} + \mathrm{R}) \cos \phi \\
\alpha_{,} &= \mathrm{T} \sin \phi' + \mathrm{T}_{,} \\
\beta_{,} &= \mathrm{T} \cos \phi'
\end{aligned}$$

The equations of motion in the upper medium are:

$$\begin{aligned}
\frac{d^2 \alpha}{d t^2} &= \Sigma \left\{ \phi r + \frac{\phi' r}{r} (\delta x\, \delta \alpha + \delta y\, \delta \beta) \right\} \overline{\delta x + \delta \alpha} \\
&= \Sigma \left(\phi r + \frac{\phi' r}{r} \delta x^2 \right) \delta \alpha + \Sigma \frac{\phi' r}{r} \delta x\, \delta y\, \delta \beta \\
\frac{d^2 \beta}{d t^2} &= \Sigma \left(\phi r + \frac{\phi' r}{r} \delta y^2 \right) \delta \beta + \Sigma \frac{\phi' r}{r} \delta x\, \delta y\, \delta \alpha
\end{aligned}$$

The values thus substituted will, of course, have to be replaced by others, when the particle under consideration is near the common surface of the media. And

$$\delta\alpha=(\delta I-\delta R)\sin\phi+\delta I_{,}$$

but if we adopt the particular values of $I, R,$ &c. which we may do since all values have the same *form*, we have the following results :

$$\delta I = a\cos(ex+fy+ct+e\delta x+f\delta y)-a\cos\overline{ex+fy+ct}$$

$$= -a\cos(ex+fy+ct)\,2\sin^2\left(\frac{e\delta x+f\delta y}{2}\right)-a\sin(ex+fy+ct)\sin(e\delta x+f\delta y)$$

$$= -2I\,.\sin^2\frac{Kx'}{2}+\frac{1}{e}\frac{dI}{da}\sin Kx'$$

if we denote $e\delta x+f\delta y$ by Kx' instead of $k\varrho$.

Let us in like manner assume

$$e\delta x-f\delta y=Kx''$$
$$e_{,}\delta x+f\delta y=Kx_{,}$$

then
$$\delta R=-2R\sin^2\frac{Kx''}{2}+\frac{1}{e}\frac{dR}{dx}\sin Kx''$$

$$\delta T=-2T\sin^2\frac{Kx_{,}}{2}+\frac{1}{e_{,}}\frac{dT}{dx}\sin Kx_{,}$$

$$\delta I_{,}=Ae^{-mx-m\delta x}\cos(fy+ct+\eta+f\delta y)-Ae^{-mx}\cos\overline{fy+ct+\eta}$$

$$=Ae^{-mx}\left\{(e^{-m\delta x}\cos f\delta y-1)\cos\overline{fy+ct+\eta}-e^{-m\delta x}\sin f\delta y\sin\overline{fy+ct+\eta}\right\}$$

$$=-I_{,}(1-e^{-m\delta x}\cos f\delta y)+\frac{1}{f}\frac{dI_{,}}{dy}e^{-m\delta x}\sin f\delta y$$

$$\delta T_{,}=-T_{,}(1-e^{-m_{,}\delta x}\cos f\delta y)+\frac{1}{f}\frac{dT_{,}}{dy}e^{-m_{,}\delta x}\sin f\delta y$$

11. Now for a particle at a distance from the common surface $\delta I_{,}$ and $\delta R_{,}$ vanish ; in such cases the values of $\delta\alpha$ and $\delta\beta$ are

$$\delta\alpha=\sin\phi\left\{-2I\sin^2\frac{Kx'}{2}+\frac{1}{e}\frac{dI}{dx}\sin Kx'+2R\sin^2\frac{Kx''}{2}-\frac{1}{e}\frac{dR}{dx}\sin Kx''\right\}$$

$$\delta\beta=\cos\phi\left\{-2I\sin^2\frac{Kx'}{2}+\frac{1}{e}\frac{dI}{dx}\sin Kx'-2R\sin^2\frac{Kx''}{2}+\frac{1}{e}\frac{dR}{dx}\sin Kx''\right\}$$

By substituting these values in the equations in art. 10, and at once omitting terms of the form $\Sigma M\delta x\delta y$, we obtain

$$\frac{d^2\alpha}{dt^2}=\Sigma\left\{\phi r+\frac{\phi' r}{r}\overline{\delta x'^2\cos^2\phi+\delta y'^2\sin^2\phi}\right\}\times\left\{\sin\phi\left(-2I\sin^2\frac{Kx'}{2}\right)\right\}$$

$$+\Sigma\left\{\phi r+\frac{\phi' r}{r}\overline{\delta x''^2\cos^2\phi+\delta y''^2\sin^2\phi}\right\}\left\{\sin\phi\,2R\sin^2\frac{Kx''}{2}\right\}$$

$$+\Sigma\frac{\phi' r}{r}(\delta y'^2-\delta x'^2)\sin\phi\cos\phi\left(\cos\phi.\overline{-2\,\mathrm{I}\sin^2\frac{\mathrm{K}x'}{2}}\right)$$

$$-\Sigma\frac{\phi' r}{r}(\delta y''^2-\delta x''^2)\sin\phi\cos\phi\left(-2\,\mathrm{R}\cos\phi\sin^2\frac{\mathrm{K}x''}{2}\right)$$

Now if the law of force be that of the inverse square of the distance,

$$\phi\,r=\frac{\mathrm{S}}{r^3}$$

$$\frac{\phi' r}{r}=-\frac{3\,\mathrm{S}}{r^5}$$

$$\therefore\ \phi\,r+\frac{\phi' r}{r}\delta x'^2=\mathrm{S}.\frac{\delta x'^2+\delta y'^2+\delta z'^2-3\,\delta x'^2}{r^5}$$

And $\quad \Sigma\left(\phi r+\frac{\phi' r}{r}\delta y'^2\right)2\sin^2\frac{\mathrm{K}x'}{2}\quad$ has been already designated c^2

$$\therefore\ c^2=2\,\mathrm{S}\,\Sigma\,\frac{\delta x'^2+\delta z'^2-2\,\delta y'^2}{r^5}\sin^2\frac{\mathrm{K}x'}{2}$$

but

$$\Sigma\sin^2\frac{\mathrm{K}x'}{2}\ \frac{(\delta y'^2-\delta z'^2)}{r^5}=0$$

$$\therefore\ c^2=2\,\mathrm{S}\,\Sigma.\frac{\delta x'^2-\delta y'^2}{r^5}\sin^2\frac{\mathrm{K}x'}{2}$$

$$=2\,\mathrm{S}\,\Sigma.\frac{\delta x''^2-\delta y''^2}{r^5}\sin^2\frac{\mathrm{K}x''}{2}$$

and

$$\Sigma\left(\phi r+\frac{\phi' r}{r}\delta x'^2\right)\sin^2\frac{\mathrm{K}x'}{2}=2\,\mathrm{S}\,\Sigma\frac{2(\delta y'^2-\delta x'^2)}{r^5}\sin^2\frac{\mathrm{K}x'}{2}$$

$$=-2\,c^2$$

hence by substitution we obtain

$$\frac{d^2\alpha}{dt^2}=-c^2\sin\phi\,\mathrm{I}\,(\sin^2\phi-2\cos^2\phi)+c^2\sin\phi\,\mathrm{R}\,(\sin^2\phi-2\cos^2\phi)-3c^2\sin\phi\cos^2\phi\,\mathrm{I}+3\,c^2\sin\phi\cos^2\phi\,\mathrm{R}$$

$$=-c^2\sin\phi\,\mathrm{I}+c^2\sin\phi\,.\,\mathrm{R}$$

$$=-c^2(\mathrm{I}-\mathrm{R})\sin\phi$$

$$=-c^2\,\alpha$$

$$\frac{d^2\beta}{dt^2}=-c^2\,\beta$$

12. This result is obviously correct, and hence we may with confidence apply the same process to the more complicated case, that in which the quantities $\mathrm{I}_{,}$ and $\mathrm{R}_{,}$ appear, and for which the equations of motion must be found, by taking into account the forces which result from particles on both sides of the surface.

As a preliminary step, we will write down the values of $\delta\alpha$, $\delta\beta$, $\delta\alpha_{,}$ and $\delta\beta_{,}$ They are

$$\delta \alpha = (\delta I - \delta R) \sin \phi + \delta I_,$$

$$= -2 I \sin \phi \sin^2 \frac{K x'}{2} + 2 R \sin \phi \sin^2 \frac{K x''}{2} + \frac{d I}{d x} \cdot \frac{1}{e} \sin \phi \sin K x' - \frac{d R}{d x} \cdot \frac{1}{e} \sin \phi \sin K x''$$

$$- I_, (1 - e^{-m \delta x} \cos f \delta y) + \frac{d I_,}{d y} \cdot \frac{1}{f} e^{-m \delta x} \sin f \delta y$$

$$\delta \beta = (\delta I + \delta R) \cos \phi$$

$$= \cos \phi \left\{ -2 I \sin^2 \frac{K x'}{2} + \frac{1}{e} \frac{d I}{d x} \sin K x' - 2 R \sin^2 \frac{K x''}{2} + \frac{1}{e} \frac{d R}{d x} \sin K x'' \right\}$$

$$\delta \alpha_, = \delta T \sin \phi' + \delta T_,$$

$$= \sin \phi' \{ c \cdot \cos (\overline{e_, x_, + \delta x_, + f y_, + \delta y_, + c t + g}) - c \cos \overline{e_, x_, + f y_, + c t + g} \}$$

$$+ C e^{-\overline{m_, x_, + \delta x_,}} \cos (\overline{f y_, + \delta y_, + c t + h + \eta_,}) - C e^{-m_, x_,} \cos \overline{f_, y_, + c t + h + \eta_,}$$

$$= -2 T \sin \phi' \sin^2 \frac{K x''}{2} + \frac{d T}{d x} \frac{1}{e} \sin \phi' \sin K x'' - T_, (1 - e^{-m_, \delta x} \cos f d y) + \frac{d T_,}{d y} \frac{1}{f} e^{-m_, \delta x} \sin f d y$$

$$\delta \beta_, = \delta T \cos \phi'$$

$$= -2 T \cos \phi' \sin^2 \frac{e_, \delta x'}{2} + \frac{1}{e_,} \frac{d T}{d x} \cos \phi' \sin e_, \delta x$$

$$= -2 T \cos \phi' \sin^2 \frac{K x''}{2} + \frac{1}{e_,} \frac{d T}{d x} \cos \phi' \sin K x''$$

13. The equation of motion in the upper surface parallel to the axis of x is

$$\frac{d^2 \alpha}{d t^2} = \Sigma \left(\phi r + \frac{\phi' r}{r} \delta x^2 \right) \delta \alpha + \Sigma \frac{\phi' r}{r} \delta x \, \delta y \, \delta \beta$$

$$+ \Sigma \left(\phi r' + \frac{\phi' r'}{r'} \delta x'^2 \right) (\alpha_, - \alpha) + \Sigma \frac{\phi' r'}{r'} \delta x' \delta y' (\beta_, - \beta)$$

retaining the limitations to Σ.

But $\quad \Sigma \left(\phi r + \frac{\phi' r}{r} \delta x^2 \right) \delta \alpha =$

$$\Sigma \left(\phi r + \frac{\phi' r}{r} \overline{\delta x'^2 \cos^2 \phi + \delta y'^2 \sin^2 \phi} \right) \sin \phi \times \left\{ -2 I \sin^2 \frac{K x'}{2} + \frac{1}{e} \frac{d I}{d x} \sin K x' \right\}$$

$$+ \Sigma \left(\phi r + \frac{\phi' r}{r} \overline{\delta x''^2 \cos^2 \phi + \delta y''^2 \sin^2 \phi} \right) \sin \phi \times \left\{ 2 R \sin^2 \frac{K x''}{2} - \frac{1}{e} \frac{d R}{d x} \sin K x'' \right\}$$

$$+ \Sigma \left(\phi r + \frac{\phi' r}{r} \delta x^2 \right) \times \left\{ - I_, . (1 - e^{-m \delta x} \cos f \delta y) \right\}$$

$$= -\frac{c^2}{2} \sin \phi (\sin^2 \phi - 2 \cos^2 \phi) (I - R)$$

$$+ \frac{M}{e} \sin \phi (\sin^2 \phi - 2 \cos^2 \phi) \left(\frac{d I}{d x} - \frac{d R}{d x} \right) - D I_,$$

where $$M = \Sigma \left(\phi r + \frac{\phi' r}{r}\,\delta y^2\right)\,\sin K x'$$

$$D = \Sigma \left(\phi r + \frac{\phi' r}{r}\,\delta x^2\right)\,\left(1 - e^{-m\,\delta x}\cos f \delta y\right).$$

$$\Sigma \frac{\phi' r}{r}\,\delta x\,\delta y\,\delta\beta = \Sigma \frac{\phi' r}{r}(\delta y'^2 - \delta x'^2)\sin\phi\cos^2\phi \times \left\{-2\,I\sin^2\frac{K x'}{2} + \frac{1}{e}\frac{d\,I}{d\,x}\sin K x'\right\}$$

$$- \Sigma \frac{\phi' r}{r}\cdot(\delta y''^2 - \delta x''^2)\sin\phi\cos^2\phi \times \left\{-2\,R\sin^2\frac{K x''}{2} + \frac{1}{e}\frac{d\,R}{d\,x}\sin K x''\right\}$$

$$= -\frac{3\,c^2}{2}\sin\phi\cos^2\phi\,(I - R) + 3\frac{M}{e}\sin\phi\cos^2\phi\,\left(\frac{d\,I}{d x} - \frac{d\,R}{d\,x}\right)$$

By adding this term to that which we have just found, the sum is

$$-\frac{c^2}{2}\sin\phi\,(I - R) + \frac{M}{e}\sin\phi\left(\frac{d\,I}{d x} - \frac{d\,R}{d\,x}\right) - D\,I_{,}\,.$$

$$\Sigma\left(\phi r' + \frac{\phi' r'}{r'}\,\delta x'^2\right)(\alpha'_{,} - \alpha) =$$

$$\Sigma\left(\phi r' + \frac{\phi' r'}{r'}\,\delta x'^2\right)\left\{\overline{T\sin\phi' + T_{,}}_{\,x+\delta x,\,y+\delta y} - \overline{I - R}\sin\phi - I_{,}\right\}$$

$$= \Sigma\left(\phi r' + \frac{\phi' r'}{r'}\,\delta x'^2\right)\left\{\delta\,T\sin\phi' + \delta\,T_{,} + T\sin\phi' + T_{,} - \overline{I - R}\sin\phi - I_{,}\right\}$$

$$= \Sigma\left(\phi r' + \frac{\phi' r'}{r'}\,\delta x'^2\right)(\alpha_{,} - \alpha)$$

$$+ \Sigma\left(\phi r' + \frac{\phi' r'}{r'}(\delta\,x_{,}^2\cos\phi' + \delta\,y_{,}^2\sin^2\phi')\right)\sin\phi' \times \left\{-2\,T\sin^2\frac{K\,x_{,}}{2} + \frac{1}{e_{,}}\frac{d\,T}{d\,x}\sin K\,x_{,}\right\}$$

$$+ \Sigma\left(\phi r' + \frac{\phi' r'}{r'}\,\delta x'^2\right)\left\{-T_{,}(1 - e^{-m_{,}\delta x}\cos f\,\delta y) + \frac{1}{f}\frac{d\,T_{,}}{d y}\,e^{-m_{,}\delta x}\sin f\,d\,y\right\}$$

$$= Q_{,}(\alpha_{,} - \alpha) - \frac{c^2}{2}\,T\sin\phi'(\sin^2\phi' - 2\cos^2\phi') + \frac{M_{,}}{e_{,}}\frac{d\,T}{d\,x}\sin\phi'(\sin^2\phi' - 2\cos^2\phi') - D_{,}\,T_{,}$$

$$\Sigma \frac{\phi' r'}{r'} \delta x' \delta y' (\beta'_{,} - \beta) = \Sigma \frac{\phi' r'}{r'} \delta x' \delta y' \{\delta \beta_{,} + \beta_{,} - \beta\} = \Sigma \frac{\phi' r'}{r'} \delta x' \delta y' \delta \beta_{,}$$

$$= \Sigma \frac{\phi' r'}{r'} \delta x' \delta y' \delta \mathrm{T} \cos \phi'$$

$$= \Sigma \frac{\phi' r'}{r'} (\delta y_{,}'^2 - \delta x_{,}'^2) \sin \phi' \cos^2 \phi' \times \left\{ -2 \mathrm{T} \sin^2 \frac{\mathrm{K} x_{,}}{2} + \frac{1}{e_{,}} \frac{d \mathrm{T}}{d x} \sin \mathrm{K} x_{,} \right\}$$

$$= -\frac{3 c^2}{2} \mathrm{T} \sin \phi' \cos^2 \phi' + \frac{3 \mathrm{M}_{,}}{e_{,}} \frac{d \mathrm{T}}{d x} \sin \phi' \cos^2 \phi'$$

By adding this term to that just found, we get

$$\mathrm{Q}_{,} (\alpha_{,} - \alpha) - \frac{c^2}{2} \mathrm{T} \sin \phi' + \frac{\mathrm{M}_{,}}{e_{,}} \frac{d \mathrm{T}}{d x} \sin \phi' - \mathrm{D}_{,} \mathrm{T}_{,}$$

Hence

$$\frac{d^2 \alpha}{d t^2} = -\frac{c^2}{2} \left\{ \overline{\mathrm{I} - \mathrm{R}} \sin \phi + \mathrm{T} \sin \phi' \right\} + \mathrm{Q}_{,} (\alpha_{,} - \alpha) + \frac{\mathrm{M}}{e} \sin \phi \left(\frac{d \mathrm{I}}{d x} - \frac{d \mathrm{R}}{d x} \right) + \frac{\mathrm{M}_{,}}{e_{,}} \sin \phi' \frac{d \mathrm{T}}{d x} - \mathrm{D} \mathrm{I}_{,} - \mathrm{D}_{,} \mathrm{T}_{,}$$

14. From this equation we obtain, by interchanging the quantities $(\mathrm{I} - \mathrm{R}) \sin \phi$, $\mathrm{T} \sin \phi'$ &c.

$$\frac{d^2 \alpha_{,}}{d t^2} = -\frac{c^2}{2} (\overline{\mathrm{I} - \mathrm{R}} \sin \phi + \mathrm{T} \sin \phi')$$

$$+ \mathrm{Q} (\alpha - \alpha_{,}) + \frac{\mathrm{M}}{e} \sin \phi \left(\frac{d \mathrm{I}}{d x} - \frac{d \mathrm{R}}{d x} \right) + \frac{\mathrm{M}_{,}}{e_{,}} \sin \phi' \frac{d \mathrm{T}}{d x}$$

$$- \mathrm{D}_{,} \mathrm{T}_{,} - \mathrm{D} \mathrm{I}_{,}$$

By subtraction

$$\frac{d^2 \alpha}{d t^2} - \frac{d^2 \alpha_{,}}{d t^2} = (\mathrm{Q}_{,} + \mathrm{Q}) (\alpha_{,} - \alpha)$$

$$= -(\mathrm{Q}_{,} + \mathrm{Q}) (\alpha - \alpha_{,})$$

Now $\mathrm{Q}_{,} + \mathrm{Q}$ differs from c^2 by a finite quantity: hence this equation can only be satisfied by making

$$\alpha_{,} - \alpha = 0$$

$$\frac{d^2 \alpha_{,}}{d t^2} - \frac{d^2 \alpha}{d t^2} = 0$$

the second of which equations is a consequence of the first.

By adding the two equations we get

$$\frac{d^2 \alpha}{d t^2} + \frac{d^2 \alpha_{,}}{d t^2} = -c^2 (\overline{\mathrm{I} - \mathrm{R}} \sin \phi + \mathrm{T} \sin \phi')$$

$$+ (\mathrm{Q} - \mathrm{Q}_{,}) (\alpha - \alpha_{,}) - 2 \mathrm{D} . \mathrm{I}_{,} - 2 \mathrm{D}_{,} \mathrm{T}_{,}$$

$$+ \frac{2 \mathrm{M}}{e} \sin \phi \left(\frac{d \mathrm{I}}{d x} - \frac{d \mathrm{R}}{d x} \right) + \frac{2 \mathrm{M}_{,}}{e_{,}} \sin \phi' \frac{d \mathrm{T}}{d x}$$

Now the sum of the two quantities which constitute the first line is

$$-c^2(\overline{I-R}\sin\phi+T\sin\phi')-c^2(I_,+T_,)$$

And, from the nature of the functions, the last two lines of the above equation cannot, when $x=0$, give any part of the quantity $-c^2(\alpha+\alpha_,)$, for the one involves sines of the same quantities whose cosines constitute the other; hence we must have separately equal to O the two following expressions, viz.

$$(c^2-2D)I_,+\overline{c^2-2D_,}\,T_, \quad \ldots\ldots \quad (1)$$

and $$\frac{M}{e}\sin\phi\left(\frac{dI}{dx}-\frac{dR}{dx}\right)+\frac{M_,}{e_,}\sin\phi'\frac{dT}{dx} \quad \ldots\ldots \quad (2)$$

The former equation gives $I_,+T_,=0$, for $D_,$ and D are the same thing. Hence

$$\alpha+\alpha_,=\overline{I-R}\sin\phi+T\sin\phi'$$

and $$\frac{d^2(\alpha+\alpha_,)}{dt^2}=-c^2(\alpha+\alpha_,)$$

as it ought to be.

On the second of the above equations we shall make some remarks after we have deduced the equations for the motion parallel to the surface.

15. It would be rather difficult to write down the equation for β from the equation for α, I shall therefore briefly deduce it.

$$\frac{d^2\beta}{dt^2}=\Sigma\left(\phi r+\frac{\phi' r}{r}\delta y^2\right)\delta\beta+\Sigma\frac{\phi' r}{r}\delta x\,\delta y\,\delta\alpha$$

$$+\Sigma\left(\phi r'+\frac{\phi' r'}{r'}\delta y'^2\right)(\beta_,-\beta)+\Sigma\frac{\phi' r'}{r'}\delta x'\,\delta y'(\alpha_,'-\alpha)$$

Now $$\Sigma\left(\phi r+\frac{\phi' r}{r}\delta y^2\right)\delta\beta=$$

$$\left\{\Sigma\phi r+\frac{\phi' r}{r}(\delta x'^2\sin^2\phi+\delta y'^2\cos^2\phi)\right\}\cos\phi$$

$$\times\left\{-2I\sin^2\frac{Kx'}{2}+\frac{1}{e}\frac{dI}{dx}\sin Kx'-2R\sin^2\frac{Kx''}{2}+\frac{1}{e}\frac{dR}{dx}\sin Kx''\right\}$$

$$=-\frac{c^2}{2}\cos\phi(I+R)(\cos^2\phi-2\sin^2\phi)+\frac{M}{e}\left(\frac{dI}{dx}+\frac{dR}{dx}\right)\cos\phi(\cos^2\phi-2\sin^2\phi)$$

Again, if we denote $\Sigma\frac{\phi' r}{r}\delta x\,\delta y\,e^{-m\delta x}\sin f\delta y$ by F, we obtain

$$\Sigma\frac{\phi' r}{r}\delta x\,\delta y\,\delta\alpha=-\frac{3c^2}{2}\sin^2\phi\cos\phi(I+R)$$

$$+\frac{3M}{e}\sin^2\phi\cos\phi\left(\frac{dI}{dx}+\frac{dR}{dx}\right)+\frac{F}{f}\cdot\frac{dI_,}{dy}$$

by adding this term to the former we obtain

$$-\frac{c^2}{2}\cos\phi(I+R)+\frac{M}{e}\cos\phi\left(\frac{dI}{dx}+\frac{dR}{dx}\right)+\frac{F}{f}\frac{dI_,}{dy}$$

Again $$\Sigma\left(\phi\, r' + \frac{\phi' r'}{r_{,}}\,\delta y'^{2}\right)(\beta_{,}' - \beta)$$

$$= \Sigma\left(\phi' r' + \frac{\phi' r'}{r'}\,\delta y'^{2}\right)(\delta\beta_{,} + \beta_{,} - \beta)$$

$$= Q_{,}(\beta_{,} - \beta) - \frac{c^2}{2}\,T\cos\phi'(\cos^2\phi' - 2\sin^2\phi') + \frac{M_{,}}{e_{,}}\frac{dT}{dx}\cos\phi'(\cos^2\phi' - 2\sin^2\phi')$$

and $$\Sigma\frac{\phi' r'}{r'}\,\delta x'\,\delta y'\,(\alpha_{,}' - \alpha) = \Sigma\frac{\phi' r'}{r'}\,\delta x'\,\delta y'\,\delta\alpha'$$

$$= -\frac{3c^2}{2}\,T\cos\phi'\sin^2\phi' + \frac{3M_{,}}{e_{,}}\frac{dT}{dx}\sin^2\phi'\cos\phi' \div \frac{F_{,}}{f}\frac{dT_{,}}{dy}$$

which being added to the previous term gives

$$Q_{,}(\beta_{,} - \beta) - \frac{c^2}{2}\,T\cos\phi' + \frac{M_{,}}{e_{,}}\frac{dT}{dx}\cos\phi' + \frac{F_{,}}{f}\frac{dT_{,}}{dy}$$

Hence $$\frac{d^2\beta}{dt^2} = -\frac{c^2}{2}\left(\overline{I+R}\cos\phi + T\cos\phi'\right)$$

$$+ Q_{,}(\beta_{,} - \beta) + \frac{M}{e}\cos\phi\left(\frac{dI}{dx} + \frac{dR}{dx}\right) + \frac{M_{,}}{e_{,}}\cos\phi'\frac{dT}{dx}$$

$$+ \frac{F}{f}\frac{dI_{,}}{dy} + \frac{F_{,}}{f}\frac{dT_{,}}{dy}$$

Similarly $$\frac{d^2\beta_{,}}{dt^2} = -\frac{c^2}{2}\left(\overline{I+R}\cos\phi + T\cos\phi'\right)$$

$$+ Q(\beta - \beta_{,}) + \frac{M}{e}\cos\phi\left(\frac{dI}{dx} + \frac{dR}{dx}\right) + \frac{M_{,}}{e_{,}}\cos\phi'\frac{dT}{dx}$$

$$+ \frac{F}{f}\frac{dI_{,}}{dy} + \frac{F_{,}}{f}\frac{dT_{,}}{dy}$$

16. By the same mode which we exhibited for α and $\alpha_{,}$ we can shew that

$$\frac{d^2\beta}{dt^2} - \frac{d^2\beta_{,}}{dt^2} = -\overline{Q+Q_{,}}\,\overline{\beta - \beta_{,}}$$

and $$\therefore\ \beta = \beta_{,}$$

also $$\frac{d^2\beta}{dt^2} + \frac{d^2\beta_{,}}{dt^2} = -c^2(\beta + \beta_{,})$$

$$+ \frac{2M}{e}\cos\phi\left(\frac{dI}{dx} + \frac{dR}{dx}\right) + \frac{2M_{,}}{e_{,}}\cos\phi'\frac{dT}{dx}$$

$$+ \frac{2F}{f}\frac{dI_{,}}{dy} + \frac{2F_{,}}{f}\frac{dT_{,}}{dy}$$

and since

$$\frac{d^2\beta}{dt^2}+\frac{d^2\beta_{\prime}}{dt^2}=-c^2(\beta+\beta_{\prime})$$

from the nature of the case, it follows that

$$\frac{\mathrm{M}}{e}\cos\phi\left(\frac{d\mathrm{I}}{dx}+\frac{d\mathrm{R}}{dx}\right)+\frac{\mathrm{M}_{\prime}}{e_{\prime}}\cos\phi'\frac{d\mathrm{T}}{dx}$$

$$+\frac{\mathrm{F}}{f}\frac{d\mathrm{I}_{\prime}}{dy}+\frac{\mathrm{F}_{\prime}}{f}\frac{d\mathrm{T}_{\prime}}{dy}=0\text{.................}(3)$$

It only remains that we find the values of M, $\mathrm{M}_{\prime}$, F, $\mathrm{F}_{\prime}$, and substitute them in the five equations

$$(\mathrm{I}-\mathrm{R})\sin\phi+\mathrm{I}_{\prime}=\mathrm{T}\sin\phi'+\mathrm{T}_{\prime}\text{....}(1)$$

$$(\mathrm{I}+\mathrm{R})\cos\phi=\mathrm{T}\cos\phi'\text{........}(2)$$

$$\mathrm{I}_{\prime}+\mathrm{T}_{\prime}=0\text{............}(3)$$

$$\frac{\mathrm{M}}{e}\sin\phi\left(\frac{d\mathrm{I}}{dx}-\frac{d\mathrm{R}}{dx}\right)+\frac{\mathrm{M}_{\prime}}{e_{\prime}}\sin\phi'\frac{d\mathrm{T}}{dx}=0\text{............}(4)$$

and

$$\frac{\mathrm{M}}{e}\cos\phi\left(\frac{d\mathrm{I}}{dx}+\frac{d\mathrm{R}}{dx}\right)+\frac{\mathrm{M}_{\prime}}{e_{\prime}}\cos\phi_{\prime}\frac{d\mathrm{T}}{dx}+\frac{\mathrm{F}}{f}\frac{d\mathrm{I}_{\prime}}{dy}+\frac{\mathrm{F}_{\prime}}{f}\frac{d\mathrm{T}_{\prime}}{dy}=0\text{.............}(5)$$

Now, we have already shewn that

$$\frac{\mathrm{M}}{e}+\frac{\mathrm{M}_{\prime}}{e_{\prime}}=0$$

and in precisely the same manner it appears that

$$\frac{\mathrm{F}}{f}+\frac{\mathrm{F}_{\prime}}{f}=0$$

17. By substituting in equation (4) of the last article, we deduce

$$\cos\phi\,(\mathrm{I}'-\mathrm{R}')-\cos\phi'\,\mathrm{T}'=0$$

where I′, R′, T′ are the differential coefficients of I, R, and T.

But if we differentiate (2), we obtain the same result; hence equation (4) is a result of (2), and cannot be employed in our calculation.

Now

$$e=\frac{\cos\phi}{\lambda}\qquad e_{\prime}=\frac{\cos\phi'}{\lambda'}=\frac{\mu\cos\phi'}{\lambda}=\frac{\sin\phi}{\lambda\sin\phi'}\cos\phi'$$

$$f=\frac{\sin\phi}{\lambda}$$

Hence equation (4) becomes

$$\frac{\mathrm{M}}{e}\left\{\frac{\cos^2\phi}{\lambda}(\mathrm{I}-\mathrm{R})-\frac{\sin\phi\cos^2\phi'}{\lambda\sin\phi'}\mathrm{T}\right\}+\frac{\mathrm{F}}{f}\left\{\frac{\sin\phi}{\lambda}\mathrm{I}_{\prime}-\frac{\sin\phi}{\lambda}\mathrm{T}_{\prime}\right\}=0$$

Or
$$(\mathrm{I}-\mathrm{R})\frac{\cos^2\phi}{\sin\phi}-\mathrm{T}\frac{\cos^2\phi'}{\sin\phi'}=-\frac{e\,\mathrm{F}}{f\mathrm{M}}(\mathrm{I}_{\prime}-\mathrm{T}_{\prime})$$

$$=-\frac{e\,\mathrm{F}}{f\mathrm{M}}\cdot 2\,\mathrm{I}_{\prime} \text{ by means of (3).}$$

Now
$$\frac{\mathrm{M}}{e}=\frac{1}{e}\Sigma\left(\phi r+\frac{\phi' r}{r}\delta y^2\right)\sin \mathrm{K}x'$$

$$=\frac{\mathrm{S}}{e}\Sigma\cdot\frac{\delta x^2+\delta z^2-2\,\delta y^2}{r^5}\sin\delta x'$$

$$-\frac{\mathrm{F}}{f}=+\frac{\mathrm{S}}{f}\;\frac{3\,\delta x\,\delta y}{r^5}\,e^{-m\,\delta x}\sin f\delta y$$

and the quantities on the right hand sides of these two equations, are the coefficients respectively of terms which result from forces arising from a motion perpendicular to that of transmission, but extending only half through the system. There are, in fact, two terms arising from this cause, the one corresponding to the vibratory motion each, and having its value c^2 in both, and the other the term in question.

Hence we conclude, that
$$\frac{\mathrm{M}}{e}=-\frac{\mathrm{F}}{f}$$

Our equation (4) is by this means reduced to
$$\frac{\cos^2\phi}{\sin\phi}(\mathrm{I}-\mathrm{R})-\frac{\cos^2\phi'}{\sin\phi'}\mathrm{T}=2\,\mathrm{I}_{\prime}$$

and
$$(\mathrm{I}-\mathrm{R})\sin\phi=\mathrm{T}\sin\phi'-2\,\mathrm{I}_{\prime} \text{ by (1).}$$

By addition
$$(\mathrm{I}-\mathrm{R})\left(\frac{\cos^2\phi}{\sin\phi}+\sin\phi\right)=\mathrm{T}\left(\frac{\cos^2\phi'}{\sin\phi'}+\sin\phi'\right)$$

or
$$\frac{(\mathrm{I}-\mathrm{R})}{\sin\phi}=\frac{\mathrm{T}}{\sin\phi'}$$

or
$$(\mathrm{I}-\mathrm{R})\sin\phi'=\mathrm{T}\sin\phi$$

and by (2)
$$(\mathrm{I}+\mathrm{R})\cos\phi=\mathrm{T}\cos\phi'$$

$$\therefore(\mathrm{I}-\mathrm{R})\sin 2\,\phi'=(\mathrm{T}+\mathrm{R})\sin 2\,\phi$$

$$\mathrm{I}(\sin 2\,\phi'-\sin 2\,\phi)=\mathrm{R}(\sin 2\,\phi'+\sin 2\,\phi)$$

$$\mathrm{R}=-\mathrm{I}\cdot\frac{\sin 2\,\phi-\sin 2\,\phi'}{\sin 2\,\phi+\sin 2\,\phi'}$$

$$=-\mathrm{I}\frac{\tan\overline{\phi-\phi'}}{\tan\overline{\phi+\phi'}}$$

Again, by eliminating R, we obtain

$$\mathrm{I}\sin\phi'\cos\phi+\mathrm{I}\sin\phi'\cos\phi=\mathrm{T}\sin\phi\cos\phi+\mathrm{T}\sin\phi'\cos\phi'$$

$$\mathrm{T}=2\,.\,\mathrm{I}\,.\,\frac{\sin\phi'\cos\phi}{\sin\phi\cos\phi+\sin\phi'\cos\phi'}$$

$$=2\,\mathrm{I}\,.\,\frac{\cos\phi}{\cos\phi'}\left\{\frac{\sin\phi'}{\sin\phi\dfrac{\cos\phi}{\cos\phi'}+\sin\phi'}\right\}$$

$$=\mathrm{I}\,.\,\frac{\cos\phi}{\cos\phi'}\left\{1-\frac{\sin\phi\dfrac{\cos\phi}{\cos\phi'}-\sin\phi'}{\sin\phi\dfrac{\cos\phi}{\cos\phi'}+\sin\phi'}\right\}$$

$$=\mathrm{I}\,.\,\frac{\cos\phi}{\cos\phi'}\left\{1-\frac{\sin 2\phi-\sin 2\phi'}{\sin 2\phi+\sin 2\phi'}\right\}$$

$$=\mathrm{I}\,.\,\frac{\cos\phi}{\cos\phi'}\left\{1-\frac{\tan(\phi-\phi')}{\tan\overline{\phi+\phi'}}\right\}$$

These are precisely FRESNEL's results; in fact, the equation (2) corresponds with his empirical formula.

In conclusion, I have only to observe, that some of the equations involve what appears almost too accurate a substitution to be called an approximation, but which may in some extreme cases give rise to considerable deviation from the resulting formulæ. It will not, however, repay us for the labour of entering into the discussion of such points; suffice it to say, that the more *deviation* a ray suffers, the greater is the difference between the assumed and the real value of some of the forces. Except, however, the deviation be very great indeed, they cannot differ widely from each other.

EDINBURGH, *February* 4. 1839.

XXI.—*On the Composition of a New Writing-Ink, which, in resisting Chemical Deletion, promises to diminish the chance of the Falsification of Bills, Deeds, and other Documents.* *By* THOMAS STEWART TRAILL, *M. D., F. R. S. Ed. &c., Professor of Medical Jurisprudence in the University of Edinburgh.*

Read Monday, 19th February 1838.

THE preparation of my Lectures on Medical Jurisprudence involved a consideration of the means of diminishing the chances of successful forgery, and again engaged me on a subject to which, many years ago, my attention had been very painfully turned by the frequency of executions for that crime. This will scarcely appear exaggeration, when it is stated, that, in the year 1809, there were no less than thirteen executions for forgery in the county of Lancaster, where I then resided: and when it is recollected that, in the fourteen years preceding 1819, two hundred and four individuals perished on the scaffold for that offence in England and Wales, every means of discouraging so fertile a source of misery and crime must be allowed to be a subject of no trifling importance.

Many of those forgeries were no doubt counterfeits of Bank of England notes, in which the effacing of writing-ink had no concern; but Parliamentary returns shew that, out of forty convictions for forgery throughout England in 1833, no more than sixteen were connected with the Bank.

A Statistical Report, published in France in 1836, gave 292 as the number of persons *committed* for forgery in that kingdom in 1831; while in Belgium there were 39; in Spain 44; and in Britain 44. This gives, in proportion to the population of each, for

Belgium, with a population equalling	4,082,427,	1 in	104,677
France, ...	32,960,534,	......	112,877
Spain, ...	13,950,000,	......	317,045
Britain, ...	20,721,350,	......	470,990

On the Continent the crime is more frequently attempted by erasure of writing-ink than in this country; yet the facilities which Chemistry affords of *falsifying deeds, and of deleting signatures* engrossed with common ink, without leaving a trace of the writing, has often in Britain also tempted to forgery. Hence the discovery of a cheap and durable WRITING-INK, as one mode of diminishing crime, and of protecting the interests of the public, has been considered as meriting the attention both of the moralist and the legislator; and has lately been the subject of a prize offered by the National Institute of France, which has not yet, I believe, been awarded to any competitor.

The important interests involved lent an additional interest to the investigation, and engaged me in a long series of experiments, the results of which I beg leave now to submit to the Royal Society of Edinburgh, as a body well fitted to decide on the merits of my proposition; and, should the discovery be deemed worthy of that distinction, to communicate it to the public.

Before proceeding to the more immediate subject of this paper, it may not be uninteresting to review the general results of my experimental investigations on various substances, before I succeeded in forming a good *durable writing-ink*; and a knowledge of my previous failures may at least save the time of future investigators.

The substances which either wholly or nearly efface common writing-ink are chiefly,

1. Solutions of chlorine 2. Chloride of lime with a weak acid 3. Chloride of antimony 4. Dilute nitro-muriatic acid 5. Oxalic acid	entirely.
6. Diluted nitric acid 7. ... sulphuric acid 8. ... hydrochloric acid	in a great measure.
9. Solutions of potassa 10. ... of soda 11. ... of ammonia	greatly impair its colour.

These substances, with the exception of pure soda, which acts just as potassa, were tried on different inks; and resistance to their effects, either when applied singly or in succession, was considered as the criterion of the durability of the ink.

Series I. *Experiments with prepared Paper.*

It was imagined that, by first impregnating unsized paper with substances capable of affording deep coloured precipitates from metallic solutions, and then sizing the paper, that the precipitate from a metallic ink might be so fixed in the paper as to resist these chemical agents.

Unsized paper was therefore soaked in the following solutions:

A Infusion and tincture of galls.
B Prussian alkali, or ferro-cyanide of potassium.
C Common salt, or chloride of sodium.
D Phosphate of soda.
E Hydriodate of potassa, or iodide of potassium.
F Bichromate of potassa.

The paper was dried, and afterwards sized in the usual manner. Characters traced on A with sulphate of iron were black; with sulphate of copper, were yellowish-brown.

On B, with chloride of antimony, they were of a brilliant blue, which resisted chlorine, but was effaced by ammonia; with sulphate of iron, of a dark blue; with sulphate of copper, they were of a rich brown; with nitrate of cobalt, of a deep brown, which resisted alkalis, but was effaced by chlorine.

On C, with nitrate of silver, they quickly passed from white to brownish-black.

On D, with nitrate of silver, they were of yellowish-green.

On E, with acetate of lead, they were of a lively yellow; with bichloride of mercury, of a bright crimson.

On F, with acetate of lead, they were of an intense yellow.

These methods afforded no protection against some of the chemical agents; nor did the reverse of the process prove of any utility. I since find that a method very similar was proposed to the French Institute, and found equally unavailing.

Series II. *Metallic Sulphurets.*

Metallic sulphurets were totally discharged from paper by chlorine, and the substances yielding it. No advantage was derived from precipitating the sulphurets on the paper by sulphureted hydrogen.

Series III. *Sulphurets mingled with Common Ink.*

One proposition made to the French Institute, and vaunted as affording a durable ink, is the mixture of sulphuret of lead with common ink. A repetition of the process shewed the idea to be erroneous; and mixtures of other metallic sulphurets with that liquid were found to be equally useless.

Series IV. *Sulphate of Indigo.*

Sulphate of indigo has its colour destroyed by chlorine, and its above mentioned compounds; but this sulphate, precipitated by carbonate of potassa or of soda, when mixed with common ink, gave a blue fluid, which resists powerful chemical agents better than common ink. This mixed ink, however, is very easily effaced by chlorine. The precipitated colouring matter from sulphate of indigo appears to be the basis of the blue inks, which are now much in use; but I have found none of them capable of resisting chlorine.

Series V. *Iodides of the Metals.*

None of the metallic iodides, when united with many different vehicles, afforded durable inks.

On reviewing these experiments, it appeared,

1. That chlorine, and substances readily yielding it, such as chlorides of lime and antimony, totally destroy the colours of all the metallic compounds tried, except one—the beautiful blue precipitate obtained on adding chloride of antimony of the shops to the prussian alkali. This substance remains unchanged when precipitated on the prepared paper B, though immersed in chlorine, oxalic acid, or in diluted sulphuric, nitric, and hydrochloric acids; but it is instantly destroyed by potassa and ammonia. The intensity of the colour of this substance, induced me to try it as a substitute for ultramarine in oil-painting; but though its hue be exceedingly brilliant when ground with oil, I fear that it is not sufficiently permanent.

2. Oxalic acid totally effaces inks made of gallate of iron, of prussian blue, of iodides of lead and mercury, of chromate of lead, and of sulphate of indigo; but writing with this last fluid on some of the prepared papers was not wholly discharged by it.

3. Chloride of antimony destroys or weakens all the metallic inks tried, except the blue antimonial one; but writing with sulphate of indigo on paper prepared with hydriodate of potassa is not easily effaced by this agent.

4. The caustic alkalis weaken or destroy almost all the metallic inks, except one—the salt formed by adding nitrate of cobalt to the prussian alkali; but this ink is effaced by chlorine and the stronger acids.

Series VI. *Mixtures of Salts of Antimony and of Cobalt.*

The mixture of the two salts of antimony and cobalt, found to be most persistent, seemed to promise success, and became the subject of many experiments. The rich golden-yellow liquid formed by mixing nitrate of cobalt and chloride of antimony soon becomes solid, and gives out nitro-muriatic acid. When ground up with gum-water, it formed an ink, which, applied to paper prepared with prussian alkali, produces dark brown letters, which, though weakened in colour, are not destroyed by the application of an acid or an alkali; but soaking the paper first in one, and then in the other, effaces the characters entirely.

In short, I found that none of the dark coloured salts of the metals, nor their iodides, nor their sulphurets, were capable of withstanding all the chemical agents tried: most of them were destructible by a single agent, and none of them were capable of resisting two applied in succession. I was therefore constrained to turn my attention to inks with a basis of carbon.

The μελαν of the Greeks, and the atramentum of the Romans, was employed to denote not only writing-ink, but various carbonaceous substances used as pigments. The ancient writing-ink, as exhibited in the Herculaneum MSS., in Egyptian MSS. found in mummy cases, and other ancient *papyri*, by its qualities may be proved to be carbonaceous; even had we not the direct testimony of Vitruvius, PLINY, and DIOSCORIDES. VITRUVIUS describes particularly the mode of making the carbonaceous matter, by a process very similar to that which we employ for the manufacture of lamp-black. A furnace was prepared, having flues opening into a condensing chamber; resin was burnt in the furnace, and the smoke was carefully condensed in the chamber: this material, mixed with gum, was the *writing-atramentum* employed by the Romans.* PLINY gives a similar, though less minute account; † and DIOSCORIDES expressly informs us that the μελαν used in writing was formed of soot of torches or of resin, mixed with one part of gum for every three parts of the carbon.‡ A finer sort of ink, he says, was manufac-

* The atramentum described by VITRUVIUS, PLINY, and DIOSCORIDES, was employed both as writing-ink and as a pigment. The account of VITRUVIUS is as follows:

" Ingrediar nunc ad ea, quæ ex aliis generibus tractationum temperaturis commutata respiciunt colorum proprietates: et primum exponam de *atramento,* cujus usus in operibus magnas habet necessitates, ut sint notæ, quemadmodum præparentur certis nationibus artificiorum ad id temperaturæ. Namque ædificatur locus uti Laconicum, et expolitur marmore subtiliter, et lævigatur. Ante id fit Fornacula, habens in Laconicum nares, et ejus præfurnium magna diligentia comprimitur, ne flamma extra dissipetur: in fornace resina collocatur. Hanc autem ignis potestas urendo cogit emittere per nares intra Laconicum fuliginem, quæ circa parietem, et cameri curvaturam adhærescit, inde collecta passim componitur ex gummi subacto ad usum *atramenti Librarii:* reliqua tectores glutinum admiscentes in parietibus utuntur. Sin autem eæ copiæ non fuerint paratæ, ita necessitatibus erint administrandum, ne expectatione moræ res retineantur. Sarmenta aut tedæ schidiæ comburantur; cum erunt carbones, extinguantur. Deinde in mortario cum glutino tereantur, ita erit atramentum tectoribus non invenustum. Non minus si fæx vini arefacta, et cocta in fornace fuerit, et ea contrita cum glutino in opere inducetur, per quam atramenti suavem efficiet colorem; et quo magis ex meliore vino parabitur, non modo atramenti, sed etiam *Indici* colorem dabit imitari." Lib. vii. chap. x. *Ex ed. Venet. folio,* 1567, *cum commentario Danielis Barbati,* p. 246.

The *Laconicum* means a species of condensing chamber, of an hemispherical shape, placed near the furnace, for receiving and condensing the smoke. *Atramentum Librarii* et *Scriptorum* is writing-ink. *Gluten,* or glue, was prepared from the ears or genitals of bulls in ancient times. *Indicum,* perhaps indigo.

† PLINY is very explicit on this subject:

" Fit enim et fuligine pluribus modis, resina vel pice exustis. Propter quod officinas etiam ædificare, fumum non emittentes."......" Adulteratur Fornacum Balnearumque fuligine quo ad volumina scribenda utuntur."

‡ DIOSCORIDES, in the last chapter of Lib. v. περι Ὑλης Ἰατρικης, describes the composition of ink, and gives the proportions of the ingredients:

" περι Μελανος.

" The ink with which we write is compounded of the soot of torches. For every three ounces of soot, one of gum is to be added. It is also made from the soot of resin, and also from the material called *painters' black* (και τῆς προειρημενῆς ζωγραφικῆς Ασβολης). Of this black it is proper to take one *mina,* half a *litra* of gum, of ox glue and of *flos ferri* (χαλκανθου) each half an ounce."

The *painter's black* is described in a former chapter as a soot collected in the chimneys of workers in glass (ὑελεργειων). The *flos ferri* is, from his description of its colour and other qualities, evidently a sulphate of iron. We here, then, may trace the probable step which led to the use of inks composed of iron; and may observe also, that the mixture of carbon with salts of iron is not a recent proposition for the composition of ink.

tured from what he terms *painters' black*, the soot collected in *glass furnaces*, rubbed up with gum, animal glue, and sulphate of iron (χαλκανθος).

I find the antique inks of the Herculaneum and Egyptian papyri to resist chlorine, acids, and other agents which efface metallic inks. The characters also so often found on unwrought fragments of limestone, in the tombs of Upper Egypt, are written with a carbonaceous ink. These fragments appear to have been employed, just as we should a scrap of paper, to receive an impromptu composition.

These inks have resisted the ordinary effects of time exceedingly well; but they have the disadvantage of being removable by water; and they are not so well adapted for writing in a current hand (the great desideratum in modern practice) as for the slower process of the ancients. The introduction of paper instead of parchment and papyrus, probably contributed to the general employment of metallic inks; as those containing the salts of iron not only flowed readily from the pen on paper, but struck sufficiently into that material to prevent the easy ablution of the writing. But metallic inks, however convenient in other respects, are inferior to those with a base of carbon in resisting the effects of time, and still more in withstanding *deletion* by chemical means.

Carbonaceous inks appear to have been generally employed till about the middle of the fourteenth century. The high price of manuscripts rendered copyists careful in the choice of their inks; and probably it was not until the introduction of printing that they fell into disuse among copyists. BLAGDEN, however, mentions having found a manuscript of the ninth century written with an ink composed of iron with a vegetable astringent (Phil. Trans. for 1787); but, after having examined a great many manuscripts of the thirteenth, and early part of the fourteenth, centuries, I found them chiefly written with carbonaceous inks, resisting all the ordinary deleting agents. After the middle of the fourteenth century, however, the carbonaceous inks begin to disappear, and manuscripts are written with compositions very similar to our ordinary writing-inks. This was attended with small inconvenience, except the gradual fading of their colour, until the discovery of the properties of chlorine; and of late years their imperfections have been generally admitted.

Series VII. *Carbon with Gums, Gum-resins.*

1. I attempted to form a durable ink by incorporating carbon with gums and gum-resins; but they took little hold of the paper, and readily washed off when moistened. The well-known *Chinese ink* is a beautiful mixture of carbonaceous matter with glue, which we try to imitate in Europe, but uniformly fail to equal.* It is, however, destitute of some of the essential qualities of a good

* In *Annales de Chimie* for 1833, vol. liii., is a curious account of the Chinese method of making ink, extracted from a Japanese Encyclopædia in 80 octavo volumes, and from a Chinese Dictionary of

writing-ink, though it resists chemical agents. It is better suited for the pencil than the pen; it does not flow freely, and therefore is unfavourable to rapid writing; it does not sink sufficiently into the paper to resist rubbing; and it may be washed entirely from the surface of paper by the careful and repeated application of a moistened pencil.

2. Inks of carbon, incorporated with essential oils, promised to resist water. One was several years ago proposed by Mr W. Close, which is formed by dissolving 24 grains of copal in 200 grains of English oil of lavender, and grinding it with 4 grains of lamp-black. This ink resists water and several chemical agents well; but it cannot be used as a common writing-ink; for it does not flow easily from the pen; it sinks much in the paper, forming unseemly ragged lines.

3. I tried many other resinous solutions in spirit and in oils; but with no better success.

4. I obtained a better ink by mixing lamp-black with a solution of caoutchouc in coal-tar-naphtha. This process forms an ink of a good body, which flows pretty freely from the pen; but it spreads and sinks too deeply through the paper, and is disagreeable from its penetrating and sickening odour.

4. I obtained a far more elegant solution of caoutchouc in an essential oil, which spontaneously exudes from a South American tree, supposed to be a species of *Laurus*. This substance is fragrant, and readily dissolves caoutchouc; but the ink made with it has some of the disadvantages of the last sort, and would be expensive, as the oil is a rare substance. It has hitherto only been brought to us from British Guyana, under the Spanish name of *Aceite de Sassafras*, "oil of sassafras," and has been highly extolled as a remedy for rheumatism.

Series VIII. *Carbon with Animal Fluids.*

1. Lamp-black ground with the albumen of the egg united with difficulty; the ink did not flow readily from the pen, and the characters appeared coarse and unequal.

2. The white of the egg and Indian inks were not more successful.

3. Milk and urine were also tried; but though better than white of egg as vehicles, the ink made with either of them was easily effaced.

4. Serum of blood was liable to similar objections as a vehicle for the carbon; and I found that,

5. Glue succeeded no better.

Arts and Sciences, in the Royal Library at Paris. The finest would seem to be prepared from a lamp-black, obtained by the combustion of vegetable oils, particularly that of *Bignonia Tomentosa*, mixed with animal glue; the greatest quantity is prepared by collecting the soot of pine wood, received in a chamber 100 feet long, formed of paper pasted over bamboo, and divided into various compartments. The lamp-black collected in the first compartment is coarse; the finest is in the last compartment. The precautions used will shew how careful the Chinese are to obtain a lamp-black of a fine quality.

All the inks of this species, though they resisted chlorine, were loosened by weak acids, and by diluted alkaline solutions. These effects took place, even after the writing had been strongly heated, and its surface washed over with alcohol, in the hope of fixing the albumen by coagulation. Another objection to animal fluids as the vehicles of a liquid ink, is their speedy corruption and disagreeable odour.

Series IX. *Carbon with Starch.*

Solutions of starch were next employed to suspend the carbon.

1. I found great difficulty in incorporating lamp-black with starch. The ink thus formed is pale, unequal, and does not flow from the pen.
2. Inks of this kind are not improved by the addition of iodine.

Series X. *Carbon with Gluten.*

The intimate mixture of carbonaceous matter with another vegetable substance afforded the results I had hoped. I need not occupy the time of the Society with a description of the numerous preliminary trials on this combination of materials, but proceed to detail the process which appears to me capable of affording a CONVENIENT, CHEAP, AND DURABLE WRITING INK.

The qualities requisite for such an ink are: an easy process for its manufacture from cheap materials; a fluid which shall flow as freely from the pen as common ink,—which shall dry quickly,—which shall take such hold of the paper, after it is dry, as not to rub off by a friction short of what injures the texture of the paper,—which shall resist the chemical agents that efface common ink, unless they be so concentrated as to destroy the paper itself,—and which is not liable to lose its colour by time.

It is sufficiently obvious, that if an ink be only removable by means which destroy the texture of the paper, it is sufficiently entitled to the appellation of *durable;* because no ink can be conceived capable of resisting an agent that destroys the material to which it is applied; nor have we any title to expect it to resist such agent, any more than to be indestructible in the fire which consumes the paper.

The vehicle of my ink is VEGETABLE GLUTEN, DISSOLVED IN STRONG VINEGAR, OR BETTER IN PYROLIGNEOUS ACID.

This substance was suggested to me by the recollection of numerous experiments, which I had made several years ago, on solutions of gluten as varnishes, and the use of vinegar as a solvent, by a remarkable passage in PLINY, where he treats of writing-inks:

"Omne atramentum sole perficitur, librarium gummi, tectorium glutino admisto. Quod autem *aceto liquefactum*, ægre eluitur."

PLINY'S *gluten*, however, is very different from mine; it is common animal glue, but such an ink is removable by water.

I need scarcely add, that many other preparations of gluten were tried; but I give a preference to the acetous solution, because it incorporates very freely with lamp-black (the cheapest impalpable carbon), and unites all the valuable qualities of the other solutions of this substance. The first step is to prepare a good gluten from wheat-flour, by kneading a mass of the dough below a small stream of water, in order to separate the starch. On the large scale, this may be done in linen bags or by machinery, but a considerable quantity may be formed in a short time by the hand. The vicinity of a starch-work might probably afford this material cheaply. The more perfectly the starch is separated from the gluten the better will be the quality of the ink. When the gluten is kept in water for twenty-four or thirty-six hours, it dissolves more readily in the pyroligneous acid than when recently made. The proportions I employ are,—a pound and a half of this gluten to ten pounds of pyroligneous acid, of the specific gravity of 1033 or 1034. By the aid of a gentle heat, they form a saponaceous fluid of a greyish-white colour, which will keep in this state for long periods.

The colouring matter should be the best lamp-black, such as is usually sold for 2s. per pound. I find that the colour of the ink, though not its other qualities, is improved by the addition of a small quantity of fine indigo. The proportions which I employ are from eight to twelve grains of lamp-black and two grains of indigo for every fluid ounce of the vehicle. With the larger proportion, the ink is of a fine deep black; but it is thought by some persons to flow more easily from the pen, when the smaller proportion of the finest colouring matter is employed, and it seemed also to enter more deeply into the paper. When the lamp-black is of the finest quality, a smaller proportion of this very light and bulky powder will suffice than when the lamp-black is coarse, and the ink will be more equal, and flow more smoothly from the pen. It seems scarcely necessary to add, that the more completely the ingredients are incorporated, the more perfect will be the ink.

An agreeable aroma may be communicated to this ink, by soaking bruised pimento, cloves, or cassia bark, in a portion of the acid, before mixing it up with the gluten.

Those who have experienced the difficulty of triturating any carbonaceous substance with other vehicles, will be surprised at the facility with which a short triture in a mortar unites the materials of this new ink.

The following tests will enable the Society to judge how it answers the characters which I have ascribed to this composition:

1. It flows as freely as common ink from the pen.
2. It speedily becomes sufficiently dry not to be rubbed off the paper.

3. Its colour, compared to common ink, is at once full; and it is not liable to change by time.

4. It is not affected by soaking or washing with water, as long as the texture of the paper is unchanged.

5. Slips of paper written with my ink, immersed for twenty-four hours in a solution of chlorine, capable in a few minutes of effacing common ink, underwent no change; and even when immersed for seventy-two hours, the ink was not impaired in colour or adhesion.

6. Similar experiments were made with a mixture of chloride of lime and water, acidulated with sulphuric acid;

7. With diluted nitro-muriatic acid;

8. With a saturated solution of oxalic acid;

9. With diluted sulphuric acid;

10. With diluted nitric acid;

11. With diluted hydrochloric acid. In all of which it remained without change of colour.

12. What is more remarkable, immersion in pyroligneous acid does not efface it when once thoroughly dry; and it seems to become more fixed in the paper by exposure to solar light and to the air.

13. Pencilling common writing ink with liquid chloride of antimony immediately discharges it; but the same substance produces no effect on the new ink.

14. Moderately diluted solutions of caustic alkalis, produce no effect on the writing with this ink, though they injure the colour of common inks immersed in them; but when the solutions are strong, they act on the texture of the paper, and also soften the vehicle of the colouring matter, so that it may be partially rubbed off; but the softened texture of the paper becomes imbued with the carbon, producing a stain, which will shew the attempt at erasure.

15. Slips of paper, written with the new ink, have been suffered to remain in moderately strong alkaline solutions for seventy-two hours without any loss of colour; and when the paper is dried, the ink has again become as firm as before.

16. It is not affected by the alkaline carbonates, nor by any of the neutral salts hitherto applied to it; nor am I aware of any chemical agent capable of more effectually removing it from paper, than the substances already mentioned.

It may be used with an iron pen, provided that pen be washed immediately afterwards, for iron is not readily acted on by ascetic acid.

It is not applicable as our ink for writing on parchment, because it may be washed off that material like every other ink, nor can it be employed for marking linen. It is, in fact, only offered as a *writing ink*, to be used on paper,—well suited for the drawing out of bills, deeds, or wills; or wherever it is important to prevent the alteration of sums of money, or of signatures, as well as for handing down to posterity public records, in a less perishable material than common ink.

I have thus, Gentlemen, without reserve, submitted to you the results of an investigation, which, unless I deceive myself, promises to be of public utility.

Should it be found to present an obstacle to the commission of crime,—should it, even in a single instance, prevent the perpetration of an offence so injurious in its consequences to society, as the falsification of a public or a private document, the author will rejoice in the publication of his discovery, and consider that his labour has not been in vain.

P. S.—Since this communication was made to the Royal Society, I obtained in London a " prepared paper," from which it was pretended that common ink could not be removed by chemical agents; but letters written on it with common ink, were instantly effaced by chlorine and by oxalic acid.

Good wheat flour will yield from 14 to 24 per cent. of moist gluten; but the quantity appears to be exceedingly variable, according to the quality of the flour, and perhaps also to the season. In lately preparing a large quantity of my ink, for the use of THE NATIONAL BANK OF SCOTLAND, Messrs DUNCAN and FLOCKHART found one sample of wheat to afford 14.73 per cent. of gluten; another yielded no more than 8.40 per cent. Barley and rye meal seldom give more than 4 or 5 per cent. of that substance.

It has been stated that my ink, though it resist the stronger chemical agents, is liable to be washed off paper by simple water. I had discovered that, when used on *highly-sized* or very *highly-glazed* paper, it does not adhere very firmly. The reason is obvious: the size or glaze prevents the ink from really coming into contact with the paper. They form a *varnish* that defends its surface from any ink; which, to be permanent, must sink into the substance of the paper. On other paper this new ink is *not* removable by water. It answers best on a paper not too highly glazed or sized; and, with a light pen, may be even used on unsized paper.

Several forgeries have of late been successfully perpetrated on some of the Scottish banks by the following stratagem. Bank orders for small sums were obtained on some of their country branches. The blank space in the engraved bill was filled up, as usual, in writing with common ink, thus—" *Four*——— *Pounds*." The dash after the word *Four* was deleted by chemical means, and the word " *Hundred*" inserted. The fraud has been but too successful. In consequence of its detection, one most respectable bank, after instituting numerous experiments on my ink, and ascertaining its *durability*, have already had 100 gallons of it prepared by Messrs DUNCAN and FLOCKHART of this city, and sent to their numerous country branches; and have directed it to be employed to fill up all bills, orders, &c. drawn on the bank.

XXII.—*Investigation of Analogous Properties of Co-ordinates of Elliptic and Hyperbolic Sectors.* *By* William Wallace, *LL.D., F.R.S.E., F.R.A.S., M. Cambridge P.S., &c. Emeritus Professor of Mathematics in the University of Edinburgh.*

Read 15th April 1839.

The analogy between certain properties of the co-ordinates of elliptic and hyperbolic sectors, which forms the subject of this memoir, was observed by the early writers on the fluxional or differential calculus, and employed by them in its improvement. But, independently of this important application, the propositions in question are some of the most elegant theorems in geometry, and highly interesting as abstract truths.

Maclaurin, in the third chapter of the second book of his *Treatise of Fluxions*, has proved the truth of this theorem: "Supposing n to be any number, let E and $n \times$ E be elliptic sectors, which stand on arcs that begin at a vertex of either axis, and H and $n \times$ H hyperbolic sectors, which stand on arcs that begin at the extremity of the transverse axis: the algebraic equation which expresses the relation between x and z, ordinates drawn to the other axis from the extremities of the arcs, must be the very same in the two curves."

From this he has inferred, that if the relation between the ordinates x and z of the one curve be found, it may be assumed as equally true of the ordinates of the other curve, and thus the properties of the one may be deduced from those of the other.

The late Professor Playfair, in a paper in the London Philosophical Transactions for 1779, has, by the use of the symbol of imaginary quantity (viz. $\surd -1$) deduced from the geometrical properties of the circle, which are the foundation of the angular calculus, corresponding properties of the hyperbola; but I do not remember any writer who has, by a single process of legitimate geometrical or analytical reasoning, established the identity of the properties of the two curves; although such an investigation would have been a fine example of the power of modern algebraic geometry as an instrument of research, and a valuable addition to treatises on the conic sections constructed by that compendious and elegant mode of reasoning.

2. Considering it desirable that a subject so purely elementary should be brought under the dominion of the ordinary doctrines of analysis, I have endeavoured to include the properties of both curves in one set of formulæ; taking from geometry the very least assistance possible; and avoiding all use of the imaginary symbol

$\surd-1$. The result of my investigation is given in what follows: the only geometrical properties of the curves supposed known are their definitions by their foci; from which, their equations, both included in this, $x^2+cy^2=a^2$, may be easily deduced; and this other property, viz. *If semidiameters be drawn to the extremities of any chord in an ellipse or hyperbola, the sector between them will be bisected by the diameter which bisects that chord.*

3. The formulæ sought are to be deduced from the resolution of the following problem:

Problem.

Let C *be the centre of an ellipse (fig.* 1), *or hyperbola (fig.* 2); *and let* AA′ *be any diameter in the ellipse, but a transverse diameter in the hyperbola; and* BB′ *its conjugate; also let* PP‴, P′P″ *be two parallel chords in either curve; and let*

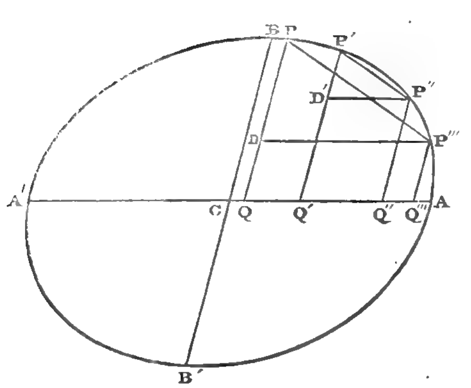
$CQ=x$,	$CQ'=x_1$,	$CQ''=x_2$,	$CQ'''=x_3$,
$PQ=y$,	$P'Q'=y_1$,	$P''Q''=y_2$,	$P'''Q'''=y_3$,

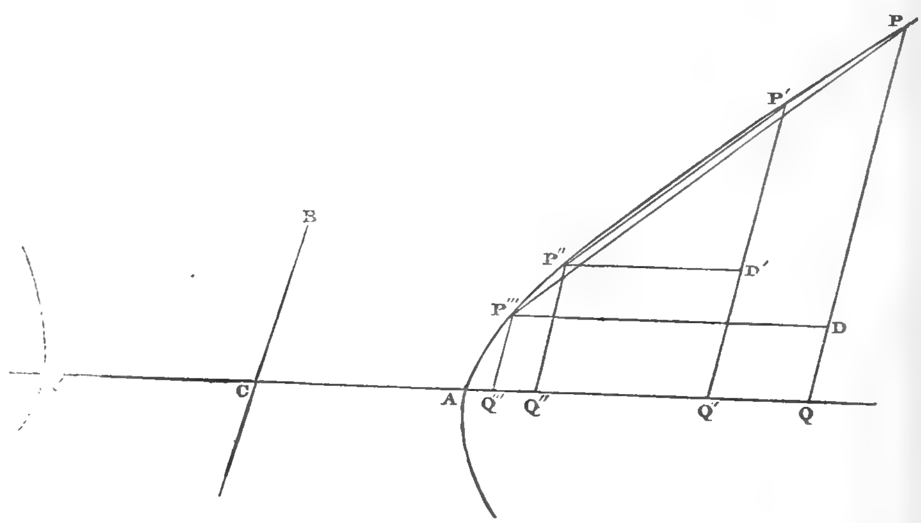
be the co-ordinates of their extremities P, P′, P″, P‴.

It is proposed to express the co-ordinates of each of the four points P, P′, P″, P‴ *by those of the other three.*

Fig. 2.

Fig. 1.





Draw P‴D, P″D′ parallel to the diameter AC. Put the semitransverse $CA=a$, the semiconjugate $CB=b$; and let $c=\pm\frac{a^2}{b^2}$, the sign + applying in the ellipse, and the sign − in the hyperbola.

From the similar triangles PDP‴, P′D′P″, we have

$$\frac{PD}{P'''D}=\frac{P'D'}{PD}.$$

Hence, in both curves, having regard to the signs of the co-ordinates

$$\frac{y-y_3}{x_3-x}=\frac{y_1-y_2}{x_2-x_1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a$$

Now in both curves,

$$x^2+cy^2=x_3^2+cy_3^2,$$
$$x_1^2+cy_1^2=x_2^2+cy_2^2;$$

wherefore, by transposition and resolution into factors,

$$(x_3-x)(x_3+x)=c(y-y_3)(y+y_3),$$
$$(x_2-x_1)(x_2+x_1)=c(y_1-y_2)(y_1+y_2):$$

and hence,

$$\frac{x_3-x}{y-y_3}=\frac{c(y+y_3)}{x_3+x};$$

$$\frac{x_2-x_1}{y_1-y_2}=\frac{c(y_1+y_2)}{x_2+x_1}.$$

By comparing these with formula (a), we obtain three others, which, with formula (a), may be expressed thus,

$$\frac{x_3-x}{y-y_3}=\frac{x_2-x_1}{y_1-y_2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{y_1+y_2}{x_2+x_1}=\frac{y+y_3}{x_3+x}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\frac{x_3-x}{y-y_3}=\frac{c(y_1+y_2)}{x_2+x_1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\frac{x_2-x_1}{y_1-y_2}=\frac{c(y+y_3)}{x_3+x}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

From these, again, there are obtained

$$(x_3-x)(y_1-y_2)=(x_2-x_1)(y-y_3), \quad . \quad . \quad . \quad . \quad (5)$$
$$(x_3+x)(y_1+y_2)=(x_2+x_1)(y+y_3), \quad . \quad . \quad . \quad . \quad (6)$$
$$(x_3-x)(x_2+x_1)=c(y_1+y_2)(y-y_3), \quad . \quad . \quad . \quad (7)$$
$$(x_3+x)(x_2-x_1)=c(y_1-y_2)(y+y_3), \quad . \quad . \quad . \quad (8)$$

By performing the multiplications here indicated, and adding and subtracting the results, we find

From (5) and (6) $\left\{\begin{array}{l} x_3y_1-x_1y_3=x_2y-xy_2, \\ x_3y_2-x_2y_3=x_1y-xy_1; \end{array}\right\}$ . . . . . . (9)

From (7) and (8) $\left\{\begin{array}{l} x_3x_1+cy_1y_3=x_2x+cyy_2, \\ x_3x_2+cy_2y_3=x_1x+cyy_1; \end{array}\right\}$ . . . . . . (10)

From these formulæ it appears, that we may interchange x_3 and x, provided that we at the same time interchange y_3 and y, and that we may also interchange x_2 and x_1, and at the same time y_2 and y_1; so that, in fact, they are reducible to two, viz.

$$x_3 y_1 - x_1 y_3 = x_2 y - x y_2 ; \quad \ldots \quad \text{(ix)}$$

$$x_3 x_1 + c y_1 y_3 = x_2 x + c y y_2 ; \quad \ldots \quad \text{(x)}$$

and here x, y; x_3 y_3 denote generally the co-ordinates of the extremities of either of the chords, and x_1, y_1; x_2 y_2 those of the extremities of the other chord.

These two equations contain the four pair of co-ordinates x, y; x_3 y_3, &c., and only the simple power of each ordinate. Therefore, each may be determined by the common process of elimination. And, to give the formulæ the greatest degree of simplicity, we must recollect that

$$x^2 + c y^2 = a^2, \qquad x_2^2 + c y_2^2 = a^2,$$

$$x_1^2 + c y_1^2 = a^2, \qquad x_3^2 + c y_3^2 = a^2.$$

The results obtained are

$$\left.\begin{aligned} a^2 x &= c y_3 (x_1 y_2 + x_2 y_1) + x_3 (x_1 x_2 - c y_1 y_2) \\ a^2 y &= x_3 (x_1 y_2 + x_2 y_1) - y_3 (x_1 x_2 - c y_1 y_2) \end{aligned}\right\} ; \quad \ldots \quad \text{A.}$$

$$\left.\begin{aligned} a^2 x_3 &= c y (x_1 y_2 + x_2 y_1) + x (x_1 x_2 - c y_1 y_2) \\ a^2 y_3 &= x (x_1 y_2 + x_2 y_1) - y (x_1 x_2 - c y_1 y_2) \end{aligned}\right\} ; \quad \ldots \quad \text{B.}$$

$$\left.\begin{aligned} a^2 x_1 &= c y_2 (x y_3 + x_3 y) + x_2 (x x_3 - c y y_3) \\ a^2 y_1 &= x_2 (x y_3 + x_3 y) - y_2 (x x_3 - c y y_3) \end{aligned}\right\} ; \quad \ldots \quad \text{C.}$$

$$\left.\begin{aligned} a^2 x_2 &= c y_1 (x y_3 + x_3 y) + x_1 (x x_3 - c y y_3) \\ a^2 y_2 &= x_1 (x y_3 + x_3 y) - y_1 (x x_3 - c y y_3) \end{aligned}\right\} ; \quad \ldots \quad \text{D.}$$

These formulæ completely resolve the problem that was proposed.

Fig. 3.

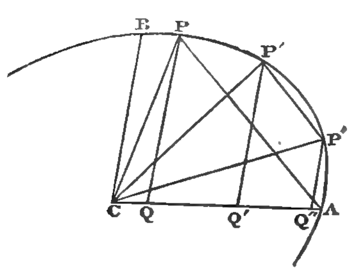
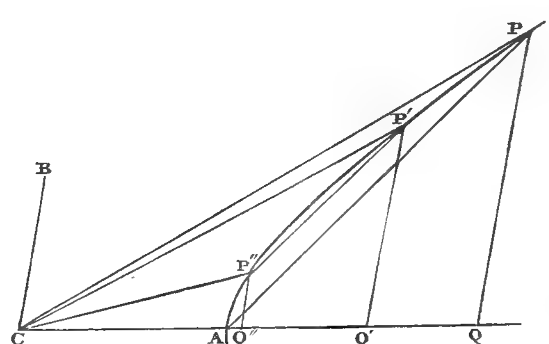


Fig. 4.



4. If one of the chords PP‴ be supposed to pass through A, one extremity of the axis CA, so that $x_3 = a$ and $y_3 = 0$, then we have

$$\left.\begin{aligned} a\,x &= x_1 x_2 - c y_1 y_2 \\ a\,y &= x_2 y_1 + x_1 y_2 \end{aligned}\right\}; \quad \ldots\ldots \quad \text{A}'.$$

$$\left.\begin{aligned} a\,x_1 &= x\,x_2 + c\,y\,y_2 \\ a\,y_1 &= x_2 y - x\,y_2 \end{aligned}\right\}; \quad \ldots\ldots \quad \text{B}'.$$

$$\left.\begin{aligned} a\,x_2 &= x\,x_1 + c\,y\,y_1 \\ a\,y_2 &= x_1 y - x\,y_1 \end{aligned}\right\}. \quad \ldots\ldots \quad \text{C}'.$$

5. Retaining the hypothesis that PA, one of the parallel chords, passes through A, the vertex of the diameter, let the semidiameters CP, CP′, CP″ be drawn. Then, by a known property of the curves, the sectors ACP″, P′CP will be equal, so that

Sector ACP = sector ACP′ + sector ACP″,

Sector ACP′ = sector ACP − sector ACP″ :

Let us now consider the co-ordinates

$$x,\ y\ ;\ x_1,\ y_1\ ;\ x_2,\ y_2,$$

as *functions* of the sectors ACP, ACP′, ACP″, and let us put

Sector ACP″ $=\alpha$, sector ACP′ $=\beta$, sector ACP $=\gamma$.

Also, let us express

x_2 by $f\ (\alpha)$, $\quad x_1$ by $f\ (\beta)$, $\quad x$ by $f\ (\gamma)$,

y_2 by $\mathrm{F}\ (\alpha)$, $\quad y_1$ by $\mathrm{F}\ (\beta)$, $\quad y$ by $\mathrm{F}\ (\gamma)$;

where the letters f and F are to be regarded as characteristics of the functions. Then, applying this notation to equations A′, B′, we have

$$\left.\begin{aligned} a\,.f(\gamma) &= a\,.f(\beta+\alpha) = f(\beta)\,f(\alpha) - c\,.\,\mathrm{F}(\beta)\,\mathrm{F}(\alpha) \\ a\,.\mathrm{F}(\gamma) &= a\,.\,\mathrm{F}(\beta+\alpha) = \mathrm{F}(\beta) f(\alpha) + f(\beta)\,\mathrm{F}(\alpha) \end{aligned}\right\}; \quad \ldots\ldots \quad \text{E}.$$

$$\left.\begin{aligned} a\,.f(\beta) &= a\,.f(\gamma-\alpha) = f(\gamma)\,f(\alpha) + c\,.\,\mathrm{F}(\gamma)\,\mathrm{F}(\alpha) \\ a\,.\mathrm{F}(\beta) &= a\,.\,\mathrm{F}(\gamma-\alpha) = \mathrm{F}(\gamma) f(\alpha) - f(\gamma)\,\mathrm{F}(\alpha) \end{aligned}\right\}. \quad \ldots\ldots \quad \text{F}.$$

These are entirely analogous to the well known formula for the cosine and sine of the sum, and of the difference of two angles; indeed the latter are comprehended in the former, and to produce them, it is only necessary to conceive the ellipse to change into a circle; in which case the symbol c, independently of the sign prefixed to it in the formula, must be regarded as positive, and its value $=1$: further, we must then exchange the functional mark f for *cos.*, the abbreviation of *cosine,* and F for *sin.* that of *sine.*

6. The reasoning throughout this memoir is perfectly general, and independent of the sign of the *modulus* c; and, in imitation of that employed to establish the calculus of angles, it may be carried on to a great extent. Indeed, pro-

ceeding from these formulæ, we might establish a complete theory of the conic sections. I shall, however, only farther extend the investigation to the co-ordinates of sectors, which are multiples of a given sector of an ellipse or hyperbola, and are contained between the transverse axis and any semidiameter. From these the whole theory of angular sections may be deduced, also the corresponding properties in an equilateral hyperbola.

7. Let a and b be the *axes* of a conic section, a being the *transverse*, and b the *conjugate*, or, in the case of the hyperbola, the *second axis*. This assumption will require no change in the formulæ. We may introduce e, the *eccentricity* of the curves instead of the second axis b, and since in the ellipse $b^2=a^2-e^2$, and in the hyperbola $b^2=e^2-a^2$; if in either curve α, β, $\alpha+\beta$, $\alpha-\beta$ denote sectors contained between a, the semitransverse, and semidiameters, the co-ordinates of whose vertices (which may be called the co-ordinates of the sectors) are respectively,

$$\begin{array}{llll} f(\alpha), & f(\beta), & f(\alpha+\beta), & f(\alpha-\beta), \\ F(\alpha), & F(\beta), & F(\alpha+\beta), & F(\alpha-\beta), \end{array}$$

In either curve,

$$\left.\begin{array}{ll} a\,.f(\alpha+\beta)=f(\alpha)\,.f(\beta)-\dfrac{a^2}{a^2-e^2}F(\alpha)\,.\,F(\beta), & \quad (1) \\ a\,.F(\alpha+\beta)=F(\alpha)\,.f(\beta)+f(\alpha)\,.\,F(\beta), & \quad (2) \\ a\,.f(\alpha-\beta)=f(\alpha)\,.f(\beta)+\dfrac{a^2}{a^2-e^2}F(\alpha)\,.\,F(\beta), & \quad (3) \\ a\,.F(\alpha-\beta)=F(\alpha)\,.f(\beta)-f(\alpha)\,.\,F(\beta). & \quad (4) \end{array}\right\} G.$$

In these formulæ no attention to the signs of the terms is now required, because they are adapted to each curve, by the circumstance of the eccentricity of the ellipse being less, and that of the hyperbola greater than the transverse axis.

By adding and substracting (1) and (3), also (2) and (4), and, to simplify, making the semitransverse $a=1$, we obtain

$$\left.\begin{array}{ll} f(\alpha+\beta)+f(\alpha-\beta)=2f(\alpha)f(\beta), & \quad (5) \\ f(\alpha+\beta)-f(\alpha-\beta)=\dfrac{2}{1-e^2}F(\alpha)\,F(\beta), & \quad (6) \\ F(\alpha+\beta)+F(\alpha-\beta)=2\,F(\alpha)f(\beta), & \quad (7) \\ F(\alpha+\beta)-F(\alpha-\beta)=2f(\alpha)\,F(\beta). & \quad (8) \end{array}\right\} H.$$

By putting $n\,\alpha$ instead of α, and α instead of β, also x for $f(\alpha)$, and y for $F(\alpha)$, these formulæ become

$$\left.\begin{array}{ll} f\{(n+1)\,\alpha\}+f\{(n-1)\,\alpha\}=2\,x\,.\,f(n\,\alpha)\,; & \quad 1 \\ f\{(n+1)\,\alpha\}-f\{(n-1)\,\alpha\}=\dfrac{2}{1-e^2}\,y\,.\,F(n\,\alpha)\,; & \quad 2 \\ F\{(n+1)\,\alpha\{+F\{(n-1)\,\alpha\}=2\,x\,.\,F(n\,\alpha)\,; & \quad 3 \\ F\{(n+1)\,\alpha\}-F\{(n-1)\,\alpha\}=2\,y\,.f(n\,\alpha). & \quad 4 \end{array}\right\} K.$$

Three of these formulæ, *viz.* the first, third, and fourth are exactly the same for the ellipse and hyperbola; in the remaining one (the second), the sign of the term that contains $F(n\alpha)$ changes, it being negative for the ellipse and positive for the hyperbola.

8. From these general formulæ, we may deduce any number of particular values, by making n equal to 0, 1, 2, 3, &c. successively. Thus, from the first and third, we find

$f(0\alpha)=1$	$F(0\alpha)=0$
$f(\alpha)=x$	$F(\alpha)=y$
$f(2\alpha)=2x^2-1$	$F(2\alpha)=2xy$
$f(3\alpha)=4x^3-3x$	$F(3\alpha)=(4x^2-1)y$
$f(4\alpha)=8x^4-8x^2+1$	$F(4\alpha)=(8x^3-4x)y$
$f(5\alpha)=16x^5-20x^3+5x.$	$F(5\alpha)=(16x^4-12x^2+1)y.$
&c.	&c.

and in this way we may proceed to any extent in deducing formulæ, which are identically the same for the ellipse and hyperbola.

9. Since the relation of every two adjoining terms of the two series $f(\alpha)$, $f(2\alpha)$, $(f3\alpha)$, &c., and $F(\alpha)$, $F(2\alpha)$, $F(3\alpha)$, &c. is exactly the same in the two curves it follows, that n being any whole number, $f(n\alpha)$, and $F(n\alpha)$ will be the same function of x and y in the two curves; this was the property particularly noticed by MACLAURIN, in the case of $f(n\alpha)$,* and employed in deducing VIETA's Theorems from a property of an equilateral hyperbola derived from the theory of logarithms. I shall now deduce the same properties from the formulæ which have been here investigated.

10. To abridge, let us denote $f(n\alpha)$ and $(Fn\alpha)$, the co-ordinates of an elliptic or hyperbolic sector (which is equal to the sector denoted by α taken n times), by the more simple symbols X_n and Y_n, then, x, and y as before, being put for the co-ordinates of the sector α, the general formulæ

$$f\{(n-1)\alpha\}+f\{(n+1)\alpha\}=2xf(n\alpha),$$
$$F\{(n-1)\alpha\}+F\{(n+1)\alpha\}=2xF(n\alpha),$$

will be expressed by fewer characters, thus,

$$X_{n-1}+X_{n+1}=2xX_n,$$
$$Y_{n-1}+Y_{n+1}=2xY_n.$$

Let z denote an arbitrary quantity, which is to be introduced as an analytical artifice to generate a series of terms; and, by giving successive values to the num-

* MACLAURIN's *Fluxions*, Article 757.

ber n in the two preceding formulæ, let there be formed two series of equations, as follows:

$$
\begin{array}{ll}
z\,X_0-2\,x\,z\,X_1+z\,X_2=0 & z\,Y_0-2\,x\,z\,Y_1+z\,Y_2=0 \\
z^2X_1-2\,x\,z^2X_2+z^2X_3=0 & z^2Y_1-2\,x\,z^2Y_2+z^2Y_3=0 \\
z^3X_2-2\,x\,z^3X_3+z^3X_4=0 & z^3Y_2-2\,x\,z^3Y_3+z^3Y_4=0 \\
\&c. & \&c.
\end{array}
$$

Put

$$
\begin{array}{l}
P=z\,X_1+z^2X_2+z\,X_3 \;\ldots\ldots\; +z^{n-1}X_{n-1}+z^nX_n+,\ \&c. \\
Q=z\,Y_1+z\,Y_2+z^3Y_3 \;\ldots\ldots\; +z^{n-1}Y_{n-1}+z^nY_n+,\ \&c.
\end{array}
$$

Then, by adding into one sum each of the two series of equations, we have

$$z\,(X_0+P)\,-2\,x\,P+\frac{P}{z}-X_1=0,$$

$$z\,(Y_0+Q)-2\,x\,Q+\frac{Q}{z}-Y_1=0.$$

From these two equations we find

$$P=\frac{X\,z-X_0z^2}{1-2\,x\,z-z^2},\qquad Q=\frac{Y_1z-Y_0z^2}{1-2\,x\,z+z^2}:$$

Now, $\qquad X_0=1,\quad X=x,\quad Y_0=0,\quad Y_1=y:$

Therefore, the values of P and Q are

$$P=\frac{(x-z)\,z}{1-2\,x\,z+z^2},\qquad Q=\frac{y\,z}{1-2\,x\,z+z^2}:$$

Let the expansion of the fraction $\frac{1}{1-2\,x\,z+z^2}$, be

$$C_0+C_1\,z+C_2z^2\ \ldots\ +C_{n-2}\,z^{n-2}+C_{n-1}z^{n-1}+C_nz^n+\&c.$$

we have now

$$
\begin{array}{lll}
 & P=x\,C_0\,z+(x\,C_1-C_0)\,z^2 \;\ldots & +(x\,C_{n-1}-C_{n-2})\,z^n+\&c. \\
 & Q=y\,C_0\,z+y\,C^1z \;\ldots & +y\,C_{n-1}\,z^n+\&c. \\
\text{But} & P=X_1\,z+X_2\,z^2 \;\ldots & +X_n\,z^n+\&c. \\
\text{and} & Q=Y_1z\ +Y_2\,z^2 \;\ldots & +Y_n\,z^n+\&c.
\end{array}
$$

Hence it follows that

$$X_n=x\,C_{n-1}-C_{n-2},\qquad Y_n=y\,C_{n-1}.\ \ldots\ldots\ \text{L.}$$

To simplify, let us put u instead of $2\,x$ in the expression $1-2\,x\,z+z^2$, so that it becomes $1-(u-z)\,z$, we have now by division $\frac{1}{1-2\,x\,z+z^2}=\frac{1}{1-(u-z)\,z}$ equal to

$$1+z\,(u-z)+z^2\,(u-z)^2\ \ldots\ +z^{n-2}\,(u-z)^{n-2}+z^{n-1}\,(u-z)^{n-1}+\&c.$$

Let A_0 and B_0 be the coefficients of z^0 in the expansions of $(u-z)^{n-1}$ and $(u-z)^{n-3}$, and let A_1 and B_1 be the coefficients of z in the expansions of $(u-z)^{n-2}$ and $(u-z)^{n-3}$, and A_2 and B_2 the coefficients z^2 in the expansions of $(u-z)^{n-3}$ and $(u-z)^{n-4}$, and in general A_p and B_p the coefficients z^p in the expansions of $(u-z)^{n-1-p}$ and $(u-z)^{n-2-p}$, then it is easy to see that

$$C_{n-1}=A_0+A_1+A_2+A_3+\&c.$$
$$C_{n-2}=B_0+B_1+B_2+B_3+\&c.$$

Now we have evidently

$$A_0=u^{n-1} \qquad B_0=u^{n-2}$$

$$A_1=-(n-2)\,u^{n-3} \qquad B_1=-(n-3)\,u^{n-4}$$

$$A_2=+\frac{(n-3)\,(n-4)}{1\,.\,2}\,u^{n-5} \qquad B_2=+\frac{(n-4)\,(n-5)}{1\,.\,2}\,u^{n-6}$$

$$A_3=-\frac{(n-4)\,(n-5)\,(n-6)}{1\,.\,2\,.\,3}\,u^{n-7} \qquad B_3=-\frac{(n-5)\,(n-6)\,(n-7)}{1\,.\,2\,.\,3}\,u^{n-8}$$

&c. &c.

$$2\,x\,C_{n-1}=u^n-(n-2)\,u^{n-2}+\frac{(n-3)\,(n-4)}{1\,.\,2}\,u^{n-4}-\frac{(n-4)\,(n-5)\,(n-6)}{1\,.\,2\,.\,3}\,u^{n-6}+\&c.$$

$$2\,C_{n-2}=2\,u^{n-2}-2\,(n-3)\,u^{n-4}+\frac{2(n-4)\,(n-5)}{1\,.\,2}\,u^{n-6}-\frac{2(n-5)\,(n-6)\,(n-7)}{1\,.\,2\,.\,3}\,u^{n-8}+\&c.$$

Observing now that $f(n\,a)=X_n=x\,C_{(n-1)}-C_{(n-2)}$,

and $F\,(n\,a)=Y_n=y\,C_{n-1}$,

we have, after replacing u by $2\,x$,

M

$$2f(n\,\alpha)=(2\,x)^n-n\,(2\,x)^{n-2}+\frac{n\,(n-3)}{1\,.\,2}(2\,x)^{n-4}-\frac{n\,(n-4)\,(n-5)}{1\,.\,2\,.\,3}(2\,x)^{n-6}+\&c.$$

$$F\,(n\,\alpha)=y\,\{(2\,x)^{n-1}-(n-2)\,(2\,x)^{n-3}+\frac{(n-3)\,(n-4)}{1\,.\,2}(2\,x)^{n-5}-\frac{(n-4)\,(n-5)\,(n-6)}{1\,.\,2\,.\,3}\,(2\,x)^{n-7}\,\&c.\}$$

These expressions for $f\,(n\,\alpha)$ and $F\,(n\,\alpha)$ are entirely independent of the quantity $c=\pm\frac{a^2}{b^2}$ they are, therefore, identically the same for the ellipse and hyperbola. And if the axes of the ellipse be supposed equal, they become the known formulæ for the cosine and sine of the multiple of an arc, which, in substance, were found by VIETA.* The angular analysis was not, however, sufficiently advanced to enable him to express them by general formulæ, otherwise than by shewing how any number of particular cases might be found.

11. That we may obtain values of the functions $f\,(n\,\alpha)$, $F\,(n\,\alpha)$ in another form, we must find a second development of the fraction $\frac{1}{1-2\,x\,z+z^2}$.

* FRANCISCI VIETA *Opera Mathematica.* *Leyden,* 1646 (pp. 295, 297).

The expression $1-2xz+z^2$ may have this form $(x-z)^2-(x^2-1)$; now in the hyperbola $x^2-1=cy^2$; therefore, denoting this last quantity by u^2, we have

$$1-2xz+z^2=(x-z)^2-u^2=(x-u-z)(x+u-z).$$

And
$$\frac{1}{1-2xz+z^2}\left\{\begin{array}{l} =\dfrac{1}{(x-u-z)(x+u-z)} \\ =\dfrac{1}{2u}\left\{\dfrac{1}{x-u-z}-\dfrac{1}{x+u-z}\right\}. \end{array}\right.$$

Now
$$\frac{1}{x-u-z}=\frac{1}{x-u}+\frac{z}{(x-u)}+\frac{z^2}{(x-u)^3}\quad\ldots\quad+\frac{z^{n-2}}{(x-u)^{n-1}}+\frac{z^{n-1}}{(x-u)^n}\ \&c.$$

And
$$\frac{1}{x+u-z}=\frac{1}{x+u}+\frac{z}{(x+u)^2}+\frac{z^2}{(x+u)^3}\quad\ldots\quad+\frac{z^{n-2}}{(x+u)^{n-1}}+\frac{z^{n-1}}{(x+u)^n}\ \&c.$$

Hence, by subtracting, and observing that $x^2-u^2=1$, and putting $\frac{2u}{1-2xz+z^2}$ for $\frac{1}{x-u-z}+\frac{1}{x+u-z}$, we have

$$\frac{2u}{1-2xz+z^2}=\left\{\begin{array}{l} 2u+\{(x+u)^2-(x-u)^2\}z+\{(x+u)^3-(x-u)\}z^2\quad\ldots \\ +\{(x+u)^{n-1}-(x-u)^{n-1}\}z^{n-2}+\{(x+u)^n-(x-u)^n\}z^{n-1}\ \&c. \end{array}\right.$$

Now the expansion of the fraction $\frac{1}{1-2xz+z^2}$ being

$$1+C_1z+C_2z^2+C_3z^3\quad\ldots\quad+C_{n-2}z^{n-2}+C_{n-1}z^{n-1}+\&c.$$

it appears, from what has been just now found, that

$$C_{n-2}=\frac{1}{2u}\left\{(x+u)^{n-1}-(x-u)^{n-1}\right\};$$

$$C_{n-1}=\frac{1}{2u}\left\{(x+u)^n-(x-u)^n\right\}.$$

And we found (Article 10, L) that

$$f(n\alpha)=x\,C_{n-1}-C_{n-2};\qquad F(n\alpha)=y\,C_{n-1}:$$

Therefore,
$$f(n\alpha)=\frac{1}{2u}\left\{(x^2-1+ux)(x+u)^{n-1}-(x^2-1-ux)(x-u)^{n-1}\right\};$$

But $x^2-1=u^2$; therefore $x^2-1+ux=\ u(x+u)$,

and $x^2-1-ux=-u(x-u)$;

And hence . . . . . $2f(n\alpha)=(x+u)^n+(x-u)^n$;

and . . . . . . $2uF(n\alpha)=\{(x+u)^n-(x-u)^n\}y$;

Or, since . . . . . $u^2=y^2c$, and $u=y\sqrt{c}$;

$$\left.\begin{array}{r} 2f(n\alpha)=(x+y\sqrt{c})^n+(x-y\sqrt{c})^n; \\ 2F(n\alpha)\sqrt{c}=(x+y\sqrt{c})^n-(x-y\sqrt{c})^n. \end{array}\right\}\quad\ldots\ldots\ldots\quad N$$

From these formulæ, putting $f(\alpha)$ for x, and $F(\alpha)$ for y, we have

$$f(n\alpha)+F(n\alpha).\sqrt{c}=\{f(\alpha)+F(\alpha).\sqrt{c}\}^n,$$

$$f(n\alpha)-F(n\alpha).\sqrt{c}=\{f(\alpha)-F(\alpha).\sqrt{c}\}^n;$$

and hence, again, n and m being any whole numbers,

$$f(\alpha)+F(\alpha).\sqrt{c}\begin{cases}=\left\{f(n\alpha)+F(n\alpha).\sqrt{c}\right\}^{\frac{1}{n}},\\=\left\{f(m\alpha)+F(m\alpha).\sqrt{c}\right\}^{\frac{1}{m}};\end{cases}$$

therefore, $$f(m\alpha)+F(m\alpha).\sqrt{c}=\left\{f(n\alpha)+F(n\alpha).\sqrt{c}\right\}^{\frac{m}{n}};$$

and, putting $n\alpha=\alpha'$, so that $m\alpha=\frac{m}{n}\alpha'$; we have

$$f\left(\frac{m}{n}\alpha'\right)+F\left(\frac{m}{n}\alpha'\right).\sqrt{c}=\left\{f(\alpha')+F(\alpha').\sqrt{c}\right\}^{\frac{m}{n}}:$$

and putting α instead of α', and again x and y instead of $f(\alpha)$ and $F(\alpha)$,

$$f\left(\frac{m}{n}\alpha\right)+F\left(\frac{m}{n}\alpha\right).\sqrt{c}=\left(x+y\sqrt{c}\right)^{\frac{m}{n}}.$$

Exactly in the same way we prove that

$$f\left(\frac{m}{n}\alpha\right)F-\left(\frac{m}{n}\alpha\right).\sqrt{c}=\left(x-y\sqrt{c}\right)^{\frac{m}{n}}.$$

In the hyperbola $x^2-cy^2=(x+y\sqrt{c})(x-y\sqrt{c})=1$,

and in general $\{f(n\alpha)+F(n\alpha).\sqrt{c}\}\{f(n\alpha)-F(n\alpha).\sqrt{c})\}=1$.

Now, as in the circle, we consider the cosine of a positive and negative angle at the centre to be equal in magnitude, and to have the same sign; but their sines to be equal in magnitude, with contrary signs, so, by analogy, in the ellipse and hyperbola, we must reckon $f(+n\alpha)=f(-n\alpha)$, but $F(+n\alpha)=-F(-n\alpha)$; and hence we have

$$\{f(n\alpha)+F(+n\alpha).\sqrt{c}\}\{f(-n\alpha)+F(-n\alpha).\sqrt{c}\}=1,$$

and $f(-n\alpha)+F(-n\alpha).\sqrt{c}=\{f(+n\alpha)+F(+n\alpha).\sqrt{c}\}^{-1}=(x+y\sqrt{c})^{-n}$.

In the same way we find $f(-n\alpha)-F(-n\alpha).\sqrt{c}=(x-y\sqrt{c})^{-n}$.

So that, on the whole, whether n be a positive, or a negative whole number, or a fraction; in each case

$$\left.\begin{aligned}f(n\alpha)+F(n\alpha).\sqrt{c}=(x+y\sqrt{c})^n;\\f(n\alpha)-F(n\alpha).\sqrt{c}=(x-y\sqrt{c})^n.\end{aligned}\right\}\quad.\ .\ .\ .\ .\ .\quad N'$$

These formulæ are perfectly definite and intelligible, when confined to the hyperbola; and numerical values being assigned to the quantities n, x, y, c; the values of $f(n\,\alpha)$, and $F(n\,\alpha)$ may be expressed in real numbers: They lose, however, this property, when extended to the ellipse or circle, by reason of the symbol $\sqrt{c}$, which, in this case, becomes $\frac{b}{a}\sqrt{-1}$ an imaginary quantity. Still, however, they are not insignificant, for they express real quantities, although under an imaginary form.*

12. We meet with the same peculiar form of expression in the elements of Algebra. Thus, the value of x being required from the two equations

$$x^2 + c\,y^2 = a^2\,; \qquad x\,y = b^2:$$

we have

$$x^2 + 2\,x\,y\sqrt{c} + c\,y^2 = a^2 + 2\,b^2\sqrt{c}\,;$$

$$x^2 - 2\,x\,y\sqrt{c} + c\,y^2 = a^2 - 2\,b^2\sqrt{c}\,;$$

and taking the square roots, we get

$$x + y\sqrt{c} = \sqrt{(a^2 + 2\,b^2\sqrt{c})}\,; \qquad x - y\sqrt{c} = \sqrt{(a^2 - 2\,b^2\sqrt{c})}\,;$$

and

$$x = \tfrac{1}{2}\{\sqrt{(a^2 + 2\,b^2\sqrt{c})} + \sqrt{(a^2 - 2b^2\sqrt{c})}\}:$$

Suppose now $c = +1$, then $x = \frac{1}{2}\{\sqrt{(a^2 + 2\,b^2)} + \sqrt{(a^2 - 2\,b^2)}\}$;

but if $c = -1$, then $x = \frac{1}{2}\{\sqrt{(a^2 + 2\,b^2\sqrt{-1})} + \sqrt{(a^2 - 2\,b^2\sqrt{-1})}\}$.

In the first case the value of x is *real*, but in the second it is illusory, because it involves the imaginary symbol $\sqrt{-1}$. We can, however, eliminate $\sqrt{-1}$; thus taking the square of the expressions for x, we have

$$x^2 = \tfrac{1}{2}\{\,a^2 + \sqrt{(a^4 +)}\sqrt{(a^4 - 4\,b^4\,c)}\,\}\,;$$

The square root of this quantity gives a real value for x, whether c be positive or negative.

The same value for x^2 may, however, be found from the proposed equations by proceeding in a different way: Thus subtracting four times the square of $x\,y\sqrt{c}\,(=b^2\sqrt{c})$ from the squares of the sides of the first equation, we have

$$x^4 - 2\,x^2\,y^2\,c + c^2\,y^4 = a^4 - 4\,b^4\,c\,;$$

and taking the square roots,

$$x^2 - c\,y^2 = \sqrt{(a^4 - 4\,b^4\,c)}\,;$$

* These important analytical expressions were found by De Moivre in 1707, and inserted in the *Philosophical Transactions* of that year; and again in the *Transactions* for 1722. They are also in his *Miscellanea Analytica*, printed at London 1730.

From this, and the first given equation, there is obtained

$$x^2=\tfrac{1}{2}\{a^2+\sqrt{(a^4-4\,b^4c)}\}.$$

The same result as was deduced from the first solution but by a different process.

13. In the preceding example the imaginary expression $\sqrt{-1}$ has been eliminated by a transformation, which has brought together two terms with opposite signs. In a similar way we shall eliminate it from formulæ (N).

By the binomial theorem, and putting A_1, A_2, A_3, &c., for the coefficients of the terms containing the first, second, third powers, &c. of y in the development of $(1+y)^n$, we have

$$(x+y\sqrt{c})^n=x^n+A_1x^{n-1}\,y\sqrt{c}+A_2x^{n-2}\,y^2\,c+A_3x^{n-3}\,y^3\,c\sqrt{c}+\&c.$$
$$(x-y\sqrt{c})^n=x^n-A_1x^{n-1}\,y\sqrt{c}+A_2x^{n-2}\,y^2\,c-A_3x^{n-3}\,y^3\,c\sqrt{c}+\&c.$$

Now from formulæ (N),

$$\left.\begin{aligned} f(n\,\alpha)&=\tfrac{1}{2}\{(x+y\sqrt{c})^n+(x-y\sqrt{c})^n\}\,;\\ \sqrt{c}\,\mathrm{F}\,(n\,\alpha)&=\tfrac{1}{2}\{(x+y\sqrt{c})^n-(x-y\sqrt{c})^n\}.\end{aligned}\right\}\quad .\;.\;.\;.\;.\quad \mathrm{O}$$

Therefore, by adding and subtracting, there is found

$$\left.\begin{aligned} f(n\,\alpha)&=x^n \qquad\quad +A_2\,x^{n-2}\,y^2\,c+A_4\,x^{n-4}\,y^4\,c^2+\&c.\\ \mathrm{F}(n\,\alpha)&=A_1x^{n-1}y+A_3\,x^{n-3}\,y^3\,c+A_5\,x^{n-5}\,y^5\,c^2+\&c.\end{aligned}\right\}\quad .\;.\;.\;.\;.\quad \mathrm{P}$$

These series will terminate when n is a whole number, otherwise they will proceed *ad infinitum*. In the circle, or ellipse, c must have the sign $-$, but in the hyperbola, the sign $+$.

John Bernoulli found these theorems in the case of the circle, and gave them in the *Leipsic Acts* for 1704, but without demonstration.* It is remarkable that, knowing them, he did not discover also De Moivre's theorem, which has been the germ of the finest discoveries in geometry.

14. In the circle, $tan\ \alpha$, the trigonometrical tangent of an angle α at its centre is equal to $\dfrac{sin\ \alpha}{cos\ \alpha}$. In the ellipse or hyperbola, if a straight line touch the curve at the vertex of the transverse axis, the segment of this line between the vertex and any semidiameter is equal to $\dfrac{\mathrm{F}\,(\alpha)}{f\,(\alpha)}$; it may therefore be considered as a function of the sector, analogous to that which the trigonometrical tangent is of its angle; and may be similarly designated by the symbol $\mathrm{T}\,(\alpha)$.

* Joannis Bernouilli, *Opera*, vol. i. pp. 387 and 511.

We have now from Formulæ O, in either curve

$$T(n\,\alpha)=\frac{1}{\sqrt{c}}\cdot\frac{(x+y\sqrt{c})^n-(x-y\sqrt{c})^n}{(x+y\sqrt{c})^n+(x-y\sqrt{c})^n};$$

or, putting t for $T(\alpha)=\frac{y}{x}$, so that $y=t\,x$,

$$\{T(n\,\alpha)\}\sqrt{c}=\frac{(1+t\sqrt{c})^n-(1-t\sqrt{c})^n}{(1+t\sqrt{c})^n+(1-t\sqrt{c})^n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{Q}$$

In the ellipse, because c is negative, so that $\sqrt{-c}=\sqrt{+c}\,.\sqrt{-1}$, the formula involves the imaginary symbol $\sqrt{-1}$: this, however, disappears when the expressions $(1+t\sqrt{c})^n$ and $\sqrt{(1-t\sqrt{c})^n}$ are expanded into series; and united by subtraction and addition: We have then, putting A_1, A_2, A_3, A_4, &c. for the coefficients of t, t^2, t^3, t^4, &c. in the expansion of the binormal $(1+t)^n$,

$$T(n\,\alpha)=\frac{A_1t+A_3c\,t^3+A_5c^2t^5+\&c.}{1\;+A_2c\,t^2+A_4c^2t^4+\&c.}; \quad . \quad . \quad . \quad . \quad \text{R.}$$

For the circle or ellipse, the terms in this formula containing the odd powers of c, *viz.* the first, third, &c. must have the sign $-$, and the remainder the sign $+$: But in the hyperbola, they must all have the sign $+$; in either case the expression for the tangent of the sector contains only *real* quantities.

15. In formula Q let us put $\frac{a}{b}$ for $\sqrt{c}$ and $\frac{a\,y}{x}$ for t, we have then in the hyperbola,

$$\frac{T(n\,\alpha)}{b}=\frac{\left\{\frac{x}{a}+\frac{y}{b}\right\}^n-\left\{\frac{x}{a}-\frac{y}{b}\right\}^n}{\left\{\frac{x}{a}+\frac{y}{b}\right\}^n+\left\{\frac{x}{a}-\frac{y}{b}\right\}^n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{S.}$$

Let us now put $\frac{x}{a}+\frac{y}{b}=r$, then, because $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$; and $\frac{x^2}{a^2}-\frac{y^2}{b^2}=\left(\frac{x}{a}+\frac{y}{b}\right)\left(\frac{x}{a}-\frac{y}{b}\right)$;

therefore, $r\left(\frac{x}{a}-\frac{y}{b}\right)=1$, and $\frac{x}{a}-\frac{y}{b}=\frac{1}{r}$: Formula S may now be expressed thus,

$$\frac{T(n\,\alpha)}{b}=\frac{r^n-\frac{1}{r^n}}{r^n+\frac{1}{r^n}}=\frac{r^{2n}-1}{r^{2n}+1}, \quad \text{and } r^{2n}=\frac{b+T(n\,\alpha)}{b-T(n\,\alpha)},$$

and putting $\frac{1}{m}$ for $2\,n$, and $\frac{1}{2\,m}$ for n;

$$r=\left\{\frac{b+T\left(\frac{\alpha}{2\,m}\right)}{b-T\left(\frac{\alpha}{2\,m}\right)}\right\}^m.$$

Suppose now m to be a large number, then, the sector $\frac{\alpha}{2m}$ will be small; and its area will have to the area of the triangle whose base is the semidiameter, and height the tangent of the sector almost a ratio of equality: now, the area of this triangle is $\frac{a}{2}\cdot \mathrm{T}\left(\frac{\alpha}{2m}\right)$, therefore, m being a very large number,

$$\frac{\alpha}{2m}=\frac{a}{2}\cdot \mathrm{T}\left(\frac{\alpha}{2m}\right) \text{ and } \mathrm{T}\left(\frac{\alpha}{2m}\right)=\frac{\alpha}{m\,a}:$$

we have now
$$r=\left\{\frac{b+\frac{1}{a}\cdot\frac{\alpha}{m}}{b-\frac{1}{a}\cdot\frac{\alpha}{m}}\right\}^m=\left\{1+\frac{\frac{2}{a}\cdot\frac{\alpha}{m}}{b-\frac{1}{a}\cdot\frac{\alpha}{m}}\right\}^m,$$

now the space α, *viz.* the area of the sector, is a finite magnitude, and m is by hypothesis a great number: therefore, the lineal quantity $\frac{1}{a}\cdot\frac{\alpha}{m}$ is small, and may be neglected in respect of the finite line b, and m being increased continually, we have

$$r=\left\{1+\frac{1}{m}\cdot\frac{2\,\alpha}{a\,b}\right\}^m.$$

16. Let us now assume that $\frac{2\,\alpha}{m\,.\,ab}=n$; and since $2\,\alpha$ and $a\,b$ are finite spaces, and m is a large number, n must be a small fraction. We have now $m=\frac{2\,\alpha}{n\,a\,b}$, and

$$r=(1+n)^{\frac{2\,\alpha}{n\,a\,b}}=\left\{(1+n)^{\frac{1}{n}}\right\}^{\frac{2\,\alpha}{a\,b}}.$$

Now, considering that r is a function of α, let e be the value of r when $2\,\alpha=a\,b$; then e will be a definite number, which may be found from this expression,

$$e=(1+n)^{\frac{1}{n}}.$$

To abridge, let us put $r_0=e^{\frac{2}{a\,b}}$, then r_0 will also be a definite number; and we shall have $r=r_0^{\alpha}$, that is, restoring the quantity denoted by r.

$$\frac{x}{a}+\frac{y}{b}=r_0^{\alpha}\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad \mathrm{T}$$

17. The co-ordinates of the sector α being x and y, let the co-ordinates of another sector $\alpha_{,}$, be $x_{,}$, and $y_{,}$, so that

$$\frac{x_{,}}{a}+\frac{y_{,}}{b}=r_0^{\alpha_{,}}.$$

Then
$$\left(\frac{x}{a}+\frac{y}{b}\right)\left(\frac{x_{,}}{a}+\frac{y_{,}}{b}\right)=r^{\alpha+\alpha_{,}}. \quad . \quad . \quad . \quad . \quad (b)$$

Let X and Y be the co-ordinates of a third sector equal to $\alpha+\alpha_{,}$, then we have
$$\frac{\mathrm{X}}{a}+\frac{\mathrm{Y}}{b}=r_0^{\alpha+\alpha_{,}}=r_0^{\alpha}\cdot r_0^{\alpha_{,}}=\left\{\frac{x}{a}+\frac{y}{b}\right\}\left\{\frac{x_{,}}{a}+\frac{y_{,}}{b}\right\}.$$

Now, considering α as a function of $\frac{x}{a}+\frac{y}{b}$, let us put
$$\alpha=f'\left\{\frac{x}{a}+\frac{y}{b}\right\}; \text{ then } \alpha_{,}=f'\left\{\frac{x_{,}}{a}+\frac{y_{,}}{b}\right\}, \text{ and } \alpha+\alpha_{,}=f'\left\{\frac{\mathrm{X}}{a}+\frac{\mathrm{Y}}{b}\right\};$$

we have now manifestly,
$$f'\left(\frac{x}{a}+\frac{y}{b}\right)+f'\left(\frac{x_{,}}{a}+\frac{y_{,}}{b}\right)=f'\left\{\left(\frac{x}{a}+\frac{y}{b}\right)\cdot\left(\frac{x_{,}}{a}+\frac{y_{,}}{b}\right)\right\}.$$

But, in any system of logarithms,
$$\log\left(\frac{x}{a}+\frac{y}{b}\right)+\log\left(\frac{x_{,}}{a}+\frac{y_{,}}{b}\right)=\log\left\{\left(\frac{x}{a}+\frac{y}{b}\right)\cdot\left(\frac{x_{,}}{a}+\frac{y_{,}}{b}\right)\right\}.$$

Hence it appears that the sectors α, $\alpha_{,}$, $\alpha+\alpha_{,}$ are related among themselves exactly as the logarithms of the quantities $\frac{x}{a}+\frac{y}{b}$, $\frac{x_{,}}{a}+\frac{y_{,}}{b}$, $\frac{\mathrm{X}}{a}+\frac{\mathrm{Y}}{b}$.

We have now this important property of the hyperbola;
Let x and y be the co-ordinates of α, a sector of a hyperbola whose transverse and conjugate semi-axes are a and b; then C *being some given number,* C α *is the logarithm of* $\frac{x}{a}+\frac{y}{b}$.

This theorem, at least one deducible from it, was first discovered by GREGORY *of St Vincent,** and was a most important step in the theory of logarithms, for it identified their construction with the quadrature of the hyperbola, a problem resolved by MERCATOR† and BROUNKER. This beautiful analogy between logarithms and hyperbolic sectors led to great improvements in their computation; these, however, came too late to be of practical use; for before it was found, the great labour of computing tables of logarithms had been accomplished. Its discovery induced the geometers of that day to regard logarithms as a geometrical theory; but Dr HALLEY shewed that although the theory of logarithms had some relations with geometry, yet it was properly a purely arithmetical theory, and as such he treated it.‡

* GREGORII a *S. Vincentio Vera Quadratura Circuli et Hyperbolæ.* Antwerp, 1647.

† MERCATOR. *Logarithmotechnia,* &c. London, 1668.

‡ *Philosophical Transactions* (No. 27), vol. i., LOWTHORPE'S *Abridgment.*

18. We have found that $\frac{x}{a}+\frac{y}{b}=e^{\frac{2\alpha}{ab}}$, and here $e=(1+n)^{\frac{1}{n}}$, and n is a very small fraction. This function e, from its singular form, might be supposed indeterminate; yet its value is readily found, by expanding it by the binomial theorem, to be

$$e=1+\frac{1}{1}+\frac{1}{1.2}+\frac{1}{1.2.3.4}+\frac{1}{1.2.3.4.5}+\&c.=2.7182818284.$$

This number e is the *Base* of NEPER's logarithms, and therefore

$$\frac{2\alpha}{ab}=\text{Nep. log}\left(\frac{x}{a}+\frac{y}{b}\right),\quad \text{and } \alpha=\frac{ab}{2}\text{ Nep. log}\left(\frac{x}{a}+\frac{y}{b}\right) \quad \ldots\ldots \quad \text{U}$$

And since $\left(\frac{x}{a}+\frac{y}{b}\right)=\frac{1}{\frac{x}{a}-\frac{y}{b}}$; Therefore also $\frac{x}{a}-\frac{y}{b}=e^{-\frac{2\alpha}{ab}}$;

and hence

$$\left.\begin{aligned}\frac{x}{a}&=\tfrac{1}{2}\left\{e^{\frac{2\alpha}{ab}}+e^{-\frac{2\alpha}{ab}}\right\},\\ \frac{y}{b}&=\tfrac{1}{2}\left\{e^{\frac{2\alpha}{ab}}-e^{-\frac{2\alpha}{ab}}\right\}.\end{aligned}\right\} \quad \ldots\ldots \quad \text{X}$$

By a process exactly similar, we might have deduced, in the case of the circle, supposing $a=b=1$, these other important formulæ:

$$\left.\begin{aligned}\cos\alpha=x&=\tfrac{1}{2}\left\{e^{\alpha\sqrt{-1}}+e^{-\alpha\sqrt{-1}}\right\}.\\ \sqrt{-1}\sin\alpha=y&=\tfrac{1}{2}\left\{e^{\alpha\sqrt{-1}}-e^{-\alpha\sqrt{-1}}\right\}.\end{aligned}\right\} \quad \ldots \quad \text{Y}$$

Here α denotes, not the sector, as in the hyperbola, but the angle of which x is the cosine, and y the sine. But the various steps of the process would be almost exactly the same as those by which the corresponding properties of the hyperbola have been found.

We have now passed, by a simple and uniform analysis, from the definitions, and the most elementary geometrical properties of the conic sections, to some of their most recondite properties. With these exponential formulæ, which were turned to great account by EULER,* and have been pronounced by LAGRANGE to be the finest analytical discoveries of the last century,† I conclude this memoir.

* *Miscellanea Berolinensia*, tome vii.; and *Introductio in Analysin Infinitum*, t. i.

† LAGRANGE. *Leçons sur le Calcul des Fonctions*, p. 114.

XXIII.—*Notice respecting the Depletion or Drying up of the Rivers Teviot, Nith, and Clyde, on the 27th November* 1838. *By* DAVID MILNE, *Esq., F. R. S. E., F. G. S.*

Read March 18. 1839.

THE Teviot, Clyde, and Nith, are well known to be among the largest rivers in the south of Scotland. In the lower parts of their course they are navigable; and all along their banks, nearly up to their sources, there are innumerable mills and manufactories, dependent on the continuous flow of their waters.

On the morning of the 27th November 1838, the channels of these rivers were, in the upper and middle parts of their course, found almost entirely empty. The thousands of water wheels, many of which had for years, without interruption, been turned by their currents, suddenly stopped. Immense quantities of fish, inhabitants of their deep and rapid streams, were destroyed by being left dry, or being caught with the hand in deserted pools; and in places where it was usually difficult for even horses to ford, it was easy for children to walk across without wetting their feet.

The phenomenon of a large river thus disappearing entirely from its channel, for many miles, was one in its own nature well calculated to excite interest. But being accompanied by the striking circumstances above mentioned, it could not fail to excite astonishment among all living on its banks, and no small degree of alarm among those who, in various ways, depended on these rivers for daily occupation and support. The subject became naturally one of general interest, and occasioned much speculation and inquiry. The interest so excited was greatly increased, on its being discovered that the phenomenon had happened not merely in one large river, but in several; and that, though these rivers were far distant from each other, the phenomenon had occurred in all, on the same day, and even about the same time of that day.

On making local inquiries, it was ascertained that the phenomenon was not without precedent. In the Scots Magazine, and other periodical repertories of remarkable occurrences, accounts were found of the disappearance of several rivers from their channels, at different periods, within the last century. These periodicals contain long and frequently renewed discussions as to the cause of the phenomenon, which show how much the public attention was interested by it, and how little the various observers, or even scientific men, were agreed as to the true explanation.

When the phenomenon was repeated last November, though on a scale much more striking than on any former occasion, the same speculations were excited, and the same diversity of opinion prevailed, as had previously existed. It is of course unnecessary to advert to all the theories which were started, but the most plausible of them may be noticed.

Some persons conceived, that the strong south-easterly wind which blew during the night of the 26th and following morning, was of itself sufficient to produce the effect. It was observed that the Teviot and the Nith flowed, for the greater part of their courses, in an easterly direction, and that the Clyde, in the higher part of its course, likewise flowed in this direction; so that the easterly gale blew up against these streams, and might have obstructed, if not stopped in some places, the flowing of their currents, and thus have cut off the usual supply of water from the lower parts of the river.

Others thought, that the frost which prevailed during the night of the 26th, and morning of the 27th, must have frozen up the springs and rivulets at the sources of the rivers; or that it, at all events, stopped the flowing of the current in those parts where there were caulds or damheads,—by forming a barrier of ice along the top of them.

Others, again, imagined, that the phenomenon might have been caused by an earthquake, or rather by the widening of the natural fissures and rents abounding in strata, which is known to take place during an earthquake. This opinion was suggested by the fact, that, in Italy, the temporary disappearance of rivulets, even of considerable size, is not an unfrequent occurrence before or during an eruption. Professor Phillips, in his article on Geology, published in the last edition of the Encyclopædia Britannica, mentions as the effect of volcanic agency in England, that, " in (the year) 1110, the Trent was dry at Nottingham for a whole day; and that, in 1158, the Thames was dry at London."—(P. 245–6.) A similar phenomenon was stated to have occurred in Scotland on the 6th January 1787, on which day the shock of an earthquake was violently felt in the parishes of Strathblane and Campsie, and a large district of country in the west of Scotland. On the same day, it appears that the Woodhead burn, which runs through the parish of Strathblane, and there turns a mill, left its channel dry for a short interval; and, what is still more remarkable, there was likewise on that day a stoppage and desiccation of the Clyde.

In farther explanation of this last theory, I may here allude to what is familiar to all geologists, that, in secondary rock districts, and especially those occupied by carboniferous strata, there is hardly an acre which is not intersected by numerous cracks or fissures, reaching from the surface to an unfathomable depth, and varying in width from a few inches to many feet. There is one of these fissures (near Musselburgh) which exceeds fifty feet in width; and throughout the Lothian coal-field, there are many hundreds of them exceeding six feet in width. Now the partial elevation of the earth's crust by subterranean action, in whatever way it takes place, must widen these natural fissures, so as to permit water to escape by them. A Fellow of this Society mentioned to me, that, in 1834, near Liverpool, when the shock was felt there of the earthquake which transmitted its vibrations through the south-eastern and midland counties of England, the slines or cutters (as these fissures are called by quarrymen) were observed to open, in a stone-quarry near Liverpool, and after a short interval to close again.

Such were the most plausible of the theories adopted to explain the remarkable desiccation of the Teviot, Clyde, and Nith, on the 27th November. But I should term them *conjectures* rather than *theories;* for they were formed without any previous ascertainment of even material facts, and therefore could not be relied on, to the least extent, as affording a true explanation of the phenomenon. I therefore deemed it to be my first business, to obtain from different persons who had themselves witnessed the circumstances, or could easily ascertain them on the spot, a full and accurate statement of what had occurred.

(1.) I shall not quote at length the reports themselves, but shall give merely a summary of the material facts and circumstances contained in them.* I may only here mention the names of the gentlemen on whose information I have relied; and I do so from a conviction, that I shall thus afford a proof to all who know these gentlemen, of the truth and accuracy of their communications.

The state and appearance of the Teviot were described in letters from

Andrew Jerdan, Esq. of *Bonjedward,* in a letter, dated 31st December 1838, addressed to Professor Forbes.

Dr Douglas Junior of *Kelso,* in a letter addressed to me, dated 21st January 1839.

Dr Wilson of *Kelso,* in two letters to me, dated 21st January 1838, and 15th March 1839.

Rev. Mr Aitken of *Minto,* in two letters to me, dated 27th January and 7th February 1839.

The temperature of the district during the 26th and 27th November, and the state of the wind and weather generally during these two days, was obtained from meteorological registers kept by the

Rev. Mr Wallace of *Abbey St Bathans,* near *Dunse.*

Mr Jerdan of *Bonjedward,* near *Jedburgh.*

Mr Dudgeon of *Spylaw,* near *Kelso.*

* The reports themselves were read to the Society, and are in the possession of the author, who will give access to them, to any one who may desire to examine them.

I may be permitted to make two short extracts, descriptive of the phenomenon generally, in order to shew more clearly the bearing of the facts as given in the summary.

The first extract I shall give is from Dr DOUGLAS's letter.

" At the mill at Maxwellheugh (immediately above the confluence of the Teviot with the Tweed), the supply of water in the dam began to diminish at 6 A. M. Nov. 27. It nearly ceased at 8 A. M. The whole water in the river was diverted by means of the cauld into the mill-lead, but the quantity was so small that the wheel could not be kept in motion. This state of things continued until 12 noon, when the flow began to be re-established, and at 1 P. M. the river had assumed its ordinary size. The miller distinctly told me, that the supply came gradually, and not *in a rush.*

" At *Hawick,* six miles above Minto, a correspondent there writes me, that, during the morning and forenoon, the mills were stopped for want of an adequate supply of water, and that near mid-day, the supply was again established, and the mills again at work. For several miles above Hawick, the river was remarkably small, and the same appearance was observed in the tributaries; but this, he observes, is of such common occurrence during frost, that it scarcely excites attention. The bed of the *Rule Water* (which joins the Teviot just at Minto), was so dry, that a friend told me he walked through the channel without wetting his feet."

Mr JERDAN, in his letter to Professor FORBES, states that, "on the morning of the 27th, which had been freezing hard during the night, and which was an uncommon cold frosty day, with a keen and bitter wind from SE., the miller, who was at his work between five and six, on setting the mill agoing, found a great rush of water, so much so, that he could hardly check it sufficiently. On going up to the sluice or grating at the head of the dam, to his astonishment he found the cauld or weir so frozen with the bitter night and high wind blowing right on it, that not a drop was going over it, and a bank or wall of ice, about 16 inches high he thought, was formed on the lip or top of the weir, by the accumulation of the floating ice, and the severity of the frost, and which kept the whole water in the pool, and so forced an additional quantity down the dam, when the floating ice was removed from the grating, which he had to do occasionally. From this obstruction, which seems to have been very effectual, the stream immediately below the weir was in a great measure dry, so much so that he could have easily gone over it. In a few hours the accumulation of water carried away the ice from one-half of the cauld, which is separated from the other by an island, and the water flowed as usual, covering all the bed. The ice on the other part of the weir, remained most part of the day.

" This easily explains the occurrence at Maxwellheugh, where, on examination, I find the water was not only dry below the cauld, probably from the cauld freezing in somewhat the same manner, but when the mill was set agoing, the pool above was drained in a short time to the level of the dam-head, because Mr Mein's, which is about three miles above it, had obstructed the current in the manner described."

Dr Wilson mentions in his first letter, that, "at Roxburgh and other places, persons crossed the water dry-shod; and a boy crossing from Trows to Heiton, caught several eels which were struggling in the shallows."

Dr Wilson adds, in his second letter, that there was in the Teviot, on the morning of the stoppage, little or no ice in the stream. "The miller at Roxburgh (he says) finding the mill stopped, looked out for ice, and discovering none, and finding his dam empty, thought it was the last day! Neither he nor any other miller would have been surprised, had the stoppage been caused by ice, which is an occurrence so common, that they are generally provided with large wooden mallets, to be used for its removal. Mr Dickson of Hawick, who was not likely to be an inattentive observer, as his large manufactory was stopped on the occasion, stated to me that there was no ice, unless a little at the edges of the pools."

These extracts are sufficient to shew generally, the nature of the phenomenon in the Teviot.

I shall now give a summary of the most important facts stated in the letters of the different gentlemen above mentioned, calculated to explain the phenomenon. It appears from these letters—

1. That there was an almost *total cessation* of current, and in many places an absolute depletion of the bed of the Teviot, in that part of its course situated betwixt Kelso and Hawick, and that the same phenomenon happened in most of the streams which joined it.

2. That, with one exception, all the mills on the river, from Kelso to Hawick inclusive, and most of the mills on the tributary streams, were stopped from want of water,—a circumstance which proves that the phenomenon arose, not from *obstructions* by ice or otherwise on the caulds or dam-heads, but from a *failure* of water in *the upper parts of the river*.

3. That when the current began to flow again, there was no sudden rush of waters, such as would have arisen from the mere stoppage of the current by obstructions, but that there was a gradual restoration of the current.

4. That the diminution and disappearance of the water took place in the upper parts of the Teviot and its tributaries, before it took place in the lower parts. (For example, it was noticed on the Rule Water at 11 P. M. on the 26th November, whilst it was not noticed in the lower parts of the Teviot till next morning).

5. That the restoration of the current took place first in the upper parts of the Teviot and its tributaries, the smallest and shallowest of the latter being the first to indicate motion.

6. That, during the night of the 26th, and morning of the 27th November,

there prevailed over the whole district drained by the Teviot and its tributaries, a keen and parching gale from the E.SE., accompanied by a frost, which reduced the temperature of the air to 25° or 26° Fahr.

7. That ice was formed during the night of the 26th November, in the higher parts of the Teviot, and especially its tributaries,—that the small streamlets, and even some of the springs at or near the sources of the river, were frozen, and were consequently stopped,—that, in some of the larger tributaries, ice formed both on the surface and at the bottom of the water, and that in one place the whole body of the stream was congealed into a solid mass,—and that across the lip or edge of one cauld on the Teviot, a barrier of ice formed from sixteen or eighteen inches high, which there completely obstructed the flowing of the current for several hours.

8. That the rivers in which the phenomenon was most observed, have an easterly direction, and are neither deep flowing rivers, nor sheltered by high banks or woods. The Tweed and the Eden, in neither of which the phenomenon occurred, are both deep and rapid, and, besides being sheltered by high and wooded banks, have not so many tributaries as the Teviot, so that their waters are not expanded over so large a surface.

9. That there was no shock of an earthquake felt about this time in the district, nor any appearance of cracks or fissures.

10. That on two former occasions, mentioned in Dr Douglas's letter, viz. in 1804 and 1824, stoppages in the Teviot occurred, both of which were in the winter season.

In reference to this last point, I may add, that I have met with accounts of several other stoppages, besides those noticed by Dr Douglas. On the 25th January 1748, the Teviot, for two miles of its course, remained dry during nine hours. On the 11th March 1785 the river was dry for two hours, and the phenomenon occurred again eight days afterwards. On the 25th January 1787 there was a stoppage for four hours. These four stoppages, it will be observed, were also during the season, when frosts and east winds are known to prevail in this country.

(2.) The next river, the stoppage of which I shall allude to, is the Clyde. On this subject I corresponded with Mr R. Logan, surgeon, at *New Lanark*, who addressed to me two letters, dated 24th December 1838, and 21st January 1839.

Mr Logan, in his first letter, states, that the watchmen of the New Lanark mills had their attention first attracted "to the state of the river about 2 A. M. on the 27th November, by the usual noise of the adjoining fall (Corra Linn) having ceased." "The wind had been blowing from the east for several days, and, about twelve at night, it rose to a stiff gale." "There was not the slightest indication of the stream having been absorbed by any fissure in the earth; and it was not till about seven in the evening that there was a sufficiency of water to turn the machinery."

In his second letter, Mr LOGAN repeats, that " the waters of the Clyde were entirely absent from 2 A. M. till 6 or 7 P. M. In the middle of the day, when I crossed by the bridge, a mile below Lanark, the stones in the bed of the river were so bare, that any one might have crossed *without wetting their feet.* I had intelligence of the river being in the same state, for at least twenty-five miles up the stream, and ten or fifteen down."

These letters describe the state of the Clyde in the *lower* part of its course. That the current was likewise arrested in the river near its source, appears from a letter handed to me by Professor FORBES, written by the schoolmaster of Crawford, through which parish the Clyde in the upper part of its course flows. He states that, in that high district, the Clyde was dried up on the morning of the 27th, and was crossed by several persons dry footed, and that the river did not regain its usual size till 3 o'clock P. M. That the rivers Daer and Powtrail, which join the Clyde in the south part of Crawford parish, as well as all the rivulets and feeders there, were remarkably low,—the cause of which was supposed by the inhabitants of the district to be the severe frost of the preceding night. This writer adds, that, during the winter of 1837–8, when the frost lasted continuously for six weeks, and when the thermometer was greatly lower than it was in November last, the Clyde could in no place have been crossed dry footed.

A letter (sent to me by Messrs CHAMBERS, publishers of Chambers' Edinburgh Journal), from the miller of Hyndford-mill, (situated about three or four miles above New Lanark), states that, " on the morning of the 27th November, the water in the Clyde was flowing in its usual manner, until about 6 A. M., when it began to subside rapidly, and shortly after almost ceased to flow. The frost was very severe at the time, accompanied with rather *high wind.* The channel of the river continued nearly dry, until between 12 and 1 o'clock P. M., when it began to flow."

This person adds, " that, during a severe frost, water freezes rapidly *at the bottom of the strongest current,* and the frozen particles accumulate *so rapidly, under certain circumstances,* as to present a complete barrier to the flowing of the water. I have (he says) several times been forced to stop the mill, on account of the opening *under the sluice* becoming *quite closed up.*"

From these different letters, it appears,

1. That the River Clyde, from its sources down to ten or fifteen miles below New Lanark, was, alongst with most of its tributaries, almost entirely dry.

2. That the total desiccation appears to have continued from nine to ten hours, and that the period when the current began to stop, to the period of its again flowing, was about sixteen or seventeen hours.

3. That, at New Lanark, the average width of the current is from thirty to forty yards, and its average depth from one and a half to two feet.

4. That, during the night of the 26th, there was a severe frost, accompanied by a stiff gale from the east, from which quarter the wind had blown for some days previously.

5. That, on the morning of the 27th November, at daybreak, there were accumulations of ice in the river, and especially on the bands of rock crossing its channel.

6. That the river was, on the 26th November, rather above its average level, and on the 28th it was flooded and muddy, though no rain had fallen in the interval.

7. That similar phenomena occurred in the tributaries of the Clyde, and that some of the minor springs were dried up, whilst the larger springs had their supply diminished.

8. That there was no appearance of any fissures near the channel of the river.

9. That the Clyde, in the upper part of its course, flows in a NE. direction.

10. That, besides this recent stoppage, four others had occurred in the Clyde between 1813 and 1837–8, all of which were in winter, and during the prevalence of severe frost.

I may add to these instances of stoppage, the one previously mentioned, which occurred in January 1787, on the day on which the shock of an earthquake was felt to the north and east of Glasgow.

(3.) The only other river the desiccation of which attracted particular attention is the NITH.

The general nature of the phenomenon, as observed in this river, may be judged of from the following extract from a letter, written to me by Mr THRESHIE of Dumfries :—

" I learn indirectly from Mr SMITH of Dalfibble, a most respectable man, that, on that day (the 27th November), he crossed the river a little below Drumlanrig, in the forenoon, and found the river so void of water, that, with a stout pair of shoes, he could have crossed without wetting his feet ; and recrossing in the afternoon of the same day, the river took his horse nearly to the belly, at the same place, and there had been no rain, or appearance of rain, and the circumstance struck him much. Farther, about four miles farther down the river, two persons crossed on foot, to their surprise, at the ordinary ferry ; and found in the afternoon the boat, as usual, necessary. Much higher up, or below these two points, no particular change seems to have been remarked."

The most distinct and detailed account I have received in regard to this river is from JAMES SHAW, head-gamekeeper to the Duke of BUCCLEUCH at Drumlanrig.

The letters which he has written me on this subject, prove him to be a zealous and most intelligent observer. Mr SHAW is already known to this Society and the public, by his experiments and speculations on the parr and on the fry of salmon, of which an account was read here last session by Mr STARK. The researches which for many years Mr SHAW has carried on to ascertain the nature and habits of these fish, have caused him to pay particular attention to the state of the rivers in his neighbourhood, on which account, he was peculiarly qualified to afford information as to any peculiarities in the state of the Nith and its tributaries.

From these letters of Mr SHAW, it appears,—

1. That, previous to 27th November, no rain had fallen in the district drained by the Nith and its tributaries for three weeks, and that the waters in these rivers were in consequence extremely low.

2. That, though the Nith was almost entirely dried up between Sanquhar and Enterkinefoot, this was owing to a stoppage of the waters, which took place in the *higher* parts of the river, and especially in its tributaries.

3. That, though the caulds and damheads on the Nith and its tributaries were encrusted with ice, the disappearance of the waters took place in parts of the rivers situated *above* the caulds.

4. That the phenomenon was most striking in those tributary streams, which flowed from the *highest* level, and the waters of which were the most expanded and exposed, by flowing over shallow channels unsheltered by trees or brushwood; and that, on the other hand, it was least developed in the Minnick, a river which has generally a temperature above that of the Nith, the Crawick, and the Euchan, which were arrested.

5. That the waters disappeared from the higher parts of the rivers, before they disappeared from the lower parts.

6. That the wind blew strongly on the 26th, and morning of the 27th, from the east, and was very keen and parching.

7. That the course of the Nith above Enterkinefoot is from the east, as well as that of its tributaries, in which the stoppage or desiccation was most observed.

(4.) I have now given the substance of the information procured by me, regarding the desiccation or disappearance of the waters from the channels of the rivers Teviot, Clyde, and Nith. Before submitting any views of an explanatory nature, perhaps I may be permitted to conclude the above narrative of facts, by mentioning some other Scotch rivers, in which the same phenomenon was observed on the 27th November, and also in former years.

I have a letter from the schoolmaster of Ettrick, stating, that, on the 27th November, the *Ettrick*, at about eight miles from its source, was dried up. It is there usually from ten to twelve inches deep. The desiccation was observed about the middle of the day, and continued some hours. The thermometer at 10 P. M. on the 26th was 28° Fahr., and at 8 A. M. next morning it was 30°.

The river *Tay* at Perth was observed, on the 27th November, to have had its waters greatly lowered;—my informant states, to the amount of from three to four feet in depth. Attention was drawn to the circumstance, in rather a curious way. A new water-engine had been, last autumn, contracted to be erected in Perth, for driving a saw-mill. It so happened, that the 27th November was the day fixed for trying the machinery, before it was taken off the hands of the contractor. The water-wheels were set in motion by the stream about 9 A. M., and they went very well for about an hour, when they suddenly stopped. This excited surprise, as the wheels had been erected and fixed about a month before, when the water in the river was particularly low; so that, when the wheels stopped, there was an apprehension of some fault in the machinery. But it was soon discovered, to the great satisfaction of the machine-maker, and I may add of his employer, that the cause of the stoppage lay not in the machinery, but in the river, the waters of which had subsided, and left the wheels high and dry. My informant (who is the owner of the saw-mill) states, that this want of water continued till 2 o'clock P. M., and that at 3 o'clock there was current enough to set the wheels again in motion. He adds, that the frost at Perth had been slight during the previous night; but in the higher grounds, at least, the frost must have been more severe, for, at Kinfauns, the thermometer had sunk through the preceding night to 28°.

These are the only other rivers in which (so far as I have heard) the same phenomenon was observed, which occurred (though much more strikingly) in the Teviot, Nith, and Clyde, on the 27th of November.

I should add, that, *on the 28th January last*, another desiccation, but to a more limited extent, happened in the Teviot and in the Ettrick. At Maxwell-heugh Mill, near Kelso, the bed of the Teviot again became dry. On this occasion, at least, the phenomenon may be accounted for, by the current having been entirely obstructed at Ormiston Mill, by a dyke or barrier of ice formed along the lip or edge of the cauld; for above Ormiston Mill, the phenomenon was not observed. On the same day the river Ettrick, near the schoolhouse of Ettrick, was likewise reduced to about one-tenth of its usual size.

With regard to stoppages in other rivers at former periods, I shall notice them very briefly, and shall do so in chronological order.

The *Kirtle*, a river which runs from Dumfriesshire into the Solway Frith, stopped, on the 17th February 1748, for five hours: it stopped again two days after for nine hours, and its channel was dry along seven miles of its course.

The *Sark* (a river which flows into the Eden near Carlisle) dried up near Phillipston, in the parish of Kirkandrew, on the 20th February 1748.

On the same day the *Liddell*, which joins the Esk near Langholm, dried up near Penton, in Kirkandrew's parish. The Liddell is there usually from sixteen to eighteen inches deep.

The *Lyne*, near West Linton, on the confines of Peeblesshire and Mid-Lothian, stopped likewise on the same day; and again on the 25th February in the same year, viz. 1748.

The *Esk*, near Langholm, stopped and remained dry for six hours, on 23d February 1748; and again, two days afterwards. The Esk at this place is, on an average, from eighteen inches to three feet deep.

The *Isla*, near Keith (Banffshire), was, on 28th January 1753, dried up for several hours.

The *Tweed*, at Peebles, was, on the 14th February 1753, entirely dried up from 6 A. M. to 6 P. M.

The *Dovern* was dry on the 2d April 1754, between the Rack and the Surry fords,—*i. e.* for about a quarter of a mile,—and continued so all day. Numbers of people crossed dry shod.

On the 9th February 1755, the river *Beauly*, near Kilmarnock and Kiltarlity, seven miles west of Inverness, became entirely dry during the prevalence of a hard frost.*

The *South Esk*, near Brechin, in Forfarshire, went dry in 1813,—the particular month I have not ascertained. My informant writes, that people might have crossed the bed of the river, in many places without wetting their feet, that the mills were all stopped, that salmon were caught with the hand in the deserted pools, and that the washerwomen who were occupied along the sides of the river in their peculiar vocation, came home with their clothes unwashed, in a state of distress and consternation.

I shall now offer some suggestions, with the view of explaining the occurrence described in the previous part of this memoir.

It must appear sufficiently obvious to any one, who has attended to the circumstances above described, that the frost which prevailed in the south of Scotland during the night of the 26th November, must have, at least, had a good deal to do, in the production of that occurrence. Indeed, the bare fact that, on all the occasions when it was observed in any part of the country, it happened during the six months intervening between November and April inclusive, strongly suggests this conclusion.

One thing appears abundantly evident, that the phenomenon on this occasion was in no way connected with an earthquake, as was at first imagined; and it is not improbable that, if all the circumstances attending the drying up of the Trent and the Thames, in the 12th century, could be ascertained, they would be found

* For this information, I am indebted to Mr GRIERSON of Dalgado, near Dumfries, who saw a notice of the present memoir in the Society's abstract of business.

not to warrant the explanation which Professor Phillips has suggested. I find that these remarkable occurrences happened, the one in October, and the other in April; so that they occurred at a time of the year, when the frost may have produced them.

But though frost appears to have had, at all events, a principal share in accomplishing the desiccation of the rivers on the 27th November last, it is at first sight not easy to see precisely the manner in which it operated. The frost then was not nearly so intense, or so long continued, as at other times, when, however, similar results did not occur. On the 27th November, the thermometer, in the south of Scotland, did not sink below 25° Fahr. But, in the previous winter, it will be remembered, that, in the south of Scotland, it sunk as low as 5°, and that, for a period of about ten days continuously, it was below 27°. Yet none of the larger rivers stopped during the continuance of this frost, and it was only observed that they were somewhat lower than usual. It is not easy, at first sight, to perceive how the frost acted, so as to produce on this one occasion, phenomena which it failed to produce on other occasions, when it was both more intense and more protracted.

1. It has been suggested, that the ice formed on the lip or edge of the damheads, *arrested* the current, and thus dried up the channel in *inferior* parts of the river. At Ormiston Mill, on the Teviot, a dyke or barrier of ice, from sixteen to eighteen inches high, traversed the entire width of the river, and for several hours prevented the current from flowing towards Kelso. It was observed also, that the formation of this icy barrier was greatly assisted by a strong easterly wind, which, by raising a spray on the lip of the damhead, allowed the water along the line of the cauld to be rapidly reduced to the temperature of the wind.

The effect here described would no doubt be produced, and the explanation may be sufficient to account for the drying up of the Teviot in one part of its course; and perhaps it might answer also for some of the other rivers which run towards the east, and that are crossed by damheads above the places where the stoppage occurred. But the explanation is much too partial and too local to be the true one.

(1.) For, according to it, the same phenomenon should occur almost *every* winter, as there is often a frost accompanied by a high wind, which, whatever be its direction, would blow against the stream of one or more rivers in some part of the country. Now this phenomenon, so far from happening every winter, is of very unfrequent occurrence.

(2.) But, in the next place, the explanation fails entirely in regard to those parts of the rivers where there are no damheads; and it has been seen, that, in the Slitrig, the Euchan, and a number of other rivers, the waters disappeared in those parts of their course which are situated *above* any damhead.

(3.) But even where there are damheads, this explanation is unsatisfactory; for, if the current was there obstructed, the water should have *accumulated* above them, and, so far from a want of water being experienced by the millers in their mill-leads, there should have been an unusual supply. But all the mills, except one, on the different rivers and their tributaries, were stopped from want of water.

(4.) In the fourth place, if the waters were thus merely arrested for a time, and not abstracted or excluded from the channels, there would be a prodigious overflow, or, in other words, a *tremendous flood*, after the obstruction was removed; and, moreover, the water would not begin to flow again gradually, but would burst forth with sudden and irresistible impetuosity. These effects, however, are inconsistent with what was observed on nearly all the rivers where the phenomenon occurred. The Clyde only is said to have been a little flooded and muddy on the 28th November,—which can be accounted for by the sudden thaw which, by that time, had taken place. The flow had recommenced, however, the preceding evening, and it came on gradually.

2. Another theory to explain the phenomenon, might be founded on the remarkable fact, that, during the night of the 26th, and morning of the 27th November, there was ice formed in large quantities at the *bottom* of the rivers, which must, to a certain extent, have obstructed the flow of the water.

(1.) But it appears to me, that this would have had a very opposite effect from what occurred; for, if the velocity of the current is generally diminished throughout the whole course of the river, must it not happen that the waters will be *less rapidly drained off*, and thus, so far from the waters entirely disappearing from the bed of the river, or even diminishing in it, they would appear considerably swollen.

(2.) This effect would be rendered all the more striking, as, by the supposition, the bottom of the river is *raised* by the formation of ice on it. So long as the river continues to draw its wonted supplies from the springs which feed it, the circumstance of ice forming on its bottom, so far from tending to lower the level of the current, would tend only to elevate it.

(3.) Besides, if this were the true explanation, the phenomenon ought to happen *every winter*, and indeed almost *every frost* which occurs; because I believe there are few frosts, during which ice does not form at the bottom, with even more readiness than on the surface, of running streams.

I may here mention, that the formation of ice at the bottom of running streams, has been frequently observed in the continental rivers, and especially the Elbe. But it was never known that they stopped running, or even diminished in volume, at this period. On the contrary, they were then observed to be generally swollen.

Perhaps I may be permitted to dwell for a moment on the remarkable circumstance, that ice should ever begin to form at the bottom of water. It is known that *ice* is specifically lighter than *water;* and that *water* itself, after it is cooled down to 39° of Fahr., becomes specifically lighter, and rises to the surface. It is, therefore, not easy at first to see how water should, in any case, begin to freeze *from the bottom;* but, nevertheless, there are well authenticated cases of ice forming at the bottom, even when there was no appearance of any at the surface.

This is one of the multifarious subjects which has obtained the attention of M. Arago. His views will be found in a paper, the translation of which appeared in Professor Jameson's Philosophical Journal for 1833. The phenomena referred to in M. Arago's paper, occurred in the Rhine and the Aar. The facts were fortunately observed by scientific individuals, who watched the gradual formation of the ice in the bed of the river, and ascertained the temperature not only of the air and the ground, but also of the water at the bottom and at the surface of the current. The water was found always to have been cooled down a few degrees below the freezing point; and the water at the bottom, at the surface, and in the middle, always possessed a general uniformity of temperature. M. Arago states, that the phenomenon had been observed never to occur in *stagnant* water, but always in running streams, and in those places chiefly where the waters flowed over a bottom bristled with stones, weeds, or other rough substances. The explanation of the formation of ice in these circumstances, is sufficiently simple. From contact with the air, the upper surface of the water is cooled down to the freezing point, or to any given point below it. If the water is stagnant, these cold particles, from their less specific gravity, remain on the surface; but the effect of a current is to intermix the particles belonging to the bottom and the surface respectively, in consequence of which, the whole is reduced to the temperature fitted for congelation. But, it may be asked, why, in some cases, congelation should begin at the bottom and not at the surface of the stream? There are two reasons for this: (1.) From the obstruction which stones and other objects on the bottom give to the current, the velocity is less near the bottom than at the surface, and thus congelation is more possible at the bottom than at the surface; for water, when in rapid motion, is found not to freeze readily. This, then, is one reason why, in streams, ice should form first at the bottom. (2.) The presence of certain bodies in an aqueous medium, is, in all cases, known to facilitate crystallization; than which, a better example cannot be given than water saturated with sugar, which will remain in a state of solution till a piece of wood, or even thread or twine, is dropped into it, when instantly crystals of sugar-candy are formed round the foreign body.

Such is, in general terms, the explanation given by M. Arago of the *ground-ice,* which has been observed in the German rivers, and which may be seen any winter in most of our Scotch rivers. It is the kind of ice referred to in the letter of the miller on the Clyde, already read, who states that it frequently becomes so thick as to fill up the interstice between the bottom of his sluice and the channel of his dam. I believe that this is the origin also of that kind of ice having much the appearance of melted snow, which often floats down our rivers in great abundance, and which, in Dumfriesshire and Roxburghshire, is known by the name of "grue." It is not retained long enough at the bottom to be formed into a solid cake of ice, but by its lighter specific gravity, is able to separate itself from the weeds or mud at the bottom, and rise to the surface of the current.

But to return from this digression, let me repeat, that the formation of this ground-ice cannot, by obstructing the flow of the stream, have the effect of lowering its level, and far less of drying it up. The very opposite effect would follow.

3. It must be obvious from these remarks, that no explanation can be the true one, which assumes that the quantity of water in the rivers on this occasion remained the same, and was only stopped or arrested in its flow. We must seek for some explanation which involves, as a principal element, the actual abstraction or withdrawal of the current from the bed or basin of the river. Now, this condition would be accomplished, if, whilst the supply of water at their sources was stopped, the channels in the lower parts of the river remained open, so as to permit the main current to run off, and thus drain the bed of the river. If the frost acted in such a way, as to seal up and stop the flow of the springs at the fountain-head, and yet allow the stream in the deeper channels to flow as before, then it will not be difficult to see how the phenomenon in question occurred. It is proper, therefore, to inquire whether this theory is at all supported by observation, and whether it can be justified on known principles.

That this proposed explanation is, to a certain extent at least, warranted by observation, is manifest from the letters above quoted; in all of which it is stated, that the rivulets, and even the *springs*, were frozen up and arrested in their flow on the night of the 26th and morning of the 27th November.

That this should have been the case, when the thermometer sunk in the course of that night to 26° or 27° Fahr., is quite intelligible. The only apparent difficulty is to discover in what way the water of the river could be frozen at the sources, whilst it continued to flow in the deeper channels. It has been shewn that the thermometer at 3 P. M. on the 26th stood at 32°, that at 10 P. M. it sank to 26°, and that by 10 A. M. on the 27th November it had returned to 32°. It may be assumed then, that, on the night of the 26th, there was no frost of any degree of severity which lasted longer than eight or ten hours. If, then, the stoppage of the supplies to the rivers in question was affected by the frost, was there any thing which enabled it to produce, during the short period of its continuance, a greater effect on the small streamlets at the main sources of the rivers, than on the main body of the current in the larger channels? It is not difficult to perceive how the frost on the night of the 26th November, acted so as to produce this effect. It is stated in the letters above quoted or referred to, that, during that night, there was a strong gale from the east; a very unusual circumstance when the temperature is so low as 26°. Now, it is well known that cold, when accompanied by wind, acts very differently than when the air is calm. Bodies exposed to a refrigerating breeze, will be cooled much more rapidly, than when exposed to air of the same temperature in a state of repose. The rate of cooling is, in fact, increased, exactly in proportion to the velocity of the wind. Professor LESLIE has shewn, that, whatever is the ordinary rate of cooling of a body over which a stream of colder air of a certain velocity is blowing, this rate of cooling will be doubled if the velocity of the wind be doubled, and quadrupled if the wind blow with four times its previous violence; or, in other words, it will be reduced to the temperature of the air in one-half or one-fourth of the time which it would otherwise require. "We thence gather (says Sir JOHN LESLIE), that even a moderate wind will quadruple the waste of heat, and that a vehement hurricane is capable of increasing the rate of dissipation perhaps fifteen or twenty times. Hence also the keen impression of frosty winds on our feelings, and their prodigious effects in chilling the surface of the ground. We thus perceive in a strong light the vast utility of shelter."*

The effect, therefore, of the wind on the night of the 26th, must have been to cool down to its own temperature, the objects and places exposed to it with great rapidity. If, at 10 P. M., the temperature of this wind was 26°, we may assume that, by this time, the surface of the ground which it swept over in full force, had been cooled down to at least 29°,—a degree of cold at which water in drains, streamlets, morasses, and shallow pools, would be entirely consolidated. But it is obvious that this effect would, by that time, have been produced only in exposed or unsheltered places. In the glens and valleys, especially those which were wooded, the surface would not be so rapidly cooled; and if they lay in a north and south direction, there would be additional means of shelter. In proof of this observation, if any were wanting, I might refer to the state of the thermometer at Drumlanrig (in a low sheltered district), and at Leadhills, situated about 1000 feet higher. On the night of the 26th the thermometer at Leadhills was 27°, at Drumlanrig 34°. On Tuesday morning, at Leadhills it was 29°, at Drumlanrig 30°; at noon, at Leadhills 31°, at Drumlanrig 35°.

We thus see two distinct and separate reasons why the rivers should, on the night of the 26th November, have had their sources frozen up and arrested, whilst their currents in the main channels continued to flow with scarcely diminished rapidity. In the first place, the mere difference of height subjected the sources to a greater degree of cold, than prevailed in the lower parts of the river's course,

* Experimental Inquiry into Heat, p. 284.

it being well known that the atmospheric temperature diminishes in a certain ratio with the height.* In the next place, the temperature of the ground in the higher and more exposed districts would, from the violence of the wind blowing on them, approximate much more rapidly to that of the air than it would do in the lower and more sheltered districts. So that, when the former had fallen to 27° or 28°, the latter would scarcely have reached the freezing point. From these united causes, the water oozing through marshy ground, or trickling along open drains, or in the channels of streamlets on the hills and muirs, would be suddenly congealed and arrested, whilst the larger body of water in the bed of the rivers would continue to flow on unobstructed, and thus effect an entire drainage of the channel.

It is the last of these causes to which, more particularly, I ascribe the phenomenon which has formed the subject of this paper. It is in itself sufficiently simple, and depends on the plainest principles.

The nature of this cause, serves also to explain the unfrequency of the phenomenon. The severe frosts in this country are seldom accompanied with gales of wind. Our gales are generally from the S. or SW., bringing with them warm vapours, which are exclusive of frost; and even when we have easterly gales, there is seldom sufficient dryness to admit of a very low temperature. But, whatever be the cause, it is an undoubted fact, that, in this country, during the prevalence of severe frost, the air is comparatively calm. Under these circumstances, although water at the sources of the rivers must always be more rapidly cooled than in the lower parts of their course, the difference in their respective rates of cooling cannot be nearly so great as in a gale of wind, which affects only the elevated and unsheltered districts. In an ordinary frost, the streams, and especially the springs at the sources of the rivers, are seldom frozen without there being ice formed in the current of the main channel. When the frost continues a sufficient length of time, ice will be formed on the surface of the current where it is deep and sluggish, and at the bottom as well as at the top where it is shallow and rapid. In this way the flowing of the current becomes obstructed; so that, in these circumstances, the river would actually appear more full than usual, were it not that, in consequence of the freezing of the streams at or near the sources of the river, much of the supply of water to it, is cut off. In the case of an ordinary frost, therefore, that is, when it acts with pretty nearly equal effect, both on the sources and on the main current, there will be no drainage of the channel. It is only when the frost is accompanied with a high wind, that it is enabled to affect the sources before it has had time to freeze, the larger bodies of water having a rapid motion, flowing at a lower level, and in a sheltered situation.

There is only one other point necessary to be adverted to, in order to com-

* The temperature sinks 1° of Fahr. for about every 350 feet.

plete the account of this phenomenon. The frost and gale of wind from the east which produced it, ceased on the forenoon of the 27th November. This change appears to have been brought about by the occurrence of a severe storm, or rather a hurricane, which came from southern latitudes. This is the storm alluded to at the outset of the paper, during which the barometer reached a great depression. In fact there were two storms, one following close upon the other, and which reached the British Islands on the 26th and 28th respectively, moving in a northerly direction. On these days, they descended low enough in the atmosphere to sweep over the surface of our islands. That they had previously affected the upper regions of the atmosphere, is shewn by the fact, that, so early as the night of the 25th, the barometer began to sink all over the British Islands, and, notwithstanding the prevalence of the frost and easterly gale of the 26th, which are calculated to elevate the mercury, the barometer continued to sink constantly and regularly until the 29th, when the most violent part of the storm occurred. On the 25th and 26th, therefore, it may be assumed, that the higher regions of the earth's atmosphere over this portion of the globe had become loaded with warm vapour brought by the storm, the upper part of which was in advance of that part sweeping along the surface of the globe; and hence, on the forenoon of the 27th, by which time the storm had approximated to this part of the earth's surface, the temperature suddenly rose, and the easterly gale as suddenly moderated.

Had it not been for the advent of these two storms to this part of the globe, at the exact period now mentioned, our rivers, instead of remaining dry for only twelve or fourteen hours, might have continued in that state for a much longer period, to the inconvenience and injury of many thousands of persons, dependent on the flow of their waters for employment and subsistence.

XXIV.—*Notice of Two Storms which swept over the British Islands during the last week of November* 1838. *By* David Milne, *Esq., F. R. S. E., F. G. S.*

Read 15th April 1839.

Previously to the 25th and 26th November 1838, there had prevailed in Great Britain and Ireland, for more than a week, a steady wind from the NE., accompanied with frosts, a progressively rising barometer, and tolerably clear weather. The same sort of weather existed on the Continent, and over a large portion of northern Europe, both on sea and land.

This state of things was changed, by the arrival of two storms from southern latitudes, which passed over the British isles during the last week of November. These two storms, until they reached this part of the globe, were separate. The first one reached the British seas, about thirty-six hours before the other. But the second moved with about double the velocity of the first, and overtook the first somewhere about the north of Ireland and south-west of Scotland. Accordingly, in the southern parts of England, there were distinct indicia of two different storms, each having its own period of arrival, veering, and cessation;—whilst towards the north, these indicia became gradually less distinguishable, and were at length significant of only a general gale.

It is my purpose in this paper, to state some of the most prominent signs and effects of these storms, with the view of shewing the direction in which they travelled,—the rate of their progressive motion,—and the range which each of them appears to have had over the surface of the globe. I shall also add some remarks as to whether they had a rotatory motion.

On the 25th and 26th November, the easterly wind still continued, and, on the last of these two days, it was accompanied, in the south of Scotland especially, by severe frost. By this time, the first of the two storms I am about to describe, had reached our atmosphere, though it affected only the upper regions of it. The barometer had already begun to fall, notwithstanding the severe frost and easterly wind, which, as is well known, have the effect of elevating the mercury. Hence it is obvious, there must have been in the higher parts of our atmosphere, some causes which more than counteracted the effect of the frost and east wind existing in the lower regions, and, on the whole, to produce a sensible diminution in the weight of the atmosphere.

It will be seen by the following table, constructed from registers kept in different parts of the United Kingdom, that the sinking of the barometer on the 25th November took place every where, and was therefore produced by no merely local cause, but by one which affected the entire mass of the atmosphere in this part of the globe. This table shews the *time* when the barometer began to sink in different parts of the United Kingdom; and as the places are chronologically arranged, we can pretty nearly determine in what quarter the sinking commenced, and in what direction the tendency to sink was propagated.

At Adare Abbey* (near Limerick) Barometer began to sink on 25th Nov. between	1 A. M. and 9 A. M.
... Aberavon (near Bristol)	9 A. M. and 10 P. M.
... Farnborough (near Bagshot)†	10 A. M. and 11 P. M.
... London (Somerset House)	9 A. M. and 3 P. M.
... Greenwich Observatory	3 P. M. and 9 A. M. on 26th.
... Liverpool	noon and noon on 26th.
... Sunderland	9 A. M. and 9 A. M. on 26th.
... Abbey St Bathan's (Berwickshire)	10 A. M. and 3 P. M.
... Castle Toward (Firth of Clyde)	9 A. M. and 6 P. M.
... Edinburgh Observatory	about 5 P. M.
... Inchkeith, Bell Rock, Pladda, Mull of Kintyre, Isle of Man	9 A. M. and 9 P. M.
... Cameron House (Loch Lomond)	10 A. M. and 10 P. M.
... Kingussie (Inverness-shire)	about 7 P. M.
... Kinfauns (Perthshire)	8½ P. M. and 9½ A. M. on 26th.
... Kinnaird Head (Aberdeenshire)	9 P. M. and 9 A. M. on 26th.
... Inverness	8½ P. M. and 9½ A. M. on 26th.

This table, so far as it goes, shews that the sinking of the barometer, and consequently the change in our atmosphere which produced that sinking, commenced in the south, and that it was propagated towards the north or north-east at the rate of from twelve to sixteen miles an hour.

Thus it appears that, in Scotland, on the evening of the 25th, the barometer began to sink; and I may now add, that it continued to sink, except for a short interval to be afterwards mentioned, till the 29th, when it reached the lowest depression. Its fall on the 25th and 26th was every where rapid; but notwithstanding this, there still prevailed in the lower atmospheric regions of Britain, on the 26th, and even on the morning of the 27th November, an easterly wind and severe frost, the well known concomitants of a high and rising barometer,—shewing clearly that the upper regions of the atmosphere were in a very different state, from those parts contiguous to the earth's surface.

* This register is kept by Viscount ADARE. From the 25th to the 30th November, observations were made on the barometer, thermometer, and direction of the wind, several times during each day, and on the 28th every half hour.

† The observations at Farnborough were made by the Honourable and Reverend CHARLES HARRIS (a son of Lord MALMESBURY). They were sent by him to Professor FORBES, accompanied by an extract from the Adare register, and Professor FORBES obligingly put them into my hands.

That portion of the storm (I speak now of the first storm) which swept the surface of the globe, impinged on the south coast of Cornwall about noon on Monday the 26th November. The gale which sprung up there was so violent, as to drive ships from their anchors, and wreck several. I find from the registers kept at Penzance, Truro,* Fowey, Falmouth, and Milford (published in Lloyd's List and other maritime papers), that this storm commenced there with the wind at E.,—that in the afternoon it veered to SE.,—and that, by 11 P. M., it was blowing due S. At day-break next morning, the wind had got a little to the west of south. By noon on the 27th, at the above places, it had shifted to due W., and in the afternoon of that day, it varied between W.SW. and NW. There was much thunder and lightning at Portsmouth, Plymouth, and other sea-port towns.

That this storm was a most severe one in those places which it reached, will be seen by the following account, dated Penzance, 27th November. "Last night a gale came on from the S.SE., which veered to S., and this morning increased to a *Hurricane*, and a heavier sea we have not witnessed for many years. At 3 P. M. to-day the storm abated, and the wind veered to NW., which will soon cut down the sea." At Falmouth, the wind veered to W.NW. about noon on the 27th, and moderated. In the evening, at Penzance, it fell *calm;* and along the whole coast, the wind at night moderated,—the storm having passed away, as we shall immediately find, to the northward.

This storm lasted, therefore, in the south coast of England, little more than twenty-four hours, and ended at and near Penzance, with a wind blowing in a direction exactly opposite to that with which it begun. This is one circumstance which suggests the idea that the storm had a rotatory motion, according to the theory of REDFIELD, lately illustrated in Colonel REID's popular work on the Law of Storms. As to this point, more immediately; meanwhile we may trace the progressive motion of the storm.

The gale commenced at Cork about 11 A. M. on the 26th, with the wind at S.SE. It did not reach Dublin till about half-past 3 P. M. The register at Farnborough, near Bagshot, shews its arrival there to have been at night on the 26th, with the wind at S.SE., it having been previously at E. It begun in the Isle of Man on the morning of the 27th, with the wind also at S.SE, and at 9 P. M. it had there veered round to due S. A correspondent at Carlisle has sent me an extract of a register, from which I observe that the gale was felt there in the morning of the 27th November blowing SE. by S. From the northern Lighthouse registers, extracts of which Mr STEVENSON has kindly afforded me, I learn that the storm reached Lismore (on the west coast of Scotland) on the evening of the

* At Truro, there were light breezes from W.SW. at 9 A. M. on 26th. Shortly before noon, the wind chopped suddenly round to east, and blew a gale. From the other places above mentioned there were similar accounts received.

27th,* and that it did not reach Tarbet Ness and Cape Wrath till the following day, viz. the 28th, *i. e.* about a day after it had entirely ceased in the southern part of England.

Thus, then, by noting the different places at which this gale successively arrived, and marking the time taken in passing from one place to another, we find that it had a progressive motion to the north, altogether independent of the direction of the wind; and that the rate at which it travelled to the north, was about ten or eleven miles an hour.

There are other facts which confirm this inference, and help us moreover to approximate to the probable track or path which this storm followed, in its course northwards. I have mentioned that the wind veered or shifted during the gale from E. to W. As this veering occurred at every place comprehended within the limits of the storm, the period of its occurrence becomes an element in the calculation, as important as the periods of its arrival and cessation. Now, I find that the veering from SE. to S.SW., which happened in Cornwall during the forenoon of the 27th, did not happen at Dublin till the evening of the 27th, and that the storm ceased there during the night. At Plymouth the wind veered to the westward about 9 P. M. on the 27th. I find also from the Lighthouse returns, that the westerly gusts, which may be considered the expiring breath of the storm, and which were felt at Truro and Penzance shortly after *noon* on the 27th, did not begin at Pladda (off the coast of Ayr) till the *night* of the 28th.

These, and other similar data too minute to be detailed at length, lead to the conclusion that this gale travelled northwards up the Irish Channel, and at a rate of about ten or eleven miles an hour, a result exactly the same as is brought out by the previous calculation.

The storm must have come, then, from southern latitudes. This inference is fully confirmed by the accounts brought by ships that were navigating the seas, off the coasts of France and Spain.

I find that, at Royan, near the mouth of the Garonne, a storm commenced on the 21st, with the wind at E.NE. On the 22d it veered to SE., and ultimately to S., after which it moderated, and the gale ended at Royan on the 23d. In the north part of the Bay of Biscay, at the mouth of the Loire, the gale continued on the 23d.† Off Capes Ortegal and Finisterre, (on the north-west coast of Portugal), there was a severe gale, which dismasted several vessels.‡ Going still

* This statement is confirmed by other registers. At Cameron House, on Loch Lomond, an accurate register is kept by Mr SMOLLETT of the wind and weather; from which, it appears that the gale commenced there on the evening of the 27th, with the wind at E.NE., accompanied by snow.

† On the 23d November, the *William and Robert* was seen waterlogged in Lat. 48° and Long. 3°.

‡ The names and exact positions of these vessels may here be stated. The *Ellen* experienced a heavy gale from S.SW. in Lat. 43° 10′ and Long. 10° 13′. The *Everton* of *Dundee* encountered it in Lat. 44° 51′ and Long. 10° 12′.

farther south, I find there was a storm at Gibraltar on the 21st, which veered to due W. at night, and by which several vessels were wrecked.

It is extremely probable, that it was one and the same storm which was felt at all these places,—being at Gibraltar on the 21st, travelling northwards on the 22d through Portugal, traversing the Bay of Biscay on the 23d and 24th, and arriving at the British Islands on the 26th. This inference rests not merely on the circumstance of its having the track, which its progress in our own country would lead us to expect, but of its having also travelled at very nearly the same rate which belonged to the storm that passed through the British Islands. For, reckoning the distance betwixt Gibraltar and Great Britain 1000 miles, we find that it travelled northwards at a rate of about nine miles an hour, whilst, as previously shewn, the storm in this country moved progressively northwards, at the rate of about ten or eleven miles an hour.

As to the question whether this storm had a *rotatory* motion, I confess that the facts which have come within my reach have not enabled me to form a very decided opinion; but, on the whole, I am inclined to think that it was rotatory, and that the rotatory movement was from right to left, or, to use Colonel Reid's simile, contrary to the hands of a watch. I have already alluded to one circumstance which supports this view, viz. the veering of the wind from SE. to NW. This occurred at Penzance, and I may now add, that the same thing was observed in the Scilly isles on the 27th November. In the morning of that day, the wind there was SE., in the evening it was blowing NW. According to the rotatory theory, the inference from these facts would be, that the centre of the storm passed in its course northwards near the Scilly Islands. If the wind was rotating from E. to W. in the north semicircle, then it is obvious that all the places situated to the east of the centre, and within the range of the storm, would have the wind successively SE., S., and SW.,—whilst all the places west of the centre, would find the wind veering round in the opposite direction, viz. NE., N., and NW. This corollary was to a certain extent confirmed; for, at Penzance, Truro, and other places in the south of England, the wind veered with the sun, according to the seamen's phrase. But, at Limerick, and other places on the west of Ireland, it veered in the opposite way. On the evening of the 26th, it there came round from the eastward to north, and afterwards to NW.

But the argument which chiefly influences me to adopt the rotatory theory, is the slow progressive motion of the storm, compared with the velocity of the wind. The progressive motion of the storm was, we have seen, something between nine and eleven miles an hour. Now the motion of the wind in the storm, could not have been less than fifty or sixty miles an hour. The wind *in* the storm had therefore a velocity and a direction independent of, and different from, the velocity and direction of the storm itself.

There is another circumstance which favours the idea that the wind in this and other similar storms was blowing in circular tracks, as in a whirlwind. If the storm had a northerly or north-easterly progressive movement, as well as an independent gyratory movement, the wind blowing from the south or south-west should be stronger than the wind from any other quarter, because, in that case, the progressive and rotatory movements would coincide. On the other hand, the northerly wind of the storm ought, for the same reason, to be the weakest. This inference was fully verified, by what was observed during this storm on the 26th and 27th November. It was the gusts from the south and south-west which were the most violent.

With regard to the extent and range of the storm, the data collected are not such as to enable me to speak very precisely. I find that, on the 23d November, when the storm was traversing the Bay of Biscay, a vessel from Mirimichi to Liverpool was dismasted by it in Lat. 48 and Long. 24. This would shew that the storm had a diameter of at least 900 miles. On the 24th and 25th November, a vessel from Liverpool to Batavia encountered this storm, about 600 miles to the west of the Land's End.* This first storm appears to have lost much of its force before it reached Scotland, the western parts of which only were affected by it, and that but slightly.

That the storm just described, was quite distinct from the one which I shall next notice, is evident from the fact of the wind having entirely died away before this second storm commenced, and of its having then sprang up from a totally different quarter, and that not till after an interval of several hours. The following account is given of the way in which, at *Southampton*, the first storm ended and the second commenced. About 2 A. M. on 28th, the wind " moderated, and a light air sprung up from the westward. But it did not last long; for it came a whole *spout* of wind from the S.SE., and then S., and now (in the evening) it is blowing hard from S.SW., with every appearance of a dirty night." A similar account is given from a correspondent at *Penzance:* " After nightfall (on the 27th), the weather almost suddenly fell nearly calm, and a most beautiful appearance the moonlight had, till that luminary set, and for some time after. But about five or six o'clock this morning (the 28th), the storm came on with redoubled fury, and the sea raged so furiously that nothing could brave its power. Our quay, particularly the newly erected part, is in imminent danger, and will, we fear, be prostrated before the tide ebbs." In like manner, at Truro, it is stated that the wind, which shifted about noon on the 27th to W.NW., moderated in the afternoon of that day. A light breeze from W.NW. continued till 4 A. M. on the 28th, when the wind suddenly shifted round from SE., and " blew a hurricane." At Hull the first gale abated at day-light on the 28th, and during the whole of

* The position of these vessels was shewn to the Society, on a large map of the Atlantic.

that day the weather continued moderate; but, in the evening of the 28th, the second storm commenced with great fury.

Even as far north as Liverpool, a distinct interval was observed between the departure of the first and arrival of the second storm; a transient calm having occurred there before the second gale commenced on Wednesday afternoon. I may add, as an additional proof of these two storms being quite distinct, that the barometer in all parts of the British islands underwent a temporary rise. On the 26th November it rose, at Adare Abbey, in the morning, 1–10th of an inch; at Fairnborough (near Bagshot), half a tenth, in the forenoon of that day; at Greenwich Observatory, 1–100th part between 9 A. M. and noon. At Sunderland, the barometer, after sinking to 28.50 at 2 P. M. on the 27th, rose to 29.07 at 9 A. M. on the 28th, when it recommenced falling. At Kinfauns it rose, on the 27th, 32–100, between 9½ A. M. and 8½ P. M. At Kingussie (fifty miles south-west of Inverness), it rose on the 27th, betwixt 4 and 8 P. M., 26–100th of an inch. At Inverness, it rose about half an inch, during the night of the 27th and morning of the 28th.

It is not my purpose to describe the *effects* of this second storm, by relating the damage occasioned by it, except in so far as these may afford an estimate of its violence, and indicate the places where that violence was greatest. It was the south and south-west parts of England, and the whole of Ireland, which were most severely dealt with. Parts situated to the east were comparatively little affected. At Lyme Regis, in Devonshire, the storm was so furious, as to blow off not only tiles but immense sheets of lead from the roofs of houses. In Plymouth, London, Bristol, Liverpool, Dublin, Belfast, it threw down stalks of chimnies innumerable, unroofed many houses, and blew down several which were building. At Teignmouth (near Exeter), as at other places in the south-west of England, the gale came on with the wind at south-east. The following is an account of its effects and progress at Teignmouth, written on the spot. At 1 P. M. on the 28th, "it blew a perfect gale, the sea running mountains high,—when all of a sudden the wind chopped round to south and south-west, and blew a tremendous hurricane. Its effects upon the sea, at the time tumbling in from the eastward, presented a curious sight,—the top of each wave hurled up into the air in one raging foam. Near the time of high-water, the sea made a breach over East Teignmouth church-yard wall, as well as the Baths, and rushed down the streets. In some parts of the town, the water was nearly two feet deep. Every thing was at a stand-still,—no business doing at the custom-house, bank, or other public places." At Ardglass, on the north-east coast of Ireland, two lighthouses were blown down, one of them a new one scarcely finished, and containing about 400 tons of stones. In the south of England, the hail was driven with such violence, as to destroy some millions of panes of glass in the conservatories. The lightning is described as having been, in London and Portsmouth, peculiarly vivid, and the thunder awfully loud. Some of the largest trees in the parks about London were uprooted. Considerable injury was done by lightning, not only in London, but in most of the towns of the south of England.

I am inclined to think that this, the first great storm of the bygone winter, was, on the whole, not so severe as the storm which occurred on the first week of January 1839. But if the lowness of the barometer be any criterion, it was in some parts of the country even more violent. In London, Liverpool, Wigtonshire, Ayrshire, the barometer was lower on the 29th November, than it was on the 7th January. In London it was 4–10ths lower. In Edinburgh it was pretty much the same on both occasions, viz. 27.7, which is the mean of all the observations.

This storm was experienced first on the south-west coast of Ireland. I learn from the meteorological register kept at Adare Abbey (near Limerick), that it began there about 2 A. M. on the 28th. It reached Cork about 3 or 4 A. M. the same morning;* Penzance, Truro, and Falmouth, about 5 A. M.; Milford, about 7 A. M.; Plymouth, about 9 A. M.; Fairnborough (near Bagshot), about 10 A. M. It did not reach Coloony, in the north-west of Ireland, till about noon on the 28th.

At all these places there had been, as previously mentioned, if not a calm, light airs from the westward. But at the hours just specified, the wind suddenly sprung up from the south-east, blowing with great violence.

This storm had a much wider range, and it endured for a longer period, than the one previously described. It was not till the forenoon of the 30th, that it ended in the south of England, when, as will be immediately seen, it passed, like its precursor, to the northward,—continuing, therefore, rather more than two days before it ceased in that part of the island.

The progressive movement of this storm, was more rapid than that of the first. I have said that it began near Limerick at 2 A. M. on the 28th. It reached Dublin and Liverpool about 1 P. M.; Glasgow at 3 P. M.; Kirkcaldy (Firth of Forth) between 4 and 6 P. M.; and Redheugh coast-guard station, near St Abb's Head, at 6 P. M. At all these places, it begun in nearly the same way, viz. with the wind from SE. At Cuxhaven (at the mouth of the Elbe) the frosts did not give way till the night of the 28th, and next morning the gale commenced there with the wind from SW. From these data, it results, that the storm travelled in a north or N.NE. direction, at the rate of about twenty miles an hour.

This inference, from the time when the storm begun at different places, is confirmed by observing the time of its veering from SE. to SW. or S.SW., at these and other places. The following table presents a number of places, chronologically arranged, where this veering successively occurred.

* The St Patrick steam-vessel, which left Liverpool on the 27th, was wrecked on the Irish coast at 5 A. M. on the 28th November. She was overwhelmed by the first gusts of the second storm.

At Adare Abbey (Limerick), wind veered from	SE. by E. to SE. by S.	on 28th Nov.	at noon.	
... Falmouth, Fowey, &c.	SE. to S.SW.		at 2 P. M.	
... Penzance and Milford,	? to W.SW.		about 3 P. M.	
... Farnborough (Bagshot),	SE. to SW.		about 4 P. M.	
... Portsmouth,	S. to SW. & W.SW.		about 9 P. M.	
... Greenwich Observatory,	SE. by S. to SW. by S.		after 2 P. M. and before 9 A. M.	
... Liverpool,	SE. to SW.		in evening.	[on 29th.
... Dublin, Killough, Dundalk, &c.	SE. to SW.		at night.	
... Calf of Man,	SE. to SW.		after 9 P. M.	
... Paris Observatory,	S.SE. to S.SW.		after 9 P. M. and before 9 A. M.	
... Hull,	S.SW. to SW.	on 29th Nov.	at 4 A. M.	[on 29th.
... Carlisle,	S. to SW.		betwixt 11 A. M. and 3 P. M.	
... Redheugh Coast-guard station (St Abb's Head),	SE. to SW.		betwixt 11 A. M. and 3 P. M.	
... Kirkaldy (Firth of Forth),	SE. to SW.		in afternoon.	

This table tends to shew, that the veering in the storm took place successively in a N.NE. direction, and that it advanced at the rate of about seventeen miles an hour,—which does not differ materially from the previous calculation.

A similar result is obtained, by marking the period at which the barometer reached its lowest point, assuming that this depression was occasioned by the storm.

At Adare Abbey (Limerick),	Barometer was lowest on the 28th Nov. at 4 P. M.*
... Markree, near Coloony, (NW. of Ireland),	at 11 P. M.
... Farnborough (Bagshot),	after 2 P. M. and before 10 A. M. on 29th Nov.
... London, Royal Society's observer (Somerset House),	29th Nov. at $2\frac{1}{2}$ A. M.
... London, Troughton and Simm's observer (Fleet Street),	28th Nov. between 5 and 8 P. M.
... Paris Observatory,	after 9 P. M. and before 9 A. M. on 29th.
... Wisbeach (Cambridgeshire),	after 4 P. M. and before noon on 29th.
... Bonjedward (Jedburgh),	29th Nov. at noon.
... Edinburgh College (Professor Forbes),	at 12 h. 15 m.
... Glasgow Observatory (Professor Nichol),	at 12 h. 20 m.
... Cameron House (Loch Lomond)	between 10 A. M. and 10 P. M.
... Leith (A. Mackenzie),	at 12 h. 24 m.
... Clangregor Castle, near Stirling,	at 1 P. M.
... Thurston (Dunbar),	about 2 or 3 P. M.
... Kingussie (Inverness-shire),	at 3 P. M.
... Kinfauns (Firth of Tay),	about $8\frac{1}{2}$ P. M.
... all the Scotch Lighthouses north of Isle of May latitude,	after 9 P. M. and before 9 A. M. on 30th.

This table does not specify the moment of greatest depression, at *all* the places, so precisely as could be wished for;—but, so far as it goes, it shews that there was an interval of about twenty-six hours, between the greatest barometrical depres-

* In this table it has not been thought necessary to reduce the time.

sion in the south of England, and the greatest depression in Perthshire,—so that it was propagated in a northerly direction, at the rate of about sixteen miles an hour.

As to the period when the storm ceased, I have mentioned that it was in the forenoon, or rather in the morning of the 30th, in the south of England. It seems to have ceased in the evening of that day, in the south of Scotland. When it ended in the west and north-west coast of Ireland, the wind was blowing from about W.NW. But in England and Scotland, it was then blowing from the west, or a point to the south of west. This difference in the direction of the wind, at different places, is not only consistent with the theory of rotation, but is an important confirmation of it. For if the centre of the stormy circle passed to the west of the British islands, then it would be a segment only of the circle which swept over them, and, in that case, the wind would not, at the end of the storm, blow in a direction exactly opposite to that with which it began. At Holyhead, Liverpool, Applegarth (Dumfriesshire), and Catrine works (Ayrshire), the storm ended with the wind at SW. or W.SW. At Limerick it was W. by N. At Barrahead (one of the Hebrides), and at Lismore (off the coast of Argyleshire), it varied from W. by N. to W.NW., so that these last-mentioned places were probably not far from the storm's centre, which nowhere, however, impinged on the British islands.

If this view of the matter were correct, viz. that the centre of the stormy circle was to the west of the British islands, it is evident that the veering ought to have been more rapid on the west coast of Ireland, than in places situated more to the eastward; and farther, that the same angle of veering should have required a longer period. This inference is fully confirmed by the registers. At Adare Abbey, where the direction of the wind was carefully observed, and registered every half hour, the wind veered 133° in twenty-four hours. At Penzance, during the same period, the wind veered 112°; at Fairnborough, 73°; at the Greenwich Observatory, 79°; at Rhins of Islay, on the south-west coast of Scotland, 67°; at Kinfauns, 90°; at Paris only 35°.

It farther appears that, on the west coast of Ireland, the storm passed away much more quickly, than in places situated farther eastward. At Limerick, it does not seem to have continued longer than twenty-eight and a half hours;—in London and its neighbourhood, it lasted fully two days;—in Paris, three days.

There is still another test of the correctness of the above view, which is available. If the most violent part of the stormy circle, lay to the west of the British islands, the depression of the barometer ought, during the storm, to have been greatest in places situated to the west. This inference is also remarkably confirmed by the fact, as the following table shews.

Table arranging Places where the Barometer stood at equal heights, or nearly so, during the Storm of 28*th and* 29*th November* 1838.

1	2	3	4	5
Paris Observatory, 29.04	London (R. Soc.) 28.68 .... (Fleet St.) 28.70 Greenwich Observ. 28.67 Farnborough, 28.70 Means = 28.69 Range = .03	Wisbeach . 28.54 Startpoint (Orkney) 28.44 Means = 28.49 Range = .10	Falmouth . 28.30 Aberavon . 28.38 Cardiff . 28.39 Means = 28.35 Range = .09	Liverpool . 28.15 Applegarth (Dumf.) 28.12 Isle of May L. H. 28.16 Bell Rock L. H. 28.17 Kinnaird-head 28.12 Means = 28.14 Range = .05
6	**7**	**8**	**9**	**10**
Bonjedward, 28.08 Abbey St Bathan's, 28.08 Buchanness L. H. 28.10 Means = 28.08 Range = .04	Ettrick School 28.00 Edinb. College 27.99 ...... Observat. 28.02 Kinfauns . 28.06 Girdleness L. H. 28.04 Tarbetness L. H. 27.99 Means = 28.01 Range = .05	Isle of Man . 27.97 Catrine (Ayrsh.) 27.89 Corsewall L. H. 27.95 Glasgow Observ. 27.94 Loch Lomond . 27.91 Lismore L. H. 27.93 Kingussie . 27.91 Dunnethead . 28.00 Means = 27.94 Range = .11	Dublin . 27.60 Barra L. H. . 27.80 Rhins of Islay 27.79 Cape Wrath L. H. 27.77 Means = 27.69 Range = .20	Coloony (Sligo) 27.55 Adare Abbey 27.49 Means = 27.52 Range = .06 *Depression.* At Paris . 29.04 ... Adare . 27.49 Difference = 1.55

In this table, the places where the lowest depression of the barometer was observed, have been so arranged, as to group together those where the barometer stood at the same height, or very nearly so. The height of the barometer has, in this table, been reduced to the level of high-water mark, as well as in most instances to the temperature of 32°. There is every reason to believe, that the observations at all the places, except at the lighthouses, indicate the lowest point which the barometer reached. At the lighthouses, the statement is given in the returns only at intervals of twelve hours,—viz. 9 A. M. and 9 P. M. The depression in all of these returns is therefore a little too high. But the error in those used in the above table, will hardly exceed 1–10th.

On examining the above table, it will be seen, that the places arranged as now explained, lie in lines or narrow bands, which traverse the British islands in a north or N.NE. direction,—that all the bands are parallel, or nearly so,—and that in each band, the depression becomes greater towards the west.

I think, therefore, that the barometrical observations confirm very strongly the inference, drawn from other data, that the most violent part of this storm was situated very considerably to the west of the British islands, and that the storm travelled in a N.NE. direction.

The next observations to be noticed, have a special bearing on the *magnitude* or extent of the storm, and the track which it followed over the surface of the globe.

It has been shewn, that the storm must have extended far to the west of the British islands. It can also be shewn, that it stretched considerably to the eastward. The meteorological register kept at the Paris observatory, under the superintendence of M. Arago, records a storm which begun there on the morning of the 28th November, and continued till the 1st December. During this period, the wind veered (between 9 P. M. on the 27th of November, and 9 A. M. on the 1st December), from SE. to SW. The most violent part of the storm occurred between noon on the 28th and 3 P. M. on the 30th; and the barometer reached its lowest depression sometime between 3 P. M. on the 28th and 9 A. M. on the 29th. In all these respects, the coincidence with the registers at Greenwich, and other places in England and Ireland, is so complete, that there cannot be a doubt that it was the same storm which was recorded at all of them. At Croom's-hill, Greenwich, where the direction and strength of the wind are noted twice a-day, the register shews that the gale commenced on the evening of the 28th November, and ended on the afternoon of the 30th, during which interval the wind veered from E. to SW. The most violent part of the gale was there comprised betwixt the afternoon of the 28th and night of the 29th. The Paris and Greenwich registers thus also afford additional proofs of the truth of the remark before made, that the amount of veering, and the rate of progressive movement, diminished towards the eastward.

The storm thus reaching beyond Paris with its eastern limb, and having its central parts situated to the west of Ireland, must have had a radius of at least 550 miles, and extended therefore more than half across the Atlantic ocean. It is natural to suppose, that, if this inference be correct, ample evidence of it should be found in Lloyd's List, and other records of maritime disasters.

On examining the Shipping Gazette, and other papers, I find that, at Royan, near the mouth of the Garonne, there was a storm on the 27th, 28th, and 29th. It there begun at S. and veered to W.SW. on the 28th, accompanied by a very heavy sea. At Oporto, the storm commenced on the 24th, and continued on the 27th, by which time the wind had veered to W.SW., and dismasted several vessels there. At Lisbon (180 miles south of Oporto), the storm commenced on the night of the 23d, with the wind at S.; it veered to SW. on the following day, and caused wrecks on the coast. So violent was this gale off the north-west coast of Portugal on the 28th November, that the Falmouth steam-packet was obliged to take shelter during that day and the next, in Vigo Bay. The storm ended there on the night of the 29th, and the steamer proceeded on her voyage to England on the morning of the 30th. Proceeding still farther south, I find notice of a storm at Madeira on the 23d, which drove a number of ships from their anchorage, and caused others to slip their cables and run to sea. Now, it will be remembered, that it was about two or three in the morning of the 28th, that the storm commenced in the south coast of Ireland, which is distant about 1500 miles from Madeira, and about 1000 from Lisbon.

Assuming that this was one and the same storm which visited all these places, then it travelled about the rate of nineteen miles an hour, which agrees very nearly with the rate of its progressive motion in this country.

I have shewn, that it is highly probable that the most violent part of this storm lay considerably to the west of the British islands; in which case, if it assumed a circular form, it must have stretched far across the Atlantic. We find, accordingly, abundant evidence of a very violent storm in those parts of the Atlantic where we would expect such a storm to have been. Without, however, entering into a proof merely of the range or extent of the storm, I shall proceed at once to furnish proofs of its rotatory movement.

On the 28th November, the *John and Mary*, when near the Scilly Isles, was dismasted about 5 P. M. by a hurricane from W.SW. On the same day, the *George IV.*, on her voyage from St Michael's, in Lat. 47° 10′ and Long. 9°, encountered the storm, and was laid by it on her beam ends. The wind with her varied from S.SW. to W.SW. On the previous day, a vessel from Cardiff to Malta had, in Lat. 49° and Long. 14°, lost her bowsprit, bulwarks, &c., with a man washed overboard. The position of the vessels just mentioned will be seen, on referring to a map, to have been scarcely beyond the meridian of the British islands. The next case to be mentioned is that of a vessel situated more than half-way across the Atlantic. The schooner *Brandon*, of Liverpool, from New Orleans to Glasgow, lost both her masts in a tremendous *north-west* gale, on 28th November, in Lat. 42° 45′, and Long. 32° 34′ W.

Now, assuming that the storm which the *Brandon* encountered, was the one which, on the same day, was raging in the British islands, the Bay of Biscay, and Portugal, we have this most important point established, that, when the wind in this storm was, in Lat. 42° and Long. 32°, blowing NW.,—it was at Oporto blowing SW. or W.SW.;—at Royan, S.;—at Paris, S.SE.;—at Greenwich, SE.;—in Perthshire, E. by S.;—at Coloony, in the north-west of Ireland, "steadily from the *east*;"—at Limerick, SE. by E. But this is not all; for, on the 28th, 29th, and 30th, the "Great Western" steam-ship happened to be on her passage from New York to England, and, from her log, I extract the following statements.* On the 28th, she was in Lat. 42° 34′, and Long. 52° 1′. The wind with her at that place was "westerly," the weather "moderate and cloudy, with *heavy swell from N.NE.*" On the 29th she reached Lat. 43° 57′, and Long. 46° 59′. The entry on her log for that day is, that the *wind* was "variable and south-westerly;" and the entry in the *weather* column is, "light breezes and dark hazy weather, with a heavy *N.NE. swell.*" This heavy N.NE. swell is just

* Published in the Shipping and Mercantile Gazette, 10th December 1838.

what we should expect to have been raised in this part of the Atlantic, by a storm rotating from the east round by the north. Such a swell could have been generated only by a N.NE. gale in that part of the Atlantic situated to the N.NE. of the position of the Great Western on the 28th and 29th November. Observe next the entry in her log of 30th December, when she reached Lat. 45° 23′, and Long. 41° 59′. " Squally unsettled weather. *Strong gales.* Heavy snow-squalls. High cross sea." This shews, that the steamer was entering the storm. Does the direction of the wind agree with this view? Entirely so,—for the wind with the steamer, was now blowing strongly from *W.NW.* On the 1st December she reached Lat. 46° 8′, and Long. 37° 22′, at which place a hard gale still blew from W.NW. The entry in her log for that day is, " Wind and sea increasing;—hard gales;—heavy snow-squalls;—high irregular sea." On the 2d December, the entry in the wind column is NW.ly,—and in the weather column, " wind decreasing, fresh gales, hail squalls, confused sea." She had got then to Lat. 47° 27′, and Long. 33° 20.′ On the following day, the wind was still north-westerly, but the entry in her log states only " fresh breezes." On the 4th December, the wind had become south-westerly, so that she had then got entirely out of the storm, which had passed away to the northward. It was only the outskirts of the storm,—its rear-guard circles, that the steamer, fortunately for her, encountered on 1st December. It is a strong confirmation of the above statement, that the *Sarah Birkett*, in Lat. 46°, Long. 22° had the gale severely from NW. on the 1st and 2d December. She was bound for England, and thus sailed for two days in the SE. quadrant of the storm.

These data, I think, very clearly prove, that, in this storm, the wind was blowing in bands which formed an entire circle, whilst these bands had on the whole a progressive motion towards the north. That the central parts of the storm passed probably within 200 miles west of Cape Clear, is suggested by the circumstance that the brig *Thomas Tucker*, on the morning of the 28th, encountered the storm about thirty or forty miles west of that Cape, with the wind blowing furiously from the SE. At noon on that day, the wind shifted to SW., from which quarter, after a short lull, it shifted to the W.NW., blowing as furiously as before. This vessel was wrecked on the Cape.* The *Barossa* transport, about 200 miles more to the south, encountered the storm on the 27th November. In the morning the wind blew from the SE.; in the evening it was SW., when she was so damaged that she was forced to put back to England. The consequence was, that she sailed with the storm, and continued in its south-west quadrant for two days, when she got out of it in the English Channel.

We are entitled, therefore, to conclude, that all the material facts hitherto collected, strongly support the opinion, that the storm in question was one of those rotating aërial bodies, of a figure more or less circular, which Redfield and Reid have described. At the same time, it must be confessed, that there are dif-

* Shipping Gazette of 13th December 1838.

ficulties involved in the theory, which it requires farther observation to clear up. One of these difficulties regards the velocity of the rotatory movement of the storm, which is found to diminish towards the circumference of the stormy circle. But the reverse might be expected, if the aërial particles belonged to a body which was impelled round a common axis by the influence of some law or force affecting the whole. It is quite obvious, that if there was nothing to interfere with the operation of this force, the rings of wind near the axis of rotation would whirl round in the same time with the most distant rings, and, therefore, with a proportionally smaller velocity. But it is not difficult to see, that there are circumstances which must interfere with the operation of the force above assumed. (1.) The most distant rings are of course retarded by the friction of the atmosphere, through which the storm is rotating and progressing,—as well as by the surface of the sea or land over which it is sweeping. The effect of this retardation on the outskirts of the storm must be, to a certain extent, propagated to the interior rings. (2.) Farther, it is obvious, that the rotatory movement of parts distant from the axis, will be counteracted by the centrifugal tendency which rotation produces. (3.) Lastly, it is uncertain, whether the aërial column rotates under the influence of a force acting equally on every part of it, or acting only on a central portion. If the latter alternative is made out by observation, all difficulty will vanish, because, in that case, it is evident that the rotation of the more distant bands may be accounted for, simply by their being in contact with the revolving axis.

Of what the central parts of such storms are composed, and how they are generated, are totally separate questions, which, in the present state of meteorology, may not be readily answered. But that there is every probability of there being a revolving axis sufficient to put the circumambient air in motion, is clear from the analogous phenomena of water-spouts, or "storm pillars," and "whirl pillars," as the German meteorologists term them. Professor Oersted, in his memoir on these aërial bodies, states, that they are sometimes many hundred, and even occasionally above a thousand, feet in diameter. Such a column of air, reaching to the height of several thousand feet (which is the observed height of several water-spouts), circulating with great rapidity, must soon produce an extensive gyratory movement in the atmosphere to a great distance, and thus exhibit most of the phenomena of Redpath's stormy circles. The analogy between water-spouts and storms of wind, is made still more obvious, by the fact mentioned by Professor Oersted, that, in Europe, these water-spouts have been generally observed to move in a direction from SW. to NE., being very nearly the direction of the best traced European and American storms.

It has been thought, that the formation of an aërial axis of gyration may be easily accounted for, by the mutual action of two currents of air, flowing in opposite and parallel or nearly parallel directions. These currents would, of course, form an eddy, which, in the form of a rotating body, will advance in the direction of the strongest current. The NE. wind flowing from the North Pole would, whilst in contact with the SW. wind, which is also as constantly flowing, must produce eddies of large extent, and violent in proportion to the strength of the opposing winds. So far, we see, if these eddies are the true cause of gyratory storms, why they should have both a progressive and rotatory motion. Why these circles should always advance towards the north is not so clear,—for this would imply that the southerly wind always obtained the mastery. Nor is it equally clear why these stormy circles should, as Colonel Reid also alleges, revolve in a manner contrary to the hands of a watch,—for this would imply that the southerly wind always flows on the east side of the northerly current. On the west side of the Atlantic Ocean, this last assumption is at least not improbable, as the Gulf-stream, in its progress northwards, will generally carry alongst with it an atmospheric current, whilst the continent of North America is as natural a conductor of cold winds flowing in an opposite direction. In this way, perhaps, an explanation may be found of the fact, that storms generated in the Atlantic, and impinging on the British islands, take a north or north-easterly direction, and rotate always from east to west. But it may be matter of doubt, whether this rule can be applied generally to the whole northern hemisphere, in the way Colonel Reid proposes.

Nor should it be taken for granted, that the axis of revolution, forming the nucleus of the supposed stormy circle, is produced merely by two opposite aërial currents acting on each other. It *is* possible, that this very simple solution of the problem may be the correct one, and that electricity and other active forces which generally accompany storms are *effects* rather than *causes*. But the only method of arriving at certainty on this point, is by precise and extensive observation.

It will be observed, with what remarkable regularity the depression of the barometer, during the gale last above described, diminished towards the westward. The lines or bands of equal depression, of which ten have been given in the foregoing table, are all parallel, or very nearly so, to each other, and have evidently the same direction as that taken by the storm itself in its progress northwards.

That the barometer should be lowest at those places situated nearest to the centre of the storm, appears to be not only quite consistent with the principles before explained, but to be unintelligible on any other assumption. The explanation most generally given of the fall of the barometer during a gale, is, that the air, when put into rapid motion over the surface of the globe, necessarily acquires a centrifugal tendency. Now, on the assumption that the outer bands of wind in the stormy circle revolve in the same time with the inner bands, the velocity of the former being in that case greater, their centrifugal force ought also to be greatest, so that the barometer would fall more in the outskirts than near the centre of the storm,—a result which, as we have seen, would be inconsistent with the fact. It is only on the supposition that there is a central axis of revolution, which causes the bands of wind that are nearest to it to revolve more rapidly than those which are more distant, that a diminished atmospheric pressure can there be brought about.

But there are other causes which may contribute to the observed effect. By the rapid rotation of the central parts, and consequent centrifugal force produced there, the air acquires a certain degree of attenuation, which must diminish the pressure on the mercurial column. This view is strongly supported by the fact, that the barometer has been observed to sink in the immediate vicinity of water-spouts,* which, by their vertiginous motion, must necessarily attenuate the atmosphere in contact with them.

The reasons now assigned or suggested for a fall of the barometer during storms, are applicable to them generally, from whatever quarter of the globe they proceed. But when a storm comes from the southward, it brings alongst with it a warm temperature, which speedily diminishes the weight of the atmosphere. The column of air, on being heated, expands and rises, flowing off laterally into cooler parts, the effect of which is immediately to lessen the atmospheric pressure on all places within the column, and to increase the pressure on places situated beyond its verge.

When the two storms described in this paper approached the British islands, all the causes now noticed, probably combined to depress the barometer. Previous to their arrival in these latitudes, there had prevailed for some days a strong north-easterly gale, which had caused the barometer to continue high. The whirling columns of warm air, as they advanced northwards, had therefore to contend against the cold wind blowing in an opposite direction. Now, it is obvious, that, when these two winds met and mixed, the cold air would continue to occupy the surface, even in its retreat before the southern storms. The latter would therefore affect the higher regions of the atmosphere in Europe, and especially in Britain, before the north-east wind was entirely arrested. The upper part of the revolving column would precede and overhang the under part, which would be farther impeded in its course by the surface of the sea or land over which it traversed. From this cause, both storms must have affected the upper regions of the British atmosphere, for some days before they began to sweep over the British islands, and these upper regions being heated, would immediately cause the barometer to sink. Accordingly it has been seen, that, before the arrival of either of the storms above described, the barometer began to sink in all parts of Great Britain. It has been shewn that the track of the storm's centre was most probably about 200 miles to the west of Ireland. On its arrival there, the atmosphere would, of course, attain a maximum temperature, and at that period,

* Oersted on Water-spouts. His memoir is translated in the Edinburgh Philosophical Journal for July 1839.

or very shortly thereafter, the barometer naturally reached its lowest point. But, as the storm passed to the northward, the other causes which combined with temperature to lower the mercury also passed away, and thus allowed it to rise again.

It was observed in the second storm, as well as in the first, that it was when the wind had veered to the S. or SW. that the barometer every where reached its lowest point. At Greenwich Observatory, the wind is registered SE. by S. at 2 P. M. on the 28th, and it had veered to SW. by S. before 9 o'clock next morning, the barometer having reached its minimum point in the interval. At Paris, the wind at 9 P. M. on the 28th was S.SE., and before 9 next morning the wind had veered to S.SW., during which interval the barometer reached its minimum. At Adare Abbey, the barometer reached its minimum about 4 P. M. on the 28th, at which period the wind had veered to SW. from S. by E. Similar results are indicated by the registers of all the other places of which I have obtained extracts, viz. Kinfauns, Abbey St Bathan's, Carlisle, Castle Toward, Inveresk, &c. When the centre of the storm came nearest to any of these places, the wind, according to the rotatory theory, must (if the storm was moving N.NE.) have been blowing about S.SW., which actually was the direction of the wind at these places, when the barometer reached its lowest point. The south-westerly blasts were thus an indication to the places swept by them that the storm's centre was then passing nearest to them. Other reasons also conspire to make the barometer reach its minimum, during the prevalence of a south and south-westerly wind. The south wind is necessarily warmer than any other; and if the storm to which it belongs, happens to be advancing in a direction due north, it will have a greater velocity than any other, combining its own circular motion with the progressive motion of the storm. If the storm has a NE. direction, it will of course be the SW. wind which will have the greatest velocity.

Perhaps it might, in this view, be considered that some test of the direction of the storm in its progressive course would be indicated by the wind which blows strongest at a given place. The particular winds which, in the second of the storms above described, were the most violent at different places, are stated in the following Table:—

Places.	Strongest Wind.	Time of Strongest Wind.
Adare Abbey (Limerick) . . .	SE. by E.	On 28th Novem. about 9 A. M.
Coloony* (north of Ireland) .	E. to SE.	On do. about noon.
Castle Toward*	S.SE.	
Inveresk and Redheugh station .	S.	On 29th do.
London and Greenwich . . .	SW.	On 29th do. in afternoon.
Paris	S.SW.	On do. at night.

* The direction of the strongest wind at Coloony and Castle Toward, is taken from the way in which the trees blown down there, were lying. The direction and force of the wind at the other places are derived from meteorological registers.

If any weight be attached to the remarks above made, that the direction of the strongest wind in the storm will, generally speaking, coincide with the path of the storm, this table would indicate that, in the latitude of Paris, the path lay in a N.NE. direction, and that, as it advanced northwards, it moved more directly N., and even ultimately towards the N.NW. This inference is corroborated by the circumstance to be immediately noticed, that, though the storm was most severely felt in the south of England, it was less felt on the north-east coast of England, and scarcely at all experienced on the east coast of Scotland. There was far more damage done in the Irish than in the English Channel.

On comparing the above table, shewing the period of the strongest wind, with the one on page 475, which states the period when the barometer reached its greatest depression, it will be observed that these periods are not the same; and, according to the theory of rotation, they should not be the same. The strongest wind at Adare Abbey being the SE. wind, preceded the centre of the storm, and consequently preceded also the greatest depression of the barometer. In like manner, at London and at Paris, the strongest wind having been from SW. and S.SW. (which followed the centre of the storm), was felt more than half a day after the time of lowest barometrical depression.

I may add, that this storm, or rather the eastern segment of it which traversed the British islands, became much mitigated in violence, as it proceeded northwards. There was not half the damage done by it in Scotland, which it effected in the southern and midland counties of England and Ireland. One cause of this may be, its having overtaken in Scotland and the north of Ireland, the storm which preceded it; and as the van-guard circles of the second storm were, of course, rotating in a direction opposite to that of the rear-guard circles of the first, the two would interfere where they impinged, and thus, to a certain extent, neutralize each other. The second storm being the more violent and extensive, would of course obtain the mastery; a circumstance which explains why, in Scotland and the north of Ireland, the first storm hardly exhibited any westerly or north-westerly blasts, before it ceased. It must have been owing to this interference of the two storms, causing an annihilation of the first, and a diminution of the second, that whilst, on the west coast of Scotland, the meteorological registers shew pretty distinctly the several features of the second storm, viz. its commencement, its veering, and its cessation, the registers on the north and north-east coast contain no such information, and do not even indicate the occurrence of a storm, but merely the continuance of the previous gale, interrupted, however, by frequent gusts between E. and S. At Dunnet Head, Sumburgh Head, Pentland Skerries, and the Starting Point, the wind never veered to the west of south.

At Inverness, as I learn from the very accurate register kept by Mr Adam, Rector of the Inverness Academy, there were, on the whole of the 28th November, "light airs from the *NE.*, and clouds from *SE.*" On the 29th, the entry in the wind and weather column is, for the morning, "light wind *N. by E.*, and clouds from *S. by E.*:—in the evening, "*calm*, slight rain, and clouds from *south*." On the 30th, the air on the surface of the earth at Inverness was *calm*, but there were "clouds and showers from *west*." These entries clearly indicate that, in that northern latitude, the storm had become nearly expended, but was still faintly existing, with all its characteristic features, in the upper regions of the atmosphere.

The effect of this gale on the waters of the Atlantic caused an unusually high tide in almost all the parts in the Irish and English Channel. I find that, on Wednesday night the 28th November, Newry, a town to the north of Dublin, was inundated by the highest tide ever remembered. It was also a remarkably high tide at Strangford and at Donaghadee. On the same night, at Swansea the tide rose seven feet two inches above its proper level. At Milford, the tide rose higher than it had ever been seen before. At Plymouth the tide rose over the quays, an occurrence said to have been unprecedented. On the Thursday forenoon the tide rose in the Thames, and also at Greenock, Oban, Tobermory, and in Orkney, above the level of the quays. At Oban and Tobermory, though these places are completely land-locked, and exposed to no swell from the ocean, all loose materials lying on the quays were swept off by the mere rise of the tidal waters. The height of the tide was there the more remarkable, as it was the season not of spring but of neap tide. That this extraordinary elevation of the sea was occasioned by the suddenly diminished pressure of the atmosphere, there is no doubt. The effect of this diminished pressure, must have been to elevate the surface of the ocean, and produce a sudden accumulation of waters,—a species of wave. The accumulation would take place along the line of diminished pressure, or, in other words, in the direction of the storm. This storm-wave (for such it may not improperly be termed) moved therefore through the Atlantic in a N.NE. direction, and happening to impinge on Great Britain and Ireland about the time of high-water, caused the waters to overflow. That this wave had been produced not in the British seas, but a great distance in the Atlantic, is evident from this, that it preceded by several hours the arrival of the most violent part of the hurricane, and even the lowest depression of the barometer. Any undulation in the waters of the ocean, it is well known, is very rapidly propagated. The earthquake at Lisbon produced a wave in the Atlantic, which caused an unusually high tide on the south coasts of England and Ireland. The first shock of this earthquake took place at half-past eight in the morning. A wave produced by it, about five and a half feet high, flowed into Kinsale harbour on the afternoon of the same day, between 2 and 3 P. M.; so that this wave must have travelled at the rate of about 180 miles an hour. It is, therefore, obvious, that a wave raised in the Atlantic, by the same force which originates or accompanies a storm, may easily precede the storm, and give warning of its advent.

In the previous part of this memoir, there has been an account given of only two storms. It would appear from the various registers, of which I have obtained extracts, that a third gale invaded this part of the globe between the 3d and 5th December. It was accompanied, like the two former, with the phenomena of veering, barometrical depression, and of a northerly course, by which the two storms just described were characterized.

I have only to add, in conclusion, that, for several years, during the last week of November or first week of December, there has been a violent storm in this part of the globe.

On the 28th November 1837, there was a severe storm in Great Britain, which did considerable damage in Scotland.

On the 29th November 1836, a tremendous storm visited the south and west coasts of England, which occasioned immense damage. It carried away the Chain Pier at Brighton, partly unroofed several public buildings in Plymouth (where the tide rose three feet and a half above its proper level), and blew down 200 trees in the London parks. It moved in a NE. direction, and passed over to the northern parts of France and Germany. It was not felt in Ireland or Scotland.

On the 22d and 23d November 1824, a severe storm ravaged the southern coasts of England, and then passed over to Holland and Jutland. It raged on the night of the 22d at the Scilly Isles and Plymouth. On the 23d it reached the Nore, and occasioned much damage to the shipping in the harbours and at sea. The wind is described as having been very violent, and accompanied by abundance of lightning. The Eddystone lighthouse was greatly injured. In Sweden, extensive forests were prostrated.

On the 17th December 1747, O. S., there was a violent storm which ravaged England, and which was ascertained to have extended into Germany.

On the 1st November 1740, O. S., there was a hurricane which caused extensive damage in London.

On the 14th November 1739, O. S., there was a hurricane which did great damage in Edinburgh, blowing down several houses, and injuring St Giles's steeple.

On the 28th November 1703, O. S., a hurricane overthrew the Eddystone lighthouse, and destroyed in and near London, property to the value of two millions. The most violent part of the hurricane was from SW. to W.SW.

XXV.—*On the Diminution of Temperature with Height in the Atmosphere, at different seasons of the year.* *By* JAMES D. FORBES, *Esq., F.R.SS. L. & E., Professor of Natural Philosophy in the University of Edinburgh.*

(Read 1st April 1839).

In the year 1830, I succeeded in establishing a Register of the Thermometer at the Bonally Reservoir, which formerly supplied the city water-works, being at a distance of five miles in a direction south-west from Edinburgh. This station is on the northern acclivity of the Pentland Hills, at a height of 1100 feet above the sea. The following year I obtained corresponding observations at the village of Colinton, situated a mile and a half north of the preceding station, and above 700 feet lower. Although this difference of level be not very considerable, yet, as these comparative registers have been kept for nearly five years with pretty uniform results, some confidence is evidently due to the conclusions, even although considerable difficulties opposed themselves to obtaining registers quite free from exception. The interest attaching to them is the greater, that, although registers have been kept at Leadhills and other elevated stations, I do not recollect any strictly comparative observations in Scotland, perhaps not even in Great Britain, at two stations near one another, and differing considerably in level, from which the important meteorological element of the decrement of temperature in the atmosphere could be deduced.

The Bonally station is situated on the exposed northern acclivity of the Pentland Hills, without any kind of shelter. Its elevation above the mean level of the sea was very accurately determined by myself trigonometrically, and the thermometer hung at a height of precisely 1100 feet. The exposure was the north side of a cottage, which has since been allowed to fall to ruin. The observations were made by Mr JOHNSTON, the officer appointed by the Water Company for the inspection of their works, and by his family. I have every reason to believe that they were made and registered with perfect fidelity, although, from want of practice, they may have been occasionally erroneously entered. They were made daily at 8 A. M. and 8 P. M. The thermometer was a mercurial one, now in my possession, which, by comparison with a standard one, I find reads pretty constantly 0°.35 too high.* The readings have therefore been diminished by that quantity.

The Colinton station was at the School-house there, and the observations were carefully made and registered by my friend the Rev. R. HUNTER. The

* For one year only a spirit thermometer was employed.

height of the thermometer above the mean level of the sea, ascertained by myself, is 364 feet. The hours of observation were the same as above. The thermometer has been carefully compared with a standard, and the error in different parts of the scale not being uniform, it has been ascertained, and a corresponding correction applied.

By far the greater part of the calculation of these observations was performed by my late friend and pupil Mr John Spens, son of Dr Thomas Spens, who, had he lived, must ultimately have distinguished himself in a profession which rarely fails to reward real talent. Much of the remaining calculation was kindly undertaken by Mr John T. Harrison.

The mean temperature of each month at each station at 8 a. m. and 8 p. m. being taken, the mean difference for each month of the year for the whole period is deduced, and hence the mean for the entire period, which gives a decrement for 736 feet of ascent, amounting to 3°.27 for the morning observations, 3°.18 for the evening, or 3°.22 for both, which corresponds to 229 feet of ascent for 1° of decrement of temperature. This decrement is rather rapid, and is, no doubt, partly to be accounted for by the comparatively sheltered situation of the lower station.

The influence of the season of the year on the decrement of temperature is particularly striking, as the following Table shews; and that the discrepancies it contains are not generally errors of observation, is pretty clear, from the agreement of the morning and evening columns, and various other tests, which it is not necessary to mention.

TABLE I.

Calculation of the Mean Temperature of each Month during the Years 1831–32–33–34–35, at Bonally and Colinton, and corrected for the errors of Graduation of Thermometers.

Date.	8 A. M.					8 P. M.				
	Bonally.	Colinton.	Bonally corrected.	Colinton corrected.	Diff.	Bonally.	Colinton.	Bonally corrected.	Colinton corrected.	Diff.
Jan. 1832	35.61	39.29	35.26	38.97	3.71	35.93	38.39	35.58	38.06	2.48
... 1833	31.58	31.77	31.23	31.23	0.00	32.64	33.77	32.29	33.32	1.03
... 1834	37.74	39.32	37.39	39.00	1.61	38.06	40.16	37.71	39.86	2.15
... 1835	33.97	36.39	33.62	36.00	2.38	34.71	38.16	34.36	37.82	3.46
			137.50	145.20	7.70			139.94	149.06	9.12
		Mean	34.38	36.30	1.92		Mean	34.98	37.26	2.28
Feb. 1831	34.93	37.11	34.58	36.75	2.17	35.75	37.64	35.40	37.29	1.89
... 1832	36.31	39.00	35.96	38.68	2.72	37.00	38.45	36.65	38.12	1.47
... 1833	35.14	37.86	34.79	37.50	2.71	36.36	38.61	36.01	38.28	2.27
... 1834	36.43	37.46	36.08	37.10	1.02	37.46	39.50	37.11	39.19	2.08
... 1835	36.89	39.78	36.54	39.48	2.94	36.43	40.21	36.08	39.92	3.84
			177.95	189.51	11.56			181.25	192.80	11.55
		Mean	35.59	37.90	2.31		Mean	36.25	38.56	2.31

Table I.—continued.

Date.	8 A. M.					8 P. M.				
	Bonally.	Colinton.	Bonally corrected.	Colinton corrected.	Diff.	Bonally.	Colinton.	Bonally corrected.	Colinton corrected.	Diff.
Mar. 1831	37.61	42.42	37.26	42.24	4.98	38.45	43.00	38.10	42.82	4.72
... 1832	36.68	42.48	36.33	42.28	5.95	37.68	40.64	37.33	40.37	3.04
... 1833	34.87	38.48	34.52	38.15	3.63	35.39	37.48	35.04	37.13	2.09
... 1834	39.16	42.61	38.81	42.40	3.59	37.93	41.84	37.58	41.60	4.02
... 1835	36.26	38.35	35.91	38.02	2.11	36.48	38.42	36.13	38.10	1.97
			182.83	203.09	20.26			184.18	200.02	15.84
		Mean	36.56	40.61	4.05		Mean	36.83	40.00	3.17
Apr. 1831	41.30	45.47	40.95	45.38	4.43	42.34	45.10	41.99	45.00	3.01
... 1832	42.13	46.63	41.78	46.56	4.78	41.13	44.63	40.78	44.50	3.72
... 1833	41.37	44.67	41.02	44.55	3.53	39.90	43.83	39.55	43.68	4.13
... 1834	41.37	46.53	41.02	46.46	5.44	40.43	46.23	40.08	46.15	6.07
... 1835	40.97	45.60	40.62	45.51	4.89	40.00	45.80	39.65	45.72	6.07
			205.39	228.46	23.07			202.05	225.05	23.00
		Mean	41.08	45.69	4.61		Mean	40.41	45.01	4.60
May 1831	46.48	50.39	46.13	50.40	4.27	46.58	49.93	46.23	49.93	3.70
... 1832	45.48	50.93	45.13	50.95	5.82	45.19	49.32	44.84	49.31	4.47
... 1833	52.06	56.97	51.71	57.20	5.49	51.74	55.19	51.39	55.40	4.01
... 1834	49.41	53.97	49.06	54.10	5.04	49.90	53.80	49.55	53.95	4.40
... 1835	45.61	50.10	45.26	50.10	4.84	45.39	49.87	45.04	49.87	4.83
			237.29	262.75	25.46			237.05	258.46	21.41
		Mean	47.46	52.55	5.09		Mean	47.41	51.69	4.28
June 1831	55.23	58.97	54.88	59.24	4.36	55.77	58.27	55.42	58.52	3.10
... 1832	53.97	57.83	53.62	58.09	4.47	53.07	57.00	52.72	57.24	4.52
... 1833	53.50	57.37	53.15	57.62	4.47	52.30	55.90	51.95	56.12	4.17
... 1834	54.43	57.57	54.08	57.81	3.73	54.97	56.97	54.62	57.20	2.58
... 1835	53.37	55.43	53.02	55.64	2.62	51.90	55.27	51.55	55.48	3.93
			268.75	288.40	19.65			266.26	284.56	18.30
		Mean	53.75	57.68	3.93		Mean	53.25	56.91	3.66
July 1831	57.68	60.06	57.33	60.36	3.03	58.27	60.77	57.92	61.07	3.15
... 1832	55.16	59.03	54.81	59.30	4.49	55.52	57.84	55.17	58.09	2.92
... 1833	57.03	59.61	56.68	59.90	3.22	55.68	58.55	55.33	58.82	3.49
... 1834	56.16	59.19	55.81	59.47	3.66	57.00	59.26	56.65	59.54	2.89
... 1835	55.77	57.64	55.42	57.90	2.48	54.81	57.07	54.46	57.31	2.85
			280.05	296.93	16.88			279.53	294.83	15.30
		Mean	56.01	59.38	3.37		Mean	55.90	58.96	3.06
Aug. 1831	57.06	59.71	56.71	60.00	3.29	55.90	59.55	55.55	59.84	4.29
... 1832	55.23	57.52	54.88	57.77	2.89	53.68	55.48	53.33	55.70	2.37
... 1833	52.06	55.55	51.71	55.76	4.05	50.45	54.97	50.10	55.16	5.06
... 1834	55.84	59.87	55.49	60.17	4.68	54.48	59.19	54.13	59.47	5.34
... 1835	57.58	58.84	57.23	59.12	1.89	56.00	58.97	55.65	59.25	3.60
			276.02	292.82	16.80			268.76	289.42	20.66
		Mean	55.20	58.56	3.36		Mean	53.75	57.88	4.13
Sept. 1831	50.87	54.27	50.52	54.44	3.92	50.97	54.14	50.62	54.30	3.68
... 1832	51.77	53.87	51.42	54.02	2.60	50.63	53.40	50.28	53.54	3.26
... 1833	49.13	52.83	48.78	52.95	4.17	48.57	52.47	48.22	52.57	4.35
... 1834	50.83	52.70	50.48	52.80	2.32	50.07	52.87	49.72	52.98	3.26
... 1835	50.03	50.80	49.68	50.83	1.15	49.17	51.27	48.82	51.32	2.50
			250.88	265.04	14.16			247.66	264.71	17.05
		Mean	50.18	53.01	2.83		Mean	49.53	52.94	3.41

TABLE I.—continued.

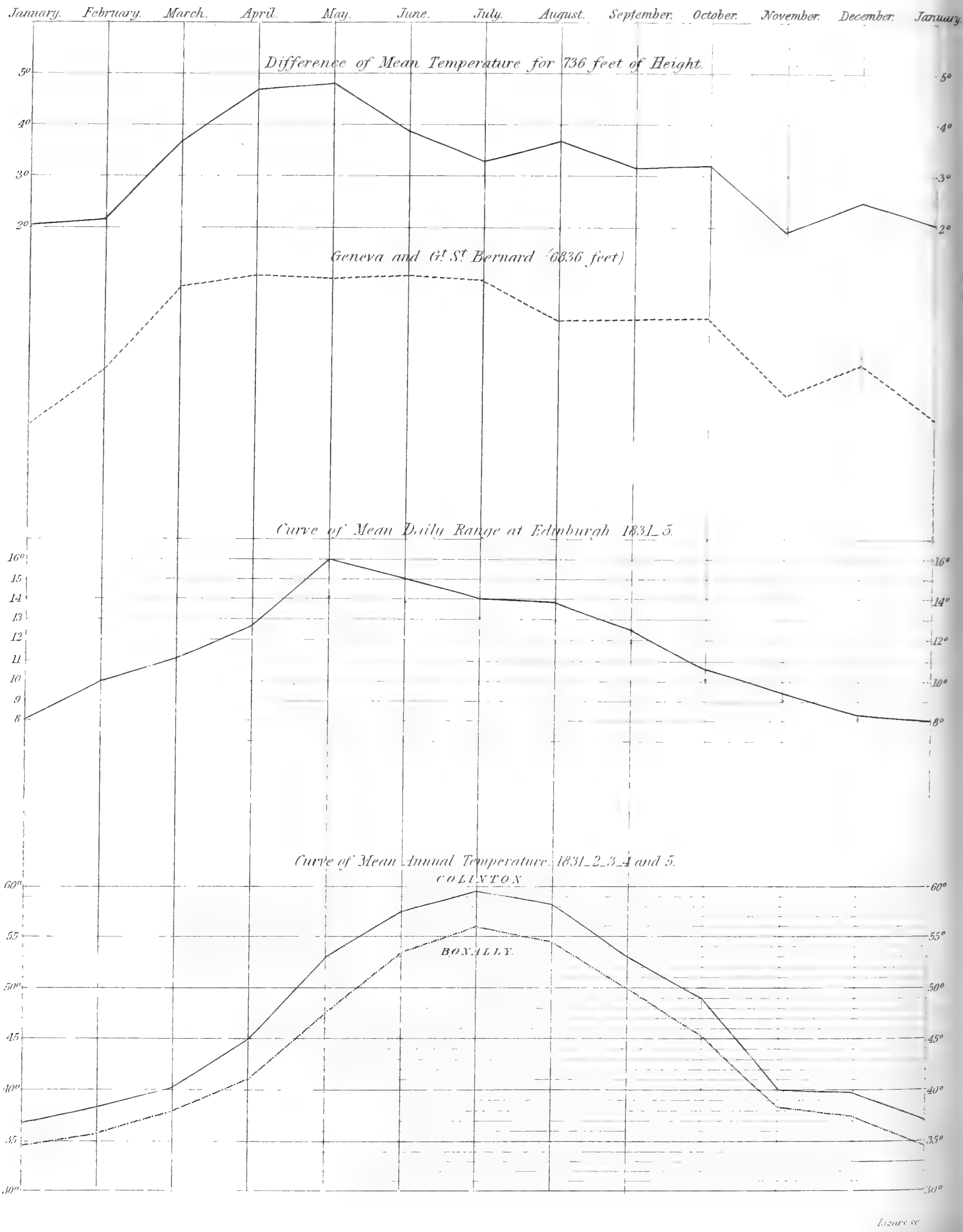
DATE.	8 A. M.					8 P. M.				
	Bonally.	Colinton.	Bonally corrected.	Colinton corrected.	Diff.	Bonally.	Colinton.	Bonally corrected.	Colinton corrected.	Diff.
Oct. 1831	49.71	53.10	49.36	53.22	3.86	50.06	52.13	49.71	52.21	2.50
... 1832	47.03	50.77	46.68	50.80	4.12	46.45	49.16	46.10	49.14	3.04
... 1833	45.61	48.06	45.26	48.03	2.77	45.32	47.45	44.97	47.40	2.43
... 1834	45.58	48.87	45.23	48.85	3.62	45.52	48.87	45.17	48.85	3.68
... 1835	42.51	44.77	42.16	44.66	2.50	41.87	44.45	41.52	44.33	2.81
			228.69	245.56	16.87			227.47	241.93	14.46
		Mean	45.74	49.11	3.37		Mean	45.49	48.38	2.89
Nov. 1831	36.73	40.03	36.38	39.73	3.35	37.73	39.50	37.38	39.19	1.81
... 1832	38.20	39.40	37.85	39.09	1.24	38.67	40.10	38.32	39.80	1.48
... 1833	38.17	40.67	37.82	40.39	2.57	38.63	39.70	38.28	39.40	1.12
... 1834	40.30	41.37	39.95	41.13	1.18	40.23	42.53	39.88	42.33	2.45
			152.00	160.34	8.34			153.86	160.72	6.86
		Mean	38.00	40.08	2.08		Mean	38.46	40.18.	1.72
Dec. 1831	39.23	41.81	38.88	41.57	2.69	38.93	41.81	38.58	41.58	3.00
... 1832	36.19	39.61	35.84	39.30	3.46	36.81	39.93	36.46	39.63	3.17
... 1833	35.87	38.42	35.52	38.09	2.57	36.10	38.53	35.75	38.20	2.45
... 1834	39.35	39.77	39.00	39.47	0.47	39.35	41.35	39.00	41.10	2.10
			149.24	158.43	9.19			149.79	160.51	10.72
		Mean	37.31	39.61	2.30		Mean	37.45	40.13	2.68

TABLE II.

General Synopsis.

MONTH.	8 A. M.			8 P. M.			Mean Diff.	Feet of Ascent for 1° Fahr.
	Bonally.	Colinton.	Diff.	Bonally.	Colinton.	Diff.		
January . . .	34.38	36.30	1.92	34.98	37.26	2.28	2.10	351
February . .	35.59	37.90	2.31	36.25	38.56	2.31	2.31	319
March . . .	36.56	40.61	4.05	36.83	40.00	3.17	3.61	204
April. . . .	41.08	45.69	4.61	40.41	45.01	4.60	4.60	159
May	47.46	52.55	5.09	47.41	51.69	4.28	4.69	157
June	53.75	57.68	3.93	53.25	56.91	3.66	3.79	194
July	56.01	59.38	3.37	55.90	58.96	3.06	3.22	229
August . . .	55.20	58.56	3.36	53.75	57.88	4.13	3.74	197
September . .	50.18	53.01	2.83	49.53	52.94	3.41	3.12	236
October . . .	45.74	49.11	3.37	45.49	48.38	2.89	3.13	235
November . .	38.00	40.08	2.08	38.46	40.18	1.72	1.90	387
December . .	37.31	39.61	2.30	37.45	40.13	2.68	2.49	296
	531.26	570.48	39.22	529.71	567.90	38.19	38.70	
General Mean	44.27	47.54	3.27	44.14	47.32	3.18	3.22	229

I have compared the results of this Table, which are projected in the uppermost curve of Plate XX., with the results obtained on the far larger scale of a difference of level of 6836 English feet, between Geneva and the Convent of St Bernard, as given by KAEMTZ in the second volume of his *Lehrbuch der Meteorologie*. Considering the different circumstances of the two, these curves (which I have purposely reduced to a similar range) approximate wonderfully. They both indi-


January. February. March. April. May. June. July. August. September. October. November. December. January.
Difference of Mean Temperature for 736 feet of Height.
Geneva and Gt. St. Bernard (6836 feet)
Curve of Mean Daily Range at Edinburgh 1831_5.
Curve of Mean Annual Temperature 1831_2_3_4 and 5.
COLINTON
BONALLY
Lizars sc

cate a most rapid increase of the Difference of Temperature between February and March, and a most rapid decline in November, the maximum being about May.

That the decrement of temperature with height is most rapid in summer, and least in winter, has been long known; * but I am not aware of any attempt to account for the law of its variation at different seasons. The following considerations will probably be found satisfactory.

If we examine the annual curves of mean temperature at Colinton and Bonally, projected in the lower part of Plate XX., we shall find that they differ in three respects, whilst there is a remarkable coincidence in their general features. (1.) The entire Bonally curve stands lower on the paper than the Colinton curve, because the mean temperature of any and every part of the year is lower. (2.) It is a *flatter* curve than the Colinton curve, or the range of the thermometer is less; consequently the *minima* differ *less* than if the two curves had been similar, and the *maxima* differ *more*. This is the reason why the decrement of temperature with height is most rapid in summer, and least so in winter. (3.) Not only is the Bonally curve lower than the Colinton one, and flatter, but it is *shifted* to the right hand, so that the maxima occur later, as well as the minima and mean temperatures. A little attention will likewise shew that a gap or vacuity must be left between the curves, greatest whilst the temperature rises, and least whilst it falls; and also that the difference of the vertical ordinates of the two curves will be greatest when they form the greatest ascending angle with the horizontal axis, and least when the descending angle is greatest, that is, as inspection shews, in May and November respectively, which agrees with the results of the uppermost curve of the plate.

The examination of these curves furnishes us with some data of the most important kind for meteorology, which it is best in the first place to state, and afterwards to consider how we can explain.

The first fact is the familiar one, that mean temperature diminishes as we ascend in the atmosphere. The second is, that the annual range diminishes as we rise, and, at a certain height, would probably sensibly vanish. The third is, that the influence of seasons begins to be felt at the plains, and is later communicated to the mountains. The two former of these facts obtain with reference to the diurnal as well as annual variation of temperature; the last appears to be in that case reversed.†

The *shift* of the annual curve, or retardation of epochs, and likewise the decreased range, is common to the strata of the air above the surface of the earth, and to those of the soil beneath it. Both ultimately, no doubt, exhibit a limit, first where the diurnal variations disappear, then the annual. The cause, how-

* See my Report on Meteorology in the first volume of the British Association Reports.

† SAUSSURE, Voyages dans les Alpes, tom. iv. § 2050, &c. See also KAEMTZ, Lehrbuch, band ii. s. 133.

ever, is very different in the two cases, the one being chiefly the result of the radiation and the other of the conduction of heat.

It is only curious that the diurnal curve seems to follow so different a law, at least in summer;—perhaps the reason is, that the direct solar radiation is more energetic in that case, and the vehicular conveyance of heat by the air (or convection) less. Thus, with respect to the process of annual heating, the earth's surface (considered as an extensive plain) is the point where the sun's rays freely transmitted by the atmosphere first become productive of any considerable warmth. That warmth is propagated slowly and progressively by conduction to the inferior strata of earth, and by convection to the superior strata of air; in either case, as I have said, a *later* and a *feebler* impress of the annual curve is found. The diurnal temperature is probably much more modified by the *direct* effects of radiation. The detached mountain tops, more exposed and less massive, receive and part with the solar heat more rapidly than the low country, presenting a complete analogy, the former with an insular, the latter with a continental climate. The summits change temperature rapidly, the extremes are less; but the changes of the heat of the plain follow later, and are more marked. This is not conjecture; many facts might be quoted to support it, but the following is sufficient, that SAUSSURE, in the part of his work already cited, finds, that, whilst the minimum temperature occurred at 4 A. M. (in the month of July) both at the Col du Géant and at Geneva, the former station had acquired the mean temperature of the day at 6 A. M., which at Geneva occurred three hours later; and, during the decline of temperature in the afternoon, the mean recurred at the Col du Géant from half an hour to an hour *sooner* than at Geneva.

There are other causes besides those just mentioned which contribute to distinguish the daily from the annual curve. Of these the more important are the more gradual character of the annual change of temperature, and the influence of humidity. The former affects our experiments by preventing the ascending and descending currents from being instantly established, in the manner that the law of specific gravity would assign; and when radiation is least intense (as in winter), and the moving power therefore small, this transfer is often impeded, and even the law of densities violated altogether. This we know to be often the case in winter and in cold climates, that the higher strata are the warmer. To place this in a clear point of view, I shall add a table shewing the number of times in each month that this has occurred, which is indicated under the column headed "Number of times negative:" considering the differences of temperature simultaneously observed at Bonally and Colinton as *positive* when the former (the higher station) was colder than the latter; and *vice versa*. I have also added the extreme positive and negative values for each month; and though here, more than any where else, errors of observation and registration are likely to have crept in, yet we cannot but be struck with the number of times in which the common law of density has been reversed, and the great excess of warmth observable at the higher station on some occasions, especially in autumn and winter. I would repeat, however, that the observation of these extremes is less likely to be invariably correct than any other part of the tables. Dividing the year into four seasons, the following summary, which includes both morning and evening observations, is, from the extent of the induction, entitled to considerable confidence.

	Mean Difference.	Times out of 100 negative.	Mean of greatest + values in each Month.	Mean of greatest — values in each Month.
SPRING (March, April, May)	4°.30	7.1 per cent.	10°.5	2°.87
SUMMER (June, July, August)	3°.58	13.5 ...	10°.3	4°.8
AUTUMN (Sept., Oct., Nov.)	2°.72	15.0 ...	9°.8	7°.4
WINTER (Dec., Jan., Feb.	2°.30	15.6 ..	9°.8	5°.9

These numbers have been obtained from the following Table, which contains the details.

TABLE III.

	8 A. M.				8 P. M.			
	No. of Observations.	No. of times negative in each Month.	Greatest + value.	Greatest — value.	No. of Observations.	No. of times negative in each Month.	Greatest + value.	Greatest — value.
Jan. 1832	31	1	16	— 1	31	4	13	— 2
... 1833	31	9	4	— 7	31	5	5	— 7
... 1834	31	7	13	— 9	31	12	10	— 7
... 1835	31	2	10	— 7	31	3	16	— 5
Feb. 1831	28	4	8	— 7	28	5	9	— 8
... 1832	29	3	9	— 7	29	9	7	— 4
... 1833	28	2	8	— 1	28	3	12	— 4
... 1834	28	8	5	— 6	28	7	15	— 7
... 1835	28	2	12	—10	28	2	12	— 6
Mar. 1831	31	1	14	— 2	31	1	14	— 2
... 1832	31	1	10	— 1	31	2	8	— 6
... 1833	31	1	7	— 1	31	1	6	— 2
... 1834	31	3	12	— 6	31	2	17	— 7
... 1835	31	7	6	— 5	31	6	7	— 2
Apr. 1831	30	2	8	— 2	30	2	6	— 3
... 1832	30	3	9	— 2	30	3	13	— 3
... 1833	30	3	7	— 1	30	0	9	0
... 1834	30	0	14	— 0	30	0	15	0
... 1835	30	2	13	— 3	30	2	17	— 2
May 1831	31	2	10	— 2	31	3	9	— 1
... 1832	31	1	9	— 3	31	1	11	— 3
... 1833	31	3	11	— 2	31	4	12	— 9
... 1834	? ...	...	...	...	...	...	...	...
... 1835	31	3	10	— 5	31	2	10	— 4
June 1831	30	0	10	0	30	5	11	— 3
... 1832	30	3	11	— 8	30	4	17	— 2
... 1833	30	2	9	— 4	30	1	8	— 1
... 1834	30	3	6	— 4	30	6	10	—12
... 1835	30	4	9	—10	30	6	15	— 9

	8 A. M.				8 P. M.			
	No. of Observations.	No. of times negative in each Month.	Greatest + value.	Greatest — value.	No. of Observations.	No. of times negative in each Month.	Greatest + value.	Greatest — value.
July 1831	31	7	9	— 2	30	5	10	— 3
... 1832	31	2	12	— 2	31	5	10	— 3
... 1833	31	3	8	— 5	31	4	8	— 3
... 1834	31	2	6	— 4	31	5	9	—12
... 1835	31	11	13	— 5	31	8	11	— 8
Aug. 1831	31	5	10	— 2	31	1	10	— 3
... 1832	31	6	7	— 4	31	6	6	— 8
... 1833	31	1	10	— 2	31	2	11	— 3
... 1834	31	2	17	— 6	31	2	15	— 5
... 1835	31	9	10	— 6	31	5	12	— 5
Sept. 1831	30	3	9	— 4	30	1	8	— 2
... 1832	30	4	12	—13	30	4	18	— 5
... 1833	30	2	15	— 2	30	1	11	— 2
... 1834	30	5	6	— 6	30	6	9	— 8
... 1835	30	11	7	— 7	30	9	11	— 6
Oct. 1831	31	0	7	...	31	2	4	— 6
... 1832	31	2	15	— 5	31	3	13	— 2
... 1833	31	5	10	— 4	31	5	12	— 7
... 1834	31	4	14	— 7	31	5	17	— 7
... 1835	31	3	5	—10	31	4	10	— 7
Nov. 1831	30	2	7	— 8	30	5	7	— 9
... 1832	30	5	5	—10	30	5	5	—10
... 1833	30	7	9	—19	30	8	12	—20
... 1834	30	10	8	— 7	30	5	12	— 6
Dec. 1831	31	2	7	— 8	31	5	7	— 9
... 1832	31	1	11	— 3	31	1	10	— 1
... 1833	31	1	7	— 3	30	5	9	— 7
... 1834	31	9	6	—13	31	10	13	— 5

The influence of humidity I believe to be very important in modifying the results. The distribution of moisture as we rise in the atmosphere varies extremely at different seasons. In spring the hills are chilled by continued condensations of moisture, whilst the plains are comparatively dry; and in autumn the reverse often occurs. I believe that the actual fall of rain on low and high grounds confirms this view, the autumnal rains being often heaviest in the plains, whilst in spring and summer the excess is amongst the hills.

The curve in Plate XX.,* representing the mean daily range for five years, is deduced from careful observations made at Edinburgh by Mr Adie, with self-registering instruments.†

* The vertical lines in the plate correspond to the middle of each month.

† The latter part of this paper has been remodelled since it was read.—*Dec.* 1839.

POSTSCRIPT.*

I am glad to find that the reasoning I have employed in page 494, to account for the *diurnal* variations of the decrement of heat in the atmosphere, is entirely confirmed by the observations of ESCHMANN, KAEMTZ, and HORNER, in Switzerland, recorded in POGGENDORFF'S *Annalen*, xxvii. 345, and in DOVE'S *Repertorium*, iii. 331. By projecting these observations graphically, I have found that the course of the diurnal curves is such as I have described it to be, the variations on the mountains *preceding* those on the plain, which, by reasoning similar to that in p. 493, will give a maximum difference in the afternoon, and a minimum in the morning. Thus the diurnal summer curve gives the minimum temperature of the day on the Rigi at $3^h\ 43^m$ A. M., and at Zurich at $4^h\ 14^m$ A. M.; whilst the maximum at the former station occurs at $12^h\ 54^m$, and at the latter at $2^h\ 28^m$ P. M. Also, it appears by projection, that the *difference* of temperature at the two stations increases from $4^h\ 31^m$ A. M. until $4^h\ 51^m$ P. M., when it attains its maximum, and then declines.

It is well known that the curves of temperature, whether diurnal or annual, may be expressed with any required degree of accuracy, by a series of terms of the form

$$A+B\sin(x+C)+D\sin(2x+E)\ \&c. \quad\ldots\ldots\ldots \quad \text{Eq. (1)}$$

where x is the time, expressed, (in the case of the diurnal curve), by the horary angle. If, then, the diurnal curves at the two stations be represented by two such series, the *difference* will universally be represented by a series of exactly the same form.

Thus, if the series written above express the ordinates of the curve (annual or diurnal) at the *lower* or *warmer* station, and the following

$$A'+B'\sin(x+C')+D'\sin(x+E')\ \&c. \quad\ldots\ldots\ldots \quad (2)$$

that at the *upper* or *colder* (in which, generally, $A' < A$, and $B' < B$, because the range is less, as well as the temperature lower, and because the second term predominates greatly over the succeeding ones),—then the difference of temperature of the two stations may always be expressed by a series of the same form (x, the time, being still the independent variable), namely,

$$a+b\sin(x+c)+\ \&c. \quad\ldots\ldots\ldots \quad (3)$$

where $a=A-A'$; $b=\sqrt{B^2+2BB'\cos(C-C')+B'^2}$ .... (4)

and $$\tan c=\frac{B\sin C-B'\sin C'}{B\cos C-B'\cos C'} \quad\ldots\ldots\ldots \quad (5)$$

and so of the others.

* Added by permission of the Council, April 1840.

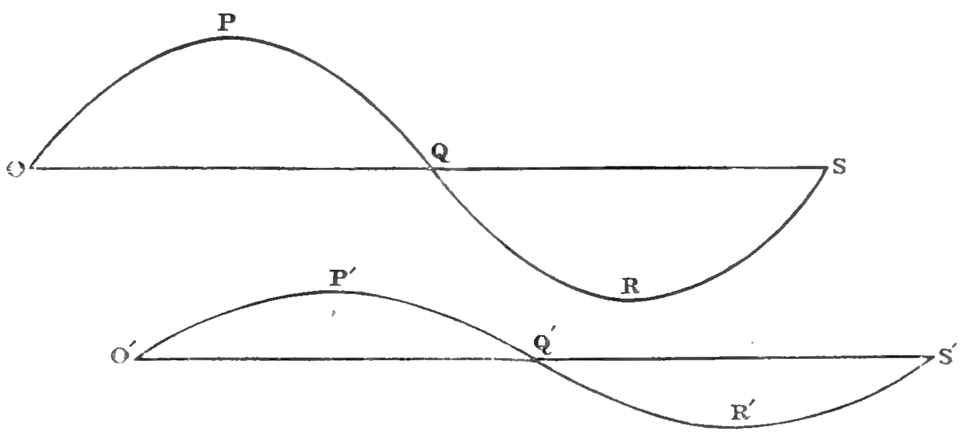
If we consider only the first two terms, which amounts to supposing the curves to become the common curve of sines, the position of maximum or minimum difference of temperature is easily found, and its relation to the elements exhibited.

The value of x (or the horary angle), which will make $a+b\sin(x+c)$ in Eq. (3) a maximum, is evidently

$$x_{,}+c=90^{\circ}\,;\ \text{or } x_{,}=90-c.$$

Whence $$\tan x_{,}=\frac{1}{\tan c}=\frac{\mathrm{B}\cos \mathrm{C}-\mathrm{B}'\cos \mathrm{C}'}{\mathrm{B}\sin \mathrm{C}-\mathrm{B}'\sin \mathrm{C}'}\,.$$

In the figure, let OPQRS represent the curve at the *lower* station, and O′P′Q′R′S′ at the *higher*; if we place the origin of the time at O, C will vanish,



and C′ will denote the acceleration or retardation of the epoch for the temperature curve at the colder, compared to that at the warmer station. Thus the above expression for the time $x_{,}$ of the maximum difference of temperature becomes

$$\tan x_{,}=\frac{\mathrm{B}'\cos \mathrm{C}'-\mathrm{B}}{\mathrm{B}'\sin \mathrm{C}'} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

When $\mathrm{C}'=0$, or, when there is no difference of epoch, $x_{,}=90$, and the maxima of the three curves coincide.

As C′ increases positively, that is, as O′ falls to the left hand of O (which is the case in the diurnal curve), $\tan x_{,}$ being negative (for whilst $\mathrm{B}>\mathrm{B}'$, as we have assumed it to be, the numerator is essentially negative), $x_{,}$ lies somewhere between 90° and 180°. It never, however, reaches the latter value, its greatest excursion being determined by the condition

$$\cos \mathrm{C}'=\frac{\mathrm{B}'}{\mathrm{B}}\,;\ \text{and therefore,}\quad \tan x_{,}=-\sqrt{\frac{\mathrm{B}^2-\mathrm{B}'^2}{\mathrm{B}'^2}}$$

When C′ becomes equal to + 180°, $x_{,}$ has resumed its value of 90°. This corresponds to a coincidence of the minimum of one curve with the maximum of the other, when b in Eq. (4) has its greatest value, which of course is $\mathrm{B}+\mathrm{B}'$, whilst its least value when $\mathrm{C}'=0$ is $\mathrm{B}-\mathrm{B}'$.

In the annual curve, on the other hand, the epochs at the colder station are retarded, O′ falls to the right of O, and C′ is negative. Tan $x_{,}$ becomes positive, and $x_{,}$ recedes from 90° towards 0°; hence the maximum difference *precedes* the greatest annual heat. The greatest recession of $x_{,}$ from 90° is determined by the limiting value

$$\tan x_{,} = \sqrt{\frac{B^2 - B'^2}{B'^2}}$$

and when $C' = -180°$, $x_{,}$ of course becomes again $= 90°$.

In the case of elevation in the atmosphere, the value of C′, or the change of epoch, is probably never extremely great, and the preceding investigation proves the direction of its influence on the period of maximum decrement of temperature, which experience confirms. But three terms would be required to be included in the expression to obtain the result with accuracy, and the successive constants may be derived from expressions similar to those for b and c.

It is interesting to observe the circumstances which modify the *precession* of the epoch of maximum difference in consequence of the *retardation* of temperature epochs at the colder station (as illustrated in the figure on the last page), and *vice versa*. The comparative velocity of displacement of the *relative* maximum and *absolute* maximum will be furnished by the value of the differential coefficient $\frac{dx_{,}}{dC'}$.

Differentiating Eq. (6),

$$\frac{dx_{,}}{\cos^2 x_{,}} = \frac{B B' \cos C' - B'^2}{B'^2 \sin^2 C'} dC'$$

and since $$\cos^2 x_{,} = \frac{1}{1 + \tan^2 x_{,}} = \frac{B'^2 \sin^2 C'}{B'^2 - 2 B' B \cos C' + B^2} \text{ by Eq. (6)}$$

$$\frac{dx_{,}}{dC'} = \frac{B B' \cos C' - B'^2}{B'^2 - 2 B' B \cos C' + B^2}.$$

When $C' = 0$, or small,

$$\frac{dx_{,}}{dC'} = \frac{B B' - B'^2}{(B - B')^2} = \frac{B'}{B - B'}, \text{ accurately or nearly. } \dots\dots\dots \quad (7)$$

which is always positive, since $B' \angle B$.

Now the value of this quantity depends entirely on the ratio of B to B′, or the approximate ranges at the two stations. As B and B′ are more disproportioned, the velocity of the motion of $x_{,}$ diminishes; and as they approximate, it increases indefinitely.

So far as the imperfect data of observation go, this tallies well with facts. Thus, for the annual curves at Colinton and Bonally, discussed in the preceding paper, we have B = 22°.4 B′ = 21°.3

$$\frac{dx_{,}}{dC'} = \frac{21.3}{1.1} = 19.4.$$

In this case, then, for every day that the epoch of the annual curve at Bonally is *retarded*, the epoch of maximum difference, or *quickest decrement* of temperature, is *accelerated* by 19.3 days. Now the data from which the two lower curves of Plate XX. are drawn, indicate a maximum temperature at Colinton on the 22d July, and a *shift* of the Bonally curve backwards of 4.6 days nearly. The corresponding anticipation of the quickest-decrement-epoch would therefore be 4.6 × 19.4 = 89 days, which corresponds with the fact, that the greatest ordinate of the differential curve (the highest curve in the plate) evidently occurs about the beginning of May.

I have examined several other curves by means of projection in a similar manner, and without pretending to any thing like a universal agreement of the formula with the partial and dispersed observations of the kind which we possess, I will content myself with stating the result of a rigorous comparison with the observations of KAEMTZ and others already quoted, with reference to the diurnal curve of decrement of temperature in the atmosphere.

I have had computed from the empirical formulæ containing four terms (POGGENDORFF, xxvii. p. 349), by which Mr KAEMTZ represents the observations, the epochs of maximum and minimum temperature at the Rigi and at Zurich, and the epochs of quickest and slowest decrement of temperature in the course of the day.

The hours and fractions stated below are reckoned from noon.

	Range.	Time of Maximum.	Time of Minimum.
		h.	h.
Diurnal temperature curve at the Rigi, . .	3.10 = B′	0.901	15.721
Diurnal temperature curve at *Zurich*, . .	5.80 = B	2.467	16.237
Differential curve, ...		4.848	16.525

Now, by Eq. (7),

$$\frac{dx_{,}}{dC'} = \frac{3.1}{2.7} = 1.15.$$

The *anticipation* of afternoon epoch at the Rigi compared to Zurich is $1^h.57$, which, multiplied by 1.15, gives $1^h.80$ for the *retardation* of the epoch of *quickest decrement*, whilst it appears to have been $2^h.38$. The morning epoch at the Rigi anticipates by $0^h.52$, and the retardation of quickest decrement is $0^h.29$.

From observations by the same industrious meteorologist for twenty-five days on the Faulhorn (POGGENDORFF, xxvii. p. 354), compared with those at Geneva and Zurich, I have obtained graphically the following results:—

	Range.	Maximum.	Minimum.
		h.	h.
(1.) Diurnal curve at Faulhorn, ...	3.9 = B′	0.70	16.55
(2.) .... Geneva, ...	6.75 = B	2.93	16.78
(3.) .... Zurich, ...	8.3 = B	2.67	17.07
Differential curve for (1) and (2), . . .		4.70	17.00
.... (1) and (3), . . .		4.10	17.80?

For the Faulhorn and Geneva, we deduce from the ranges by Eq. (7),

$$\frac{dx_{\prime}}{dC'}=\frac{3.9}{2.85}=1.4.$$

By the preceding Table:

Maximum; $x_{\prime}=1.77$ $\quad C'=2.23$ $\quad \frac{x_{\prime}}{C'}=0.79$

Minimum; $x_{\prime}=0.22$ $\quad C'=0.23$ $\quad \frac{x_{\prime}}{C'}=0.95$

For the Faulhorn and Zurich, by Eq. (7),

$$\frac{dx_{\prime}}{dC'}=\frac{3.9}{4.4}$$

By the preceding Table:

Maximum; $x_{\prime}=1.43$ $\quad C'=1.97$ $\quad \frac{x_{\prime}}{C'}=0.72$

Minimum; $x_{\prime}=0.73$ $\quad C'=0.52$ $\quad \frac{x_{\prime}}{C'}=1.4$?

It is quite evident that these results exhibit only a very rough degree of conformity with the computed value, and that the maximum and minimum epochs are not symmetrically shifted. It would be easy to point out the causes of these discrepancies, amongst which must be reckoned the very short periods during which this highly interesting class of observations has been continued, and the want of symmetry in the forms of the diurnal curve where the stations are very widely separated. This investigation may, perhaps, induce a more careful experimental inquiry into the subject under favourable circumstances. The elegant relation of the velocities of displacement of the epoch at the superior station, and of the epochs of the differential curve, is at least worthy of notice; the ratio of velocities being so low as 1 to 19 when the ranges at the two stations are as 20 to 21; whilst it rises to 64 : 100, or 2 : 3 nearly, when the ranges are (as in SAUSSURE's observations at the Col du Géant and Geneva) as 2 to 5.

In considering subterranean temperature at small depths (which, as I have observed in the body of the paper, follows a law similar to that of the annual-temperature-curves in the atmosphere), the approximation obtained from the two first terms of Eq. (3) is quite sufficient, and indeed the variation may be confined to the second alone. In this case, C′ may have a value of 180°, or even more (the only limit being the sensible extinction of the annual variation, which will probably prevent it from ever attaining the magnitude of a whole circumference) the actual opposition of the epochs of maximum and minimum temperature being observed in most soils at a depth of about 30 feet.

XXVI.—*On the Theory of Waves. Part I.* *By* The Rev. P. Kelland, *M.A., F.R.SS. L. & E., F.C.P.S., late Fellow of Queens' College, Cambridge; Professor of Mathematics, &c. in the University of Edinburgh.*

(Read April 1. 1839.)

It is my intention to undertake a series of Memoirs on that branch of Hydrodynamics which treats of the transmission of reciprocating motion. It may appear superfluous at first, that any further investigation should be bestowed on the general problem, when it is remembered that two of the most able mathematicians of the present age, M. Poisson and M. Cauchy, have devoted so much of their talents and attention to the subject. Yet, when advances are rapidly making in our experimental knowledge, we generally find that the complex and apparently general results, deduced by those who have paid the greatest attention to the science as it presented itself to their view, are in reality only particular cases, when they have to be applied to experiments in their actual form. This circumstance detracts greatly from the value of almost all *general* investigations; and in this instance it does so in a remarkable degree. The prosecution of the research has been extended in one direction—the prosecution of experimental knowledge has taken another; and thus, in order to meet the challenge of experimenters to keep pace with them, it behoves the theorist frequently to return on his path, and cross over to the line of their march.

I doubt much, however, whether such men as Laplace and Lagrange would have been induced, with the expectation of joining experiment on her lower and more trodden fields, to reconsider and remodel their investigations; nor have I any reason to hope, that such men as Poisson and Cauchy will quit the delectable atmosphere in which they are involved, of abstruse analysis, for the more humble, but not less important task of endeavouring to treat the simpler problems in a manner not made general arbitrarily to lead to the most elegant formulæ, but general to that extent, and in that mode, in which the problem in nature is so. It may seem strange, but I confess it appears to me, that this problem has hitherto fallen into the hands of men too distinguished,—of men who had attained such an elevation, that the labours of ordinary men, the trivial difficulties which impede the progress of less gigantic minds, do not form part either of their actual research, or of that which the world expects at their hands. Hence their investigations have been such as apply only to the more abstruse and captivating branches of the science, whilst the simple extension of its fundamental conditions has been almost, if not altogether, overlooked. From these impressions I have been induced to apply myself to the question. My desire is to simplify as well as to extend; not that I purpose to remodel the works of those who have gone before me, but, in certain cases, to derive conclusions previously known in support and illustration of my methods of operation. As far, however, as concerns the present memoir, although the results obtained appear to differ but slightly from those of M. POISSON or M. CAUCHY, yet, on examination, they will be found very little alike. A most essential element enters into them, which could not, from the nature of the methods employed, appear in either of theirs, viz. a variation dependent on the sphere of excursion of the molecules. Nor is this all. Both these philosophers suppose the oscillations to be complete,—indeed, one of the very first elements on which M. POISSON'S theory is based, amounts only to the expression of this circumstance. Whatever, therefore, may be the value of investigations such as his (and I consider them most valuable), they do not solve the problem which is most wanted, the problem of oscillatory motion, of large or imperfect vibrations. They push to the extreme verge the problem which they actually do contain, but leave the other branches of the subject comparatively unimproved.

The theory of the tides can derive little assistance from such speculations in the present state of our knowledge, if indeed at all. This, and many other most important divisions of philosophy, if accurately subjected to analysis, require other restrictions than those commonly imposed, and at the same time demand the removal of these. With an ultimate view to problems of this nature, I have drawn up the present memoir, which embraces the motion of regular uniformly sized waves proceeding in one direction, and solves the problem in all its generality. It contains, likewise, an approximation to the motion of a solitary wave, such as the tide wave, or, more properly, that examined by Mr RUSSELL, and designated by him the Wave of *Translation*. How far I have succeeded I am not prepared to state. My solution can be regarded only as an approximation, nor does it very accurately agree with observation; yet the agreement is to my mind very satisfactory, as it shews, as far as I have tested the formula, that the error is due to one, and only one, point: what that point may be, I do not take it on me to conjecture.

This is the whole substance of the present memoir. Since it is so restricted, I shall not be expected, at the present time, to offer a sketch of the progress of the general problem. It will suffice, that a very brief statement be made of the labours of those who have added to our theoretical knowledge of this branch of Hydrodynamics.

NEWTON, in the Second Book of his *Principia*, applied his power of abstracting the point adapted to computation from the mass around it, to the solution of the problem of oscillation in a deep fluid, and his results are still valuable, as his process is ingenious.

LAPLACE afterwards applied a regular analysis to the problem,* but his solution is of a very limited nature. LAGRANGE, in his *Mécanique Analytique*,† gives equations which contain the complete statement of the question, and all that remains is the treatment of these equations. LAGRANGE himself ‡ correctly solves the problem for fluids of small depth, but his hypothesis is very far from the truth as applied to the general problem.

It was in this state of the question that the French Institute proposed the subject as its prize essay for 1816.

M. POISSON, who, as he himself states, had for a long time been engaged on this problem, sent first a memoir to the Institute in October 1815, and afterwards a second in December. These memoirs contain an approximate solution of the motion of waves in a canal. We commence first with M. POISSON'S.§

This most important memoir starts with an approximation which (M. CHALLIS thinks) all persons who have attempted the problem have used. The assumption which gives rise to his equation (4), p. 81, also appears to me most essentially to destroy the generality of the results. As to the body of the memoir itself, it contains very little which belongs to our present matter. It is principally occupied in discussing the question of the effects which *presently* follow a disturbance in the fluid. The theoretical results of M. POISSON have been analyzed and tested by M. WEBER, with whose work I am altogether unacquainted.

The prize mentioned above was adjudged to M. CAUCHY, who was thought to have solved the question more generally. As to what the problem was I am not altogether certain. In M. CAUCHY'S memoir, printed in 1827,‖ fifteen years after it had obtained the prize, I find it stated thus:

"Une masse fluide pesante, primitivement en répos, et d'un profondeur indefinie, a été mise en mouvement par l'effet d'une cause donnée. On demande, au bout d'un temps déterminé, la forme de la surface exterieure du fluide, et la vitesse des molécules situées à cette même surface."

I have little doubt this is the form in which the question was actually proposed, and we find a sufficient cause for the detention of the two philosophers on this part of the problem, to the exclusion, or nearly so, of others. M. CAUCHY has added notes to his memoir, partly to explain and partly to enlarge on the

* Mémoires de l'Académie des Sciences, 1776.

† Mécanique Analytique, 2d Partie, Sect. xi.

‡ Ibid., Arts. 35–39.

§ Mémoires de l'Académie des Sciences, 1816.

‖ Mémoires des Savans Etrangers, tome i.

processes employed in the body of the work. In two of these, the 16th and 20th, we find solutions of the question now before us, but in both of them the approximations are used which I have before alluded to. The results will be found in equation (141) of note 16, and equation (66) of note 20. They agree closely with those obtained by M. Poisson. This memoir of M. Cauchy is of itself a large and abstruse work, and consequently, the brief notice taken of it must be understood to have reference only to those portions which belong to our present portion of the problem. To bring the history of the subject down to the present time, I have only to mention the names of Mr Challis,* Mr Earnshaw,† and Mr Green; ‡ the two former of whom solve problems nearly connected with our own, and the latter takes a very limited case of the actual problem, viz. that solved approximately by M. Lagrange.

The present must be regarded rather in the light of an introduction to a series of Memoirs, than as a complete work in itself. We treat only of motion in a canal of uniform breadth,—nor shall we take even a large portion of that problem. The mode of generating motion,—its effect on the final waves, and on the primary ones,—the variable state of the surface, owing to reflexion from the bottom and sides of the canal,—these, and the like questions, will occupy us hereafter. We have, however, a sufficiently wide field, without having recourse to these comparatively abstract points. Not to mention the application of the results to the theory of the Tides, an application becoming daily more and more tangible by the labours of Mr Lubbock and Mr Whewell, we have a vast variety of questions to resolve, more obviously belonging to the very threshold of our inquiry. What is the correct velocity of a wave in a canal of variable depth? or in one which is not shallow nor very deep, as compared with the length of a wave? Will the effect be modified if the canal be a closed one at the commencement of motion? Will the length of the wave depend on the depth, on the quantity of fluid first put in motion, or on the space over which that motion takes place, or on all these causes? How will friction modify the form of the wave and the velocity of motion? And, lastly, What will be the motion in a channel which shallows away at its sides, when the channel is broad, as is the case when reference is had to the motion of the tidal wave?

These, and like questions, are of the utmost importance, and demand a careful investigation. In my next memoir, I hope to broach at least one, in addition to that at present before us. In the mean time, we proceed to the

* Transactions of the Cambridge Philosophical Society, vols. iii. and v.

† Transactions of the Cambridge Philosophical Society.

‡ Transactions of the Cambridge Philosophical Society, vol. vi.

ANALYTICAL INVESTIGATION.

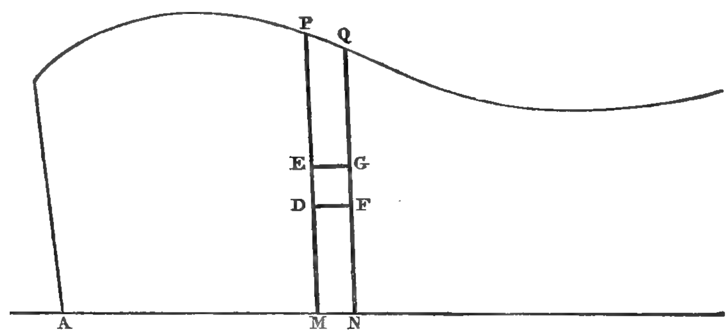
SECTION I.—UNIFORM WAVE-MOTION.

1. We commence with the determination of Wave-motion in a fluid of finite depth, on the hypothesis of *parallel sections.*

Let PQ be a portion of the surface of the fluid, PM, QN vertical planes at right angles to the direction of transmission: and let AM $= x$, MP $= z$, MD $= y$, MN $= \delta x$. We shall also retain the notation in common use according to which u or $\frac{d\phi}{dx}$ represents the velocity parallel to x, v or $\frac{d\phi}{dy}$ that parallel to y. In order to avoid unnecessary length, we must adopt without demonstration the results which have been arrived at for fluid motion in general. The demonstrations may be found in POISSON's *Traité de Mecanique,* 2d edition, tome ii. liv. 6, chap. 1; in MOSELEY's *Hydrodynamics,* chap. vii.; in PRATT's *Mechanical Philosophy,* Hydrodynamics, chap. i.; or in WEBSTER's *Theory of Fluids,* chap. x.; to all of which we shall give references, for the sake of saving trouble to the reader.

To find the motion of the portion PN.

Let p be the pressure on an unit in PM; p' that on an unit in QN; then the pressure on $\quad \text{PM} = \int_0^z dy\, p$;



the pressure on $\quad \text{QN} = \int_0^{z+\frac{dz}{dx}\delta x} dy \left(p + \frac{dp}{dx}\delta x\right)$;

$$\therefore \text{ the moving force } = \int_0^z dy p - \int_0^{z+\frac{dz}{dx}\delta x} dy \left(p + \frac{dp}{dx}\delta x\right)$$

$$= \int_0^z dy p - \int_0^z dy \left(p + \frac{dp}{dx}\delta x\right) - \int_z^{z+\frac{dz}{dx}\delta x} dy \left(p + \frac{dp}{dx}\delta x\right)$$

$$= -\delta x \int_0^z \frac{dp}{dx} dy - \int_z^{z+\frac{dz}{dx}\delta x} . p\, dy.$$

Let us suppose that all the parts in a given vertical move forwards equally at a given time; then $u = \frac{d\phi}{dx}$ is independent of y.

We have now to solve the problem of finding the motion of fluid in a tube which is continually expanding. The small difference between MP and NQ may, in this investigation, be neglected.

2. To find the descent of fluid in a tube MQ, where MP is a fixed side, but NQ is moveable by means of the pressure.

Let D be any point in the fluid; $v =$ the velocity at D in the direction of y, $u =$ that in the direction of x; then, if α be the thickness MN at the time t, $\alpha + \delta\alpha$ at the time $t+\delta t$, the quantity DG will have been pressed from δy to $\delta y \cdot \frac{\alpha}{\alpha+\delta\alpha}$.

Therefore the whole DN will be exhibited in $y \cdot \frac{\alpha}{\alpha+\delta\alpha}$;

and D will descend by a space

$$-y \cdot \frac{\alpha}{\alpha+\delta\alpha} + y$$

$$= y \frac{\delta\alpha}{\alpha+\delta\alpha}$$

$$= y \frac{\delta\alpha}{\alpha} \text{ nearly:}$$

hence the velocity of D is $\frac{y}{\alpha} \cdot \frac{d\alpha}{dt}$ (downwards)

$$\therefore \quad v = -\frac{y}{\alpha} \cdot \frac{d\alpha}{dt}.$$

3. This will, however, lead us to no result, except we assume the nature of the motion to be defined. Let us, then, make the hypothesis that the motion is a *wave*-motion. This amounts to the substitution of

$$h + a \sin\frac{2\pi}{\lambda}(ct-x) + a' \sin\frac{4\pi}{\lambda}(ct-x) + \text{ \&c.}$$

for z.

In this formula, h is the original depth;
λ the length of a wave;
c the velocity of transmission.

Now, if we retain only the first term in the variable part of this expression, we obtain

$$z = h + a \sin\theta,$$

θ being equal to $\frac{2\pi}{\lambda}(ct-x)$.

$$\therefore \quad \frac{dz}{dx} = -\frac{2\pi a}{\lambda} \cos\theta,$$

from the supposition that the velocity in the direction parallel to x is uniform through any vertical section, and that consequently u is a function of x and t only.

From this consideration, it follows that $\frac{du}{dx}$ is independent of y: and consequently $\frac{dv}{dy}$, which is equal to $-\frac{du}{dx}$,* is also independent of y.

v will in consequence assume the form $v=yf(x,t)+\phi(x,t)$; but when $y=0$, v is always $=0$, hence $v=yf(x,t)$; and, by reference to Art. 2, it appears that $v=-\frac{y}{\alpha}\cdot\frac{d\alpha}{dt}$.

But $$v=\frac{dy}{dt},$$

$$\therefore \quad \frac{dy}{dt}=-\frac{y}{\alpha}\cdot\frac{d\alpha}{dt}:$$

hence $$\frac{dz}{dt}=-\frac{z}{\alpha}\cdot\frac{d\alpha}{dt};$$

$$\therefore \quad v=\frac{y}{z}\cdot\frac{dz}{dt}.$$

Cor. $$\frac{1}{z}\frac{dz}{dt}=-\frac{1}{\alpha}\frac{d\alpha}{dt}.$$

4. Now z has been assumed to be equal to $h+a\sin\theta$.

$$\therefore \quad \frac{dz}{dt}=\frac{2\pi ca}{\lambda}\cdot\cos\theta-\frac{2\pi a}{\lambda}\cdot\cos\theta\cdot\frac{dx}{dt}$$

$$\therefore \quad v=\frac{y}{z}\frac{2\pi ca}{\lambda}\cos\theta-\frac{2\pi a}{\lambda}\cdot\frac{y}{z}\cdot\cos\theta\cdot\frac{dx}{dt}$$

$$\frac{dv}{dy}=\frac{2\pi ca}{\lambda z}\cdot\cos\theta-\frac{2\pi a}{\lambda}\cdot\frac{\cos\theta}{z}\cdot\frac{dx}{dt}$$

$$\frac{du}{dx}=-\frac{2\pi ca}{\lambda z}\cdot\cos\theta+\frac{2\pi a}{\lambda z}\cdot\cos\theta\cdot\frac{dx}{dt}$$

$$=-\frac{2\pi ca}{\lambda z}\cdot\cos\theta+\frac{2\pi a}{\lambda z}\cdot\cos\theta\cdot u.$$

Now let $$u=b\sin\theta;$$

then $$-b\frac{2\pi}{\lambda}\cdot\cos\theta=-\frac{2\pi ca}{\lambda h}\cdot\cos\theta+\frac{2\pi ab}{\lambda h}\cdot\sin\theta\cos\theta+\frac{2\pi ca^2}{\lambda h^2}\cdot\sin\theta\cos\theta;$$

but, from the hypothesis already made, the last two terms must be omitted; hence $$bh=ca;$$

$$\therefore \quad u=\frac{ca}{h}\cdot\sin\theta$$

$$v=\frac{2\pi ca}{\lambda h}\cdot y\cdot\cos.\theta.$$

5. The hypothesis relative to the value of z, by means of which the preceding results have been obtained, is that which belongs to the most simple case of wave-

* POISSON, art. 649; MOSELEY, art. 205; PRATT, art. 564; WEBSTER, art. 108. The equation is

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0.$$

motion. To solve the problem more generally, we should assume for z a series of sines of multiples of θ. If we restrict ourselves to two sines, we shall have the following equation:

$$z=h+a\sin\frac{2\pi}{\lambda}(ct-x)+e\sin 2\theta\,;$$

which gives

$$v=\frac{y}{z}\cdot\frac{dz}{dt}=\frac{y}{z}\cdot\frac{2\pi}{\lambda}\cdot\left(a\cos\theta\,.\,\overline{c-\frac{dx}{dt}}+2e\cos 2\theta\,.\,\overline{c-\frac{dx}{dt}}\right)$$

$$\frac{du}{dx}=-\frac{2\pi}{\lambda z}\left(a\cos\theta\,.\,\overline{c-\frac{dx}{dt}}+2e\cos 2\theta\,.\,\overline{c-\frac{dx}{dt}}\right).$$

The same limitation as to the extent of the series gives

$$u=b\sin\theta+f\sin 2\theta.$$

Differentiating this, and equating it to the former,

$$b\cos\theta+2f\cos 2\theta=\frac{1}{z}(a\cos\theta+2e\cos 2\theta)\left(c-\frac{dx}{dt}\right)\,;$$

$$\therefore\quad (b\cos\theta+2f\cos 2\theta)\,(h+a\sin\theta+e\sin 2\theta)$$
$$=(a\cos\theta+2e\cos 2\theta)\,(c-b\sin\theta-f\sin 2\theta),$$

or $$bh\cos\theta+2fh\cos 2\theta+\frac{ab}{2}\sin 2\theta+af(\sin 3\theta-\sin\theta)+\frac{eb}{2}(\sin 3\theta+\sin\theta)+ef\sin 4\theta$$

$$=ac\cos\theta+2ec\cos 2\theta-\frac{ba}{2}\sin 2\theta-eb(\sin 3\theta-\sin\theta)-\frac{af}{2}(\sin 3\theta+\sin\theta)-ef\sin 4\theta.$$

By equating the coefficients of $\cos\theta$ and $\cos 2\theta$, we get

$$bh=ac\,,\ 2fh=2ec\,;$$

hence we learn that, if z be known, u is known.

With respect to the terms involving sines, it is clear that no equations can be made, since the same quantities will occur again.

6. We now proceed to determine the motion parallel to the axis of x by an independent method. We will conceive the portion PQ to become solid for an instant, and calculate the force by which it is urged in the direction of the axis of x. That force will consist of two parts, totally independent of each other; the one the difference of statical pressure on the two planes PM, QN; the other the difference of the dynamical impulses on the same planes.

With respect to the latter, it may be remarked, that it must be absolutely independent of v. The real effective part of this pressure is, in fact, nothing more than the *resistance* on either of the planes due to the velocity of the fluid in a direction at right angles to it.

In order to obtain the value of this force, we shall not attempt to treat of it separately as a problem of resistances, but apply directly to it the same argument as is commonly applied to establish the theory of resistances. We have already stated that it must be independent of v: this can only be on the supposition that the force results from the impulse of the fluid behind that under consideration. We suppose, in fact, that it is the difference of the impulses on the two sides of the solid PN.

Now the expression for the pressure at any point in the mass is this,*

$$p = g\varrho(z-y) - \varrho\int dx\frac{d^2x}{dt^2} - \&c.$$

Of this expression the first term is $g\varrho(z-y)$, which is the pressure due to the action of gravity.

The second term is $-\varrho\int dx\frac{d^2x}{dt^2}$, a quantity which depends on the motion of the particles in the neighbourhood of the point.

In the theory of resistances, it is assumed that the motion is such as to admit of our writing $\frac{dx}{dt}dt$ for dx; that is, it is supposed that the motion of a particle is such as that, when it comes to another point in space after any interval, it shall move just in the same manner as those particles do which are at the present moment at that point. In other words, the motion is conceived to be *steady*, that it is always the same at any particular point of space; so that the integration for one particle during its successive stages shall be identical with the integration for different particles at the same instant.

Such an hypothesis as this is not only convenient, but appears to be absolutely correct in the case before us, for this pressure depends only on the variation of velocity of the particles *immediately* about PMN at the instant under consideration, and if one hypothesis as to the future movements of the particles gives the difference of the pressures,

$$= -g\varrho\,dy - \frac{1}{2}\varrho\,d.\left(\frac{dx}{dt}\right)^2 - \&c.$$

in going from point to point, any other hypothesis ought to give the same, provided the variation of velocity be the only thing which affects the pressure.

We may then assume as the value of p;

$$p = g\varrho(z-y) - \frac{1}{2}\varrho u^2 + \mathrm{P},$$

P being some function depending on v, but not requisite to our calculation.

Similarly, $$p' = g\varrho(z'-y) - \frac{1}{2}\varrho u'^2 + \mathrm{P}'$$

$$= g\varrho\left(z + \frac{dz}{dx}\alpha - y\right) - \frac{1}{2}\varrho\left(u^2 + 2u\frac{du}{dx}\alpha\right) + \mathrm{P}'$$

* Poisson, art. 647; Moseley, art. 203; Pratt, art. 562; Webster, art. 117. The general equation is,

$$\frac{dp}{\varrho} = \left(x - \frac{d^2x}{dt^2}\right)dx + \left(y - \frac{d^2y}{dt^2}\right)dy + \left(z - \frac{d^2z}{dt^2}\right)dz.$$

Now, the moving force on the solid PN

$$=\int_0^z dy p-\int_0^{z+\frac{dz}{dx}\alpha} dy p'$$

$$=\int_0^z dy\, g\varrho\,(z-y)-\frac{1}{2}\int_0^z dy\,\varrho\,u^2+\mathrm{Q}$$

$$-\int_0^{z+\frac{dz}{dx}\alpha} dy\, g\varrho\left(z+\frac{dz}{dx}\alpha-y\right)+\frac{1}{2}\varrho\int_0^{z+\frac{dz}{dx}\alpha} dy\left(u^2+2u\frac{du}{dx}\alpha\right)+\mathrm{Q}'$$

$$=\int_0^z dy\, g\varrho\,(z-y)$$

$$-\int_z^{z+\frac{dz}{dx}\alpha} dy\, g\varrho\left(z+\frac{dz}{dx}\alpha-y\right)+\varrho\int_0^z dy\, u\frac{du}{dx}\alpha.$$

We have omitted the part $\frac{1}{2}\varrho\int_z^{z+\frac{dz}{dx}\alpha} dy\, u^2+\mathrm{Q}+\mathrm{Q}'$ from the circumstance that it does not depend on the difference of velocity or of resistance, and, therefore, is no part of our force.

Perhaps it would be well to call p the horizontal pressure, instead of the whole pressure, as by this means we should have been spared the apparent incorrectness of omitting a part of the results; but I have preferred retaining the above, as the more usual mode of proceeding.

By integrating the above expression between limits, we obtain for the moving force

$$\frac{1}{2}g\varrho z^2-\frac{1}{2}g\varrho\left(z+\frac{dz}{dx}\alpha\right)^2+\varrho\,\alpha z u\frac{du}{dx};$$

and the mass is $\varrho\,\alpha z$; hence the accelerating force is

$$\left(\frac{du}{dt}\right)=-g\frac{dz}{dx}+u\frac{du}{dx};$$

where $\frac{du}{dt}$ is the total differential coefficient of u with respect to t.

By substituting for u its value

$$\frac{ca}{h}\cdot\sin\frac{2\pi}{\lambda}\overline{ct-x}\ \text{ or }\ \frac{ca}{h}\cdot\sin\theta$$

we obtain

$$\frac{2\pi ca}{h\lambda}\cdot\cos\theta\left(c-\frac{dx}{dt}\right)=g\cdot\frac{2\pi a}{\lambda}\cdot\cos\theta-u\frac{2\pi ac}{\lambda h}\cos\theta;$$

by equating coefficients, we obtain $\frac{c^2}{h}=g$ as the first part, and the other part is an identity.

7. We may vary the last part of the process in the following manner. Since the moving force is

$$\int_0^z p\,dy-\int_0^{z+\frac{dz}{dx}\alpha} p'\,dy,$$

it is
$$=\int_0^z (p-p')\,dy-\int_z^{z+\frac{dz}{dx}\alpha} p\,dy$$
$$=-g\varrho\int_0^z \frac{dz}{dx}\alpha\,dy-\int_z^{z+\frac{dz}{dx}\alpha} p\,dy$$
$$+\frac{1}{2}\varrho\int_0^z\left\{\left|\frac{d(x+\alpha)}{dt}\right|^2-\left(\frac{dx}{dt}\right)^2\right\}\alpha,$$
$$=-g\varrho\alpha z\frac{dz}{dx}+\frac{1}{2}\varrho\frac{2dx}{dt}\cdot\frac{d\alpha}{dt}\cdot z\,;$$

since the term $\int_z^{z+\frac{dz}{dx}\alpha} g\varrho(z-y)\,dy$ involves α^2 as a factor.

Hence the accelerating force is
$$\frac{d^2x}{dt^2}=-g\frac{dz}{dx}+\frac{dx}{dt}\cdot\frac{\frac{d\alpha}{dt}}{\alpha}.$$

But it has been shewn (Art. 2), that
$$\frac{1}{\alpha}\cdot\frac{d\alpha}{dt}=-\frac{1}{z}\cdot\frac{dz}{dt}$$
$$\therefore\quad \frac{d^2x}{dt^2}=-g\frac{dz}{dx}-\frac{dx}{dt}\cdot\frac{dz}{dt}\cdot\frac{1}{z}.$$

But
$$z=h+a\sin\frac{2\pi}{\lambda}(ct-x)$$

and
$$\frac{dx}{dt}=b\sin\frac{2\pi}{\lambda}(ct-x)$$

where b is supposed undetermined. By substituting these values, we get
$$\frac{2\pi b}{\lambda}\cdot\cos\theta\left(c-\frac{dx}{dt}\right)=ga\cdot\frac{2\pi}{\lambda}\cos\theta-\frac{\frac{dx}{dt}}{h}\cdot\frac{2\pi ac}{\lambda}\cdot\cos\theta$$

or
$$\frac{2\pi b}{\lambda}\cdot\cos\theta\,(c-b\sin\theta)=\frac{2\pi}{\lambda}ga\cos\theta-\frac{2\pi acb}{\lambda h}\cdot\sin\theta\cos\theta\,;$$

and, equating coefficients, we get
$$bc=ga\,;\ b^2=\frac{abc}{h}\,;$$

whence
$$\left.\begin{aligned}b&=\frac{ac}{h}\\ c^2&=gh\end{aligned}\right\}$$

the values which we obtained before.

8. Next, having these approximate formulæ as our guide, let us proceed to the general solution of the problem.

Retaining the same notation,

$$\frac{dp}{dx}=\varrho\left(-\left(\frac{du}{dt}\right)\right)^{*}$$

$$\frac{dp}{dy}=\varrho\left\{-g-\left(\frac{dv}{dt}\right)\right\}$$

$\left(\frac{du}{dt}\right)$, $\left(\frac{dv}{dt}\right)$ being the complete differentials of u and v divided by dt.

These equations, again, give as their result,

$$\frac{d}{dy}\left(\frac{du}{dt}\right)=\frac{d}{dx}\left(\frac{dv}{dt}\right)$$

which being combined with the equation $\frac{du}{dx}+\frac{dv}{dy}=0$, the motion will be obtained.

Now,
$$\left(\frac{du}{dt}\right)=\frac{du}{dt}+u\frac{du}{dx}+v\frac{du}{dy}$$

$$\left(\frac{dv}{dt}\right)=\frac{dv}{dt}+u\frac{dv}{dx}+v\frac{dv}{dy}.$$

But if the wave be oscillatory, we may assume for u and v a series of terms of the following form:

$$u=f(y)\,.\,\sin\frac{2\pi}{\lambda}(ct-x)$$

$$v=\mathrm{F}y\,.\cos\frac{2\pi}{\lambda}(ct-x).$$

For it is obvious, without any calculation, that, since $\frac{du}{dx}+\frac{dv}{dy}=0$, if $\frac{du}{dx}$ involve $\cos\frac{2\pi}{\lambda}(ct-x)$, v must do so too, and consequently v will contain only cosines of quantities, of which u contains sines. By substitution in the equation $\frac{du}{dx}+\frac{dv}{dy}=0$, we get the following result:

$$-\frac{2\pi}{\lambda}\cos\frac{2\pi}{\lambda}(ct-x)fy+\cos\frac{2\pi}{\lambda}(ct-x)\,\mathrm{F}'y=0$$

or
$$\mathrm{F}'y=\frac{2\pi}{\lambda}fy,\quad (1).$$

Let
$$\left.\begin{array}{l}\frac{2\pi}{\lambda}\text{ be denoted by }\alpha\\ \frac{2\pi}{\lambda}(ct-x)\text{ by }\theta\end{array}\right\}$$

then the values of $\left(\frac{du}{dt}\right)$ and $\left(\frac{dv}{dt}\right)$ become (retaining only this term of the value of u),

$$\left(\frac{du}{dt}\right)=\alpha fy\cos\theta\,(c-fy\sin\theta)+f'y\,\mathrm{F}y\sin\theta\cos\theta,$$

$$\left(\frac{dv}{dt}\right)=-\alpha\mathrm{F}y\sin\theta\,(c-fy\sin\theta)+\mathrm{F}'y\,\mathrm{F}y\cos^{2}\theta.$$

* See the references in Art. 6.

Hence the equation $\frac{d}{dy}\cdot\left(\frac{du}{dt}\right)=\frac{d}{dx}\cdot\left(\frac{dv}{dt}\right)$ gives

$$\begin{aligned}&\alpha c\,.\cos\theta f'y-2\alpha\sin\theta\cos\theta fy\, f'y+\sin\theta\cos\theta\,(\mathrm{F}y\, f''y+\mathrm{F}'y\, f'y)\\&=\alpha^2 c\,\mathrm{F}y\cos\theta-2\alpha^2\,\mathrm{F}y\, fy\sin\theta\cos\theta+\alpha\,\mathrm{F}'y\,\mathrm{F}y\sin 2\theta.\end{aligned}$$

Equating coefficients, we obtain

$$f'y=\alpha\,\mathrm{F}y \quad (2)$$

$$-\alpha fy\, f'y+\frac{\mathrm{F}y\, f''y+\mathrm{F}'y\, f'y}{2}+\alpha^2\,\mathrm{F}y\, fy-\mathrm{F}'y\,\mathrm{F}y\,\alpha=0$$

or $$-\frac{1}{\alpha}\mathrm{F}'y\,\mathrm{F}''y+\frac{1}{2}\left(\frac{1}{\alpha}\mathrm{F}y\,\mathrm{F}'''y+\frac{1}{\alpha}\mathrm{F}'y\,\mathrm{F}''y\right)=0$$

or $$\frac{1}{\alpha}\mathrm{F}y\,\mathrm{F}'''y-\frac{1}{\alpha}\mathrm{F}'y\,\mathrm{F}''y=0$$

or $$\mathrm{F}y\,\mathrm{F}'''y=\mathrm{F}'y\,\mathrm{F}''y \quad (3)$$

but $$\because\quad f'y=\alpha\,\mathrm{F}y$$

$$\mathrm{F}'y=\frac{1}{\alpha}f''y$$

and $$\alpha\,\mathrm{F}'y=\alpha^2 fy$$

$$\therefore\quad f''y=\alpha^2 fy$$

or $$\frac{d^2 fy}{dy^2}=\alpha^2.fy$$

The complete solution of this equation is

$$fy=be^{\alpha y}+b'e^{-\alpha y}$$

$$\therefore\quad \mathrm{F}'y=\alpha\left(be^{\alpha y}+b'e^{-\alpha y}\right);\ \mathrm{F}y=be^{\alpha y}-b'e^{-\alpha y} \quad (2)$$

If we substitute this value of fy in equation (3), we obtain

$$(b\,e^{\alpha y}-b'e^{-\alpha y})\,\alpha^3\,(b\,e^{\alpha y}+b'\,e^{-\alpha y})=\alpha\,(b\,e^{\alpha y}+b'e^{-\alpha y})\,\alpha^2\,(b\,e^{\alpha y}-b'e^{-\alpha y})$$

an identity.

Thus all the conditions are satisfied. Our solution, then, of the equations is

$$u=\left(be^{\frac{2\pi}{\lambda}y}+b'e^{-\frac{2\pi}{\lambda}y}\right).\sin\frac{2\pi}{\lambda}(ct-x),$$

$$v=\left(be^{\frac{2\pi}{\lambda}y}-b'e^{-\frac{2\pi}{\lambda}y}\right).\cos\frac{2\pi}{\lambda}(ct-x).$$

If it should appear more general to affix a constant to u, it will add no difficulty to the investigation.

The value of b' may be determined by supposing the origin of co-ordinates to be placed at the bottom of the fluid, so that $v=0$ when $y=0$: this process gives $b'=b$.

9. Our next step is to find p.

By the equations

$$\frac{dp}{dx} = \varrho\left(-\left(\frac{du}{dt}\right)\right)$$

$$\frac{dp}{dy} = \varrho\left(-g-\left(\frac{dv}{dt}\right)\right),$$

we get

$$dp = \frac{dp}{dx}dx + \frac{dp}{dy}dy$$

$$= \varrho\left\{-\left(\frac{du}{dt}\right).dx - \left(\frac{dv}{dt}\right)dy - g\,dy\right\}$$

$$= \varrho\left\{-b\,\alpha\cos\theta\,(e^{\alpha y}+e^{-\alpha y})\,(c-b\,.\,\overline{e^{\alpha y}+e^{-\alpha y}}\sin\theta)\right.$$
$$\left.-b^2\alpha\,(e^{\alpha y}-e^{-\alpha y})^2\sin\theta\cos\theta\right\}dx$$

$$+\varrho\left\{b\,(e^{\alpha y}-e^{-\alpha y})\,\alpha\sin\theta\,(c-b\,.\,\overline{e^{\alpha y}+e^{-\alpha y}}\sin\theta)\right.$$
$$\left.-b^2\alpha\,(e^{2\alpha y}-e^{-2\alpha y})\cos^2\theta\right\}dy$$
$$-\varrho g\,dy$$

$$\therefore\ p = -\varrho g y + \varrho\int b\,\alpha\,c\,(-\overline{e^{\alpha y}+e^{-\alpha y}}\cos\theta\,dx + \overline{e^{\alpha y}-e^{-\alpha y}}\sin\theta\,dy)$$
$$+\delta b^2\alpha\int\left\{2\sin 2\theta\,dx - (e^{2\alpha y}-e^{-2\alpha y})\,dy\right\}$$
$$= -g\varrho y + \varrho\,b\,c\,(e^{\alpha y}+e^{-\alpha y})\sin\theta$$
$$+\varrho b^2\left(\cos 2\theta - \frac{1}{2}(e^{2\alpha y}+e^{-2\alpha y})\right) + \mathrm{P}\ (4)$$

This equation contains the value of p, and the use we purpose to make of it is this. The quantity P is a function of t, and the depth of the fluid for the value of x fixed on. If this depth be called z, we have

$$p = \varrho g\,(z-y) + \varrho\,b\,c\sin\theta\,(e^{\alpha y}+e^{-\alpha y}) - \varrho\,b\,c\sin\theta\,(e^{\alpha z}+e^{-\alpha z})$$
$$+\varrho b^2\left\{-\frac{1}{2}(e^{2\alpha y}+e^{-2\alpha y}) + \frac{1}{2}(e^{2\alpha z}+e^{-2\alpha z})\right\}\ (5)$$

Now the value of $\frac{dp}{dx}$, or the expression for the force parallel to the axis of x, has been already formed, but another value of it may be obtained from this final equation. If we equate the two, we get

$$\left\{\frac{dp}{dx}\right\} = \frac{dp}{dx} + \frac{dp}{dz}\cdot\frac{dz}{dx}$$

where the quantity within brackets is the value of $\frac{dp}{dx}$ from equation (4).

But if equation (4) be written $p=\phi\,(xy)+\mathrm{P}$, and (5) $p=\phi\,(xy)-\phi\,(xz)$, we get

$$\frac{d\phi\,(xy)}{dx} = \frac{d\phi\,(xy)}{dx} - \frac{d\phi\,(xz)}{dz}\cdot\frac{dz}{dx} - \frac{d\phi\,(xz)}{dx}\quad (6)$$

hence $$\frac{d\phi(x z)}{d z}\cdot\frac{d z}{d x}+\frac{d\phi(x z)}{d x}=0 \quad (7).$$

10. To prevent confusion, however, we will at first write down all the terms of equation (6), and afterwards strike out those which occur on both sides, so as to obtain the form of equation (7).

By (4) $$\frac{1}{\varrho}\frac{dp}{dx}=-\alpha b c\cos\theta\,(e^{\alpha y}+e^{-\alpha y})+2\,.\,b^2\alpha\sin 2\theta:$$

and by (5) $$\frac{1}{\varrho}\frac{dp}{dx}=g\frac{dz}{dx}-\alpha b c\cos\theta\,(e^{\alpha y}+e^{-\alpha y}-e^{\alpha z}-e^{-\alpha z})$$

$$-\alpha b c\sin\theta\,(e^{\alpha z}-e^{-\alpha z})\frac{dz}{dx}+\alpha b^2(e^{2\alpha z}-e^{-2\alpha z})\frac{dz}{dx}:$$

hence $$-\alpha b c\cos\theta\,(e^{\alpha y}+e^{-\alpha y})+2b^2\alpha\sin 2\theta$$

$$=g\frac{dz}{dx}-\alpha . b c\cos\theta\,(e^{\alpha y}+e^{-\alpha y}-e^{\alpha z}-e^{-\alpha z})$$

$$-\alpha b c\sin\theta\,(e^{\alpha z}-e^{-\alpha z})\frac{dz}{dx}+b^2\alpha\,(e^{2\alpha z}-e^{-2\alpha z})\frac{dz}{dx}$$

that is, as we should have deduced at once from equation (7),

$$+\frac{4\pi b^2}{\lambda}\sin 2\theta=$$

$$\frac{dz}{dx}\cdot\left\{g-\frac{2\pi}{\lambda}b c\sin\theta\left(e^{\frac{2\pi}{\lambda}z}-e^{-\frac{2\pi}{\lambda}z}\right)+\frac{2\pi}{\lambda}b^2\left(e^{\frac{4\pi}{\lambda}z}-e^{-\frac{4\pi}{\lambda}z}\right)\right\}$$

$$+\frac{2\pi}{\lambda}b c\cos\theta\left(e^{\frac{2\pi}{\lambda}z}+e^{-\frac{2\pi}{\lambda}z}\right);$$

or $$\frac{4\pi}{\lambda}b^2\sin 2\theta=\frac{2\pi}{\lambda}b c\cos\theta\left(e^{\frac{2\pi}{\lambda}z}+e^{-\frac{2\pi}{\lambda}z}\right)$$

$$+\frac{dz}{dx}\left\{g-\frac{2\pi}{\lambda}b c\sin\theta\left(e^{\frac{2\pi}{\lambda}z}-e^{-\frac{2\pi}{\lambda}z}\right)+\frac{2\pi}{\lambda}b^2\left(e^{\frac{4\pi}{\lambda}z}-e^{-\frac{4\pi}{\lambda}z}\right)\right\} \quad (8).$$

Let $$z=h+\left(a\,.\,e^{\frac{2\pi}{\lambda}z}+fe^{-\frac{2\pi}{\lambda}z}\right)\sin\theta$$

$$\therefore\quad \frac{dz}{dt}=\sin\theta\frac{2\pi}{\lambda}\left(a e^{\frac{2\pi}{\lambda}z}-f.e^{-\frac{2\pi}{\lambda}z}\right)\frac{dz}{dt}$$

$$+\left(a e^{\frac{2\pi}{\lambda}z}+fe^{-\frac{2\pi}{\lambda}z}\right)\frac{2\pi}{\lambda}\cos\theta\left(c-b\left(e^{\frac{2\pi}{\lambda}z}+e^{-\frac{2\pi}{\lambda}z}\right)\sin\theta\right):$$

But from the circumstance that $v=\frac{dz}{dt}$ when $y=z$, this gives,

$$\left(1-\frac{2\pi}{\lambda}\cdot\overline{a\,e^{\alpha z}-f\,e^{-\alpha z}}\sin\theta\right)b\cos\theta\,(e^{\alpha z}-e^{-\alpha z})=$$

$$\frac{2\pi}{\lambda}\cos\theta\,(a\,e^{\alpha z}+fe^{-\alpha z})\,\{c-b\sin\theta\,(e^{\alpha z}+e^{-\alpha z})\}:$$

or

$$b\,(e^{\alpha z}-e^{-\alpha z})-\frac{2\pi}{\lambda}b\,(e^{\alpha z}-e^{-\alpha z})\,(ae^{\alpha z}-f\,e^{-\alpha z})\sin\theta$$

$$=\frac{2\pi}{\lambda}c\,(ae^{\alpha z}+fe^{-\alpha z})-\frac{2\pi}{\lambda}b\sin\theta\,(e^{\alpha z}+e^{-\alpha z})\,(a\,e^{\alpha z}+fe^{-\alpha z})$$

Hence $$f=-a \text{ and } b=\frac{2\pi}{\lambda}ac:$$

$$\therefore\quad z=h+a\,(e^{\alpha z}-e^{-\alpha z})\sin\theta$$

$$\frac{dz}{dx}=-\frac{2\pi}{\lambda}a\cos\theta\,(e^{\alpha z}-e^{-\alpha z})$$

$$+\frac{2\pi}{\lambda}a\sin\theta\frac{dz}{dx}\,(e^{\alpha z}+e^{-\alpha z})$$

$$=-\frac{\frac{2\pi}{\lambda}a\cos\theta\,(e^{\alpha z}-e^{-\alpha z})}{1-\frac{2\pi}{\lambda}a\sin\theta\,(e^{\alpha z}+e^{-\alpha z})}$$

which being substituted in equation (8), reduces it to

$$-\frac{2\pi}{\lambda}b\left(4b\sin\theta-(e^{\alpha z}+e^{-\alpha z})\,c\right)\cos\theta\left(1-\frac{b}{c}\sin\theta\,\overline{(e^{\alpha z}+e^{-\alpha z})}\right)$$

$$=\frac{b}{c}\cos\theta\,(e^{\alpha z}-e^{-\alpha z})\left\{g-\frac{2\pi}{\lambda}bc\sin\theta\,(e^{\alpha z}-e^{-\alpha z})+\frac{2\pi}{\lambda}b^2\,(e^{2\alpha z}-e^{-2\alpha z})\right\}$$

that is

$$-\frac{2\pi b}{\lambda c}\left\{4bc\sin\theta-c^2\,(e^{\alpha z}+e^{-\alpha z})-4b^2\sin^2\theta\,(e^{\alpha z}+e^{-\alpha z})+bc\sin\theta\,(e^{\alpha z}+e^{-\alpha z})^2\right\}$$

$$=\frac{b}{c}\left\{g\,\overline{e^{\alpha z}-e^{-\alpha z}}-\frac{2\pi}{\lambda}bc\sin\theta\,(e^{\alpha z}-e^{-\alpha z})^2+\frac{2\pi}{\lambda}b^2\,(e^{\alpha z}-e^{-\alpha z})\,(e^{2\alpha z}-e^{-2\alpha z})\right\}$$

or, therefore,

$$-\frac{8\pi}{\lambda}bc\sin\theta+\frac{2\pi}{\lambda}c^2\,(e^{\alpha z}+e^{-\alpha z})+\frac{8\pi}{\lambda}b^2\sin^2\theta\,(e^{\alpha z}+e^{-\alpha z})-\frac{2\pi}{\lambda}bc\sin\theta\,(e^{\alpha z}+e^{-\alpha z})^2$$

$$=g\,(e^{\alpha z}-e^{-\alpha z})-\frac{2\pi}{\lambda}bc\sin\theta\,(e^{\alpha z}-e^{-\alpha z})^2+\frac{2\pi}{\lambda}b^2\,(e^{\alpha z}-e^{-\alpha z})\,(e\qquad-e^{-2\alpha z})\quad(9)$$

Let $\theta=0$, then $z=h$, and the equation gives

$$\frac{2\pi}{\lambda}c^2\,(e^{\alpha h}+e^{-\alpha h})=g\,(e^{\alpha h}-e^{-\alpha h})+\frac{2\pi}{\lambda}b^2.(e^{\alpha h}-e^{-\alpha h})\,(e^{2\alpha h}-e^{-2\alpha h})$$

$$=g\,(e^{\alpha h}-e^{-\alpha h})+\left(\frac{2\pi}{\lambda}\right)^3a^2c^2\,(e^{\alpha h}-e^{-\alpha h})\,(e^{2\alpha h}-e^{-2\alpha h})$$

or $$\frac{2\pi}{\lambda} c^2 (e^{\alpha h} + e^{-\alpha h}) \left\{1 - \left(\frac{2\pi}{\lambda} a\right)^2 (e^{\alpha h} - e^{-\alpha h})^2\right\} = g (e^{\alpha h} - e^{-\alpha h})$$

$$\therefore \quad \frac{2\pi}{\lambda} c^2 = g \cdot \frac{e^{\alpha h} - e^{-\alpha h}}{e^{\alpha h} + e^{-\alpha h}} \cdot \frac{1}{1 - \left(\frac{2\pi}{\lambda} a\right)^2 (e^{\alpha h} - e^{-\alpha h})^2}$$

$$= g \cdot \frac{\frac{2\pi}{\lambda} h + \frac{2\pi}{\lambda} h}{2} \quad \text{nearly}$$

hence $c^2 = g h$.

11. Our result has been obtained from equation (9) by putting $\theta = o$. This mode of proceeding, of course, gives only one result; we shall, therefore, pursue another process in order to obtain a second.

Since $$z = h + a (e^{\alpha h} - e^{-\alpha h}) \sin\theta$$
$$= h + m \sin\theta \quad \text{suppose,}$$

let us substitute this value of z in the equation (9), and expand the exponential functions.

It will appear readily that

$$e^{\alpha z} + e^{-\alpha z} = e^{\alpha h} + e^{-\alpha h} + \frac{2\pi}{\lambda} m \sin\theta (e^{\alpha h} - e^{-\alpha h})$$

$$e^{\alpha z} - e^{-\alpha z} = e^{\alpha h} - e^{-\alpha h} + \frac{2\pi}{\lambda} m \sin\theta (e^{\alpha h} + e^{-\alpha h})$$

if we omit powers of $m \sin\theta$ greater than the first.

But equation (9), by means of these values, becomes

$$-\frac{8\pi b c}{\lambda} \sin\theta + \frac{2\pi}{\lambda} c^2 (e^{\alpha h} + e^{-\alpha h}) + \frac{4\pi^2}{\lambda^2} c^2 a \sin\theta (e^{\alpha h} - e^{-\alpha h})^2$$

$$+ \frac{8\pi}{\lambda} b^2 \sin^2\theta (e^{\alpha h} + e^{-\alpha h}) - \frac{2\pi}{\lambda} b c \sin\theta (e^{\alpha h} + e^{-\alpha h})^2$$

$$- \frac{8\pi^2}{\lambda^2} b c \sin^2\theta\, a (e^{\alpha h} + e^{-\alpha h}) (e^{\alpha h} - e^{-\alpha h})^2 =$$

$$g (e^{\alpha h} - e^{-\alpha h}) + g \frac{2\pi}{\lambda} a \sin\theta (e^{2\alpha h} - e^{-2\alpha h})$$

$$- \frac{2\pi}{\lambda} b c \sin\theta (e^{\alpha h} - e^{-\alpha h})^2 - \frac{8\pi^2}{\lambda^2} b c a \sin^2\theta (e^{\alpha h} + e^{-\alpha h}) (e^{\alpha h} - e^{-\alpha h})^2$$

$$+ \frac{2\pi}{\lambda} b^2 \left\{ e^{\alpha h} - e^{-\alpha h} + \frac{2\pi}{\lambda} a \sin\theta (e^{2\alpha h} - e^{-2\alpha h}) \right\}$$

$$\times \left\{ e^{2\alpha h} - e^{-2\alpha h} + \frac{4\pi}{\lambda} a \sin\theta (e^{2\alpha h} + e^{-2\alpha h}) (e^{\alpha h} - e^{-\alpha h}) \right\}$$

This equation will furnish the two results mentioned above. The first, derived from the parts which do not contain θ, is

$$\frac{2\pi}{\lambda} c^2 (e^{\alpha h} + e^{-\alpha h}) = g (e^{\alpha h} - e^{-\alpha h}) + \frac{2\pi b^2}{\lambda} (e^{\alpha h} - e^{-\alpha h}) (e^{2\alpha h} - e^{-2\alpha h}):$$

and by substituting $\alpha^2 a^2 c^2$ for b^2, and transposing it, becomes

$$\frac{2\pi}{\lambda} c^2 (e^{\alpha h} + e^{-\alpha h}) \left(1 - \overline{\frac{2\pi}{\lambda} a}\Big|^2 (e^{\alpha h} - e^{-\alpha h})^2 \right) = g\,(e^{\alpha h} - e^{-\alpha h}$$

$$\therefore \quad \frac{2\pi}{\lambda} c^2 = g \,.\, \frac{e^{\alpha h} - e^{-\alpha h}}{e^{\alpha h} + e^{-\alpha h}} \quad \frac{1}{1 - \left(\frac{2\pi}{\lambda} a\right)^2 (e^{\alpha h} - e^{-\alpha h})^2}$$

the same result as in art. 10.

The second result is

$$-\frac{8\pi}{\lambda} b c + \frac{4\pi^2}{\lambda^2} c^2 a (e^{\alpha h} - e^{-\alpha h})^2 + \frac{8\pi}{\lambda} b^2 (e^{\alpha h} + e^{-\alpha h})$$

$$-\frac{8\pi}{\lambda} b c = g \frac{2\pi a}{\lambda} (e^{2\alpha h} - e^{-2\alpha h})$$

or

$$-\frac{32\pi^2}{\lambda^2} a c^2 + \frac{4\pi^2}{\lambda^2} a c^2 (e^{\alpha h} - e^{-\alpha h})^2$$

$$+\frac{32\pi^3}{\lambda^3} a^2 c^2 (e^{\alpha h} + e^{-\alpha h}) = \frac{2\pi}{\lambda} a g (e^{2\alpha h} - e^{-2\alpha h})$$

whence $$a = \frac{1}{\frac{16\pi^2}{\lambda^2} c^2 (e^{\alpha h} + e^{-\alpha h})} \left\{ g\,(e^{2\alpha h} - e^{-2\alpha h}) - \frac{2\pi}{\lambda} c^2 (e^{\alpha h} - e^{-\alpha h})^2 + \frac{16\pi}{\lambda} c^2 \right\}$$

$$= \frac{1}{\frac{8\pi}{\lambda} g\,(e^{\alpha h} - e^{-\alpha h})} \cdot \left\{ g\,(e^{2\alpha h} - e^{-2\alpha h}) - g \frac{(e^{\alpha h} - e^{-\alpha h})^3}{e^{\alpha h} + e^{-\alpha h}} + 8g \frac{e^{\alpha h} - e^{-\alpha h}}{e^{\alpha h} + e^{-\alpha h}} \right\}$$

$$= \frac{\lambda}{8\pi} \left\{ e^{\alpha h} + e^{-\alpha h} - \frac{(e^{\alpha h} - e^{-\alpha h})^2}{e^{\alpha h} + e^{-\alpha h}} + \frac{4}{e^{\alpha h} + e^{-\alpha h}} \right\}$$

$$= \frac{\lambda}{\pi\,(e^{\alpha h} + e^{-\alpha h})}$$

These two results give us, respectively, the velocity of transmission and the height of the wave.

If h be small compared with λ, the above equations give

$$c^2 = g\,h$$

$$a = \frac{\lambda}{2\pi}$$

as an approximation.

The first result is too well known to require comment; the second, if it have any truth at all, appears to shew that the tendency of waves in shallow water is to become semicircular, measuring from the mean points to the crests.

12. Before we proceed further, it will be convenient to make a trifling alteration, both in the mode of proceeding and in the notation.

Let θ now represent $x - ct$ instead of $ct - x$, and denote u by the sum of a series of terms of the form

$$b_0 + b_1 (e^{\alpha y} + e^{-\alpha y}) \sin\theta + \&c.$$

which is the most general form of which it is susceptible for a wave motion.

Let $u\,d\,x + v\,d\,y$ be denoted by $d\,\phi$, then it is a well known theorem that

$$\frac{d\,p}{\varrho} = -g\,d\,y - d\,.\,\frac{d\,\phi}{d\,t} - \frac{1}{2}\,d\,.\,(u^2+v^2)\,.$$

By means of this equation a value of p is found, which being treated as in art. 10, will give the values of c and b_o, &c.

We proceed to the most general case.

$$u = b_o + b_1\,(e^{\alpha y} + e^{-\alpha y})\sin\theta + b_2\,(e^{2\alpha y} + e^{-2\alpha y})\sin 2\,\theta + \&c.$$

$$v = -b_1\,(e^{\alpha y} - e^{-\alpha y})\cos\theta - b_2\,(e^{2\alpha y} - e^{-2\alpha y})\cos 2\,\theta + \&c.$$

$$\phi = b_o - \frac{1}{\alpha}\left\{ b_1\,\overline{e^{\alpha y} + e^{-\alpha y}}\,.\cos\theta + \frac{b_2}{2}\,(e^{2\alpha y} + e^{-2\alpha y})\cos 2\,\theta + \dots \right\}$$

$$\frac{d\,\phi}{d\,t} = -c\,\{ b_1\,\overline{e^{\alpha y} + e^{-\alpha y}}\sin\theta + b_2\,(e^{2\alpha y} + e^{-2\alpha y})\sin 2\,\theta + \dots\}$$

$$\therefore\quad \frac{p}{\varrho} = -g\,y + c\,\{ b_1\,(e^{\alpha y} + e^{-\alpha y}\sin\theta + b_2\,(e^{2\alpha y} + e^{-2\alpha y})\sin 2\,\theta + \dots\}$$

$$-\frac{1}{2}\Big[\,b_o^2 + 2\,b_o\,\{b_1\,\overline{e^{\alpha y} + e^{-\alpha y}}\sin\theta + b_2\,(e^{2\alpha y} + e^{-2\alpha y})\sin 2\,\theta + \dots\}$$

$$+\,b_1^2\,\overline{e^{2\alpha y} + e^{-2\alpha y}} + b_2^2\,(e^{4\alpha y} + e^{-4\alpha y}) + \dots$$

$$-2\,\{b_1^2\cos 2\,\theta + b_2^2\cos 4\,\theta + \dots\}$$

$$+2\,\Sigma'\,b_r\,b_s\,(e^{\overline{r+s}\,\alpha y} + e^{-\overline{r+s}\,\alpha y})\cos\overline{r-s}\,\theta$$

$$-\,2\,\Sigma'\,b_r\,b_s\,(e^{\overline{r-s}\,\alpha y} + e^{-\overline{r-s}\,\alpha y})\cos\overline{r+s}\,\theta\,\Big] + \mathrm{P}$$

$$= -g\,y + c\,\Sigma\,b_r\,(e^{r\alpha y} + e^{-r\alpha y})\sin r\,\theta$$

$$-\frac{1}{2}\,b_o^2 - b_o\,\Sigma\,b_r\,(e^{r\alpha y} + e^{-r\alpha y})\sin r\,\theta$$

$$-\frac{1}{2}\,\Sigma\,b_r\,b_s\,(e^{\overline{r+s}\,\alpha y} + e^{-\overline{r+s}\,\alpha y})\cos\overline{r-s}\,\theta$$

$$+\frac{1}{2}\,\Sigma\,b_r\,b_s\,(e^{\overline{r-s}\,\alpha y} + e^{-\overline{r-s}\,\alpha y})\cos\overline{r+s}\,\theta + \mathrm{P}$$

the symbol Σ denoting that all the values of $b_r\,b_s$ are to be taken, so that r may $= 1, 2\dots$ and $s = 1, 2\dots$

13. Now, since
$$\frac{p}{\varrho} = f(x\,,y) - f(x\,,z)\,;$$

the first differentiation gives us
$$\frac{1}{\varrho}\frac{d\,p}{d\,x} = \frac{d\,f(x\,,y)}{d\,x}\,;$$

the second
$$\frac{1}{\varrho}\frac{d\,p}{d\,x} = \frac{d\,f(x\,,y)}{d\,x} - \frac{d\,f(x\,,z)}{d\,x}\,.$$

Hence, since the two are equal, we must have $\frac{d\,.f(x\,,z)}{d\,x} = 0$, z being considered a function of x.

Now
$$f(x, z) = -g z + c \Sigma b_r (e^{r\alpha z} + e^{-r\alpha z}) \sin r\theta$$
$$-\frac{1}{2} b_0^2 - b_0 \Sigma b_r (e^{r\alpha z} + e^{-r\alpha z}) \sin r\theta$$
$$-\frac{1}{2} \Sigma b_r b_s (e^{\overline{r+s}\,\alpha z} + e^{-\overline{r+s}\,\alpha z}) \cos (r-s)\,\theta$$
$$+\frac{1}{2} \Sigma b_r b_s (e^{\overline{r-s}\,\alpha z} + e^{-\overline{r-s}\,\alpha z}) \cos (r+s)\,\theta$$

$$\therefore \quad 0 = -g \frac{dz}{dx} + (c-b_0)\,\alpha \{ b_1 (e^{\alpha z} + e^{-\alpha z}) \cos \theta + 2 b_2 (e^{2\alpha z} + e^{-2\alpha z}) \cos 2\theta + \ldots \}$$
$$-2\alpha \{ b_1^2 \sin 2\theta + 2 b_2^2 \sin 4\theta + \ldots \}$$
$$+\alpha \Sigma' \overline{r-s}\, b_r b_s (e^{\overline{r+s}\,\alpha z} + e^{-\overline{r+s}\,\alpha z}) \sin \overline{r-s}\,\theta$$
$$-\alpha \Sigma' \overline{r+s}\, b_r b_s (e^{\overline{r-s}\,\alpha z} + e^{-\overline{r-s}\,\alpha z}) \sin \overline{r+s}\,\theta$$
$$+(c-b_0)\,\alpha \{ b_1 (e^{\alpha z} - e^{-\alpha z} \sin \theta + 2 b_2 (e^{2\alpha z} - e^{-2\alpha z}) \sin 2\theta + \ldots \} \frac{dz}{dx}$$
$$-\{ b_1^2 (e^{2\alpha z} - e^{-2\alpha z}) + 2 b_2^2 (e^{4\alpha z} - e^{-4\alpha z}) + \ldots \} \alpha \frac{dz}{dx}$$
$$-\alpha \Sigma' \overline{r+s}\, b_r b_s (e^{\overline{r+s}\,\alpha z} - e^{-\overline{r+s}\,\alpha z}) \cos \overline{r-s}\,\theta \frac{dz}{dx}$$
$$+\alpha \Sigma' \overline{r-s}\, b_r b_s (e^{\overline{r-s}\,\alpha z} - e^{-\overline{r-s}\,\alpha z}) \cos \overline{r+s}\,\theta \frac{dz}{dx}$$

or, if we adopt the latter notation, where Σ includes values for r and s, which may be identical, which Σ' does not; we get

$$0 = -g\frac{dz}{dx} + (c-b_0)\,\alpha\, \Sigma\, r\, b_r (e^{r\alpha z} + e^{-r\alpha z}) \cos r\theta$$
$$+\frac{1}{2} \alpha \Sigma \overline{r-s}\; b_r b_s (e^{\overline{r+s}\,\alpha z} + e^{-\overline{r+s}\,\alpha z}) \sin \overline{r-s}\,\theta$$
$$-\frac{1}{2} \alpha \Sigma \overline{r+s}\; b_r b_s (e^{\overline{r-s}\,\alpha z} + e^{-\overline{r-s}\,\alpha z}) \sin \overline{r+s}\,\theta$$
$$+(c-b_0)\,\alpha\, \Sigma\, b_r\, r\, (e^{r\alpha z} - e^{-r\alpha z}) \sin r\theta \,.\, \frac{dz}{dx}$$
$$-\frac{\alpha}{2} \Sigma \overline{r+s}\; b_r b_s (e^{\overline{r+s}\,\alpha z} - e^{-\overline{r+s}\,\alpha z}) \cos \overline{r-s}\,\theta \,.\, \frac{dz}{dx}$$
$$+\frac{\alpha}{2} \Sigma \overline{r-s}\; b_r b_s (e^{\overline{r-s}\,\alpha z} - e^{-\overline{r-s}\,\alpha z}) \cos \overline{r+s}\,\theta \,.\, \frac{dz}{dx} .$$

14. From this equation we shall be able to obtain a complete solution of the problem, as far as the assumed form of the velocity expresses the actual state of motion. It will readily appear that z can contain only *sines* of multiples of θ; for, if it could contain cosines, v would also contain sines, which it is supposed

not to do. Let us then assume

$$z=h+a_{,}(e^{\alpha z}\sin\theta)+a_2 e^{2\alpha z}\sin 2\theta+\&c.$$
$$+f_{,}e^{-\alpha z}\sin\theta+f_2 e^{-2\alpha z}\sin 2\theta+\&c.$$

$$\therefore\quad \frac{dz}{dt}=\alpha\,\overline{u-c}\,\{(a_{,}e^{\alpha z}+f_{,}e^{-\alpha z})\cos\theta+2(a_2 e^{2\alpha z}+f_2 e^{-2\alpha z})\cos 2\theta+\&c.\}$$

$$+\alpha\frac{dz}{dt}\{a_{,}(e^{\alpha z}-f_{,}e^{-\alpha z})\sin\theta+2(a_2 e^{2\alpha z}-f_2 e^{-2\alpha z})\sin 2\theta+\&c.\}$$

or substituting for $\frac{dz}{dt}$ its value

$$(-b_{,}\overline{e^{\alpha z}-e^{-\alpha z}}\cos\theta-b_2\,.\,\overline{e^{2\alpha z}-e^{-2\alpha z}}\cos 2\theta-\&c.)\times$$

$$\{1-\alpha\,.\,(\overline{a_{,}e^{\alpha z}-f_{,}e^{-\alpha z}}\sin\theta+2\,.\,\overline{a_2 e^{2\alpha z}-f_2 e^{-2\alpha z}}\sin 2\theta+\ldots)\}$$

$$=\alpha\{(a_{,}e^{\alpha z}+f_{,}e^{-\alpha z})\cos\theta+2(a_2 e^{2\alpha z}+f_2 e^{-2\alpha z})\cos 2\theta+\ldots\}$$

$$.\{b_0-c+b_{,}(e^{\alpha z}+e^{-\alpha z})\sin\theta+b_2(e^{2\alpha z}+e^{-2\alpha z})\sin 2\theta+\ldots\}.$$

By equating coefficients, we obtain

$$f_{,}=-a_{,},\ f_2=-a_2\ \ldots$$
$$\alpha a_{,}(b_0-c)=-b_{,},\ \ldots$$
$$2\alpha a_2(b_0-c)=-b_2,\ \ldots$$
$$\&c.\ =\ \&c.$$

$$\therefore\quad \alpha a_{,}(c-b_0)=b_{,}$$
$$2\alpha a_2(c-b_0)=b_2$$
$$3\alpha a_3(c-b_0)=b_3$$
$$\&c.\ =\ \&c.$$

So that $\quad z=h+a_{,}(e^{\alpha z}-e^{-\alpha z})\sin\theta+a_2(e^{2\alpha z}-e^{-2\alpha z})\sin 2\theta+\ldots$

and $\quad \frac{dz}{dx}\{1-\alpha(a_{,}\overline{e^{\alpha z}+e^{-\alpha z}}\sin\theta+2a_2\,.\,\overline{e^{2\alpha z}+e^{-2\alpha z}}\sin 2\theta+\ldots)\}$

$$=\alpha\{a_{,}\overline{e^{\alpha z}-e^{-\alpha z}}\cos\theta+2a_2\overline{e^{2\alpha z}-e^{-2\alpha z}}\cos 2\theta+\ldots\}.$$

15. Substituting this value in the equation, we get

$$\Big[(c-b_0)\,\alpha\{b_{,}\overline{e^{\alpha z}+e^{-\alpha z}}\cos\theta+2b_2(e^{2\alpha z}+e^{-2\alpha z})\cos 2\theta+\ldots\}$$

$$+\frac{\alpha}{2}\Sigma\,\overline{r-s}\,b_r b_s(e^{\overline{r+s}\alpha z}+e^{-\overline{r+s}\alpha z})\sin\overline{r-s}\,\theta$$

$$-\frac{\alpha}{2}\Sigma\,\overline{r+s}\,b_r b_s(e^{\overline{r-s}\alpha z}+e^{-\overline{r-s}\alpha z})\sin\overline{r+s}\,\theta\ \Big]\times$$

$$\{1-\alpha(a_{,}\overline{e^{\alpha z}+e^{-\alpha z}}\sin\theta+2a_2\overline{e^{2\alpha z}+e^{-2\alpha z}}\sin 2\theta+\ldots)\}$$

$$=-\Big[-g+\overline{c-b_o}\,\alpha\,\{b_,\overline{e^{\alpha z}-e^{-\alpha z}}\sin\theta+2\,b_2\,\overline{e^{2\alpha z}-e^{-2\alpha z}}\sin 2\theta+\ldots\}$$

$$-\frac{\alpha}{2}\Sigma\,\overline{r+s}\;b_r\,b_s\,(e^{\overline{r+s}\,\alpha z}-e^{-\overline{r+s}\,\alpha z})\cos\overline{r-s}\,\theta$$

$$+\frac{\alpha}{2}\Sigma\,\overline{r-s}\;b_r\,b_s\,(e^{\overline{r-s}\,\alpha z}-e^{-\overline{r-s}\,\alpha z})\cos\overline{r+s}\,\theta\Big]$$

$$\times\{\alpha\,(a_,\overline{e^{\alpha z}-e^{-\alpha z}}\cos\theta+2\,a_2\,.\,\overline{e^{2\alpha z}-e^{-2\alpha z}}\,.\cos 2\theta+\ldots)\}$$

the symbol Σ applying to all values of r and s from 1 to ∞.

16. But since $\qquad \frac{b_2}{b_,}=\frac{2\,a_2}{a_,}, \qquad \therefore\quad 2\,a_2=\frac{b_2}{b_,}\,a_,.$

Similarly $\qquad 3\,a_3=\frac{a_,}{b_,}\,b_3, \quad 4\,a_4=\frac{a_,}{b_,}\,b_4 \quad$ &c.

Hence, if we substitute these values in the above equation, it gives us

$$\Big[(c-b_o)\,\alpha\,\{b_,\overline{e^{\alpha z}+e^{-\alpha z}}\,.\cos\theta+2\,b_2\,.\,\overline{e^{2\alpha z}+e^{-2\alpha z}}\cos 2\theta+\ldots\}$$

$$+\frac{\alpha}{2}\,\Sigma\,(r-s)\;b_r\,b_s\,(e^{\overline{r+s}\,\alpha z}+e^{-\overline{r+s}\,\alpha z})\sin\overline{r-s}\,\theta$$

$$-\frac{\alpha}{2}\,\Sigma\,(r+s)\;b_r\,b_s\,(e^{\overline{r-s}\,\alpha z}+e^{-\overline{r-s}\,\alpha z})\sin\overline{r+s}\,\theta\Big]$$

$$\times\Big\{1-\alpha\,\frac{a_,}{b_,}\,(b_,\overline{e^{\alpha z}+e^{-\alpha z}}\sin\theta+b_2\,.\,\overline{e^{2\alpha z}+e^{-2\alpha z}}\sin 2\theta+\ldots)\Big\}$$

$$=-\Big[-g+\overline{c-b_o}\,\alpha\,\{b_,\overline{e^{\alpha z}-e^{-\alpha z}}\sin\theta+2\,b_2\,.\,\overline{e^{2\alpha z}-e^{-2\alpha z}}\sin 2\theta+\ldots\}$$

$$-\frac{\alpha}{2}\,\Sigma\,(r+s)\;b_r\,b_s\,(e^{\overline{r+s}\,\alpha z}-e^{-\overline{r+s}\,\alpha z})\cos(r-s)\,\theta$$

$$+\frac{\alpha}{2}\,\Sigma\,\overline{r-s}\;b_r\,b_s\,(e^{\overline{r-s}\,\alpha z}-e^{-\overline{r-s}\,\alpha z})\cos\overline{r+s}\,\theta\Big]$$

$$\times\frac{\alpha\,a_,}{b_,}\,\{b_,\overline{e^{\alpha z}-e^{-\alpha z}}\cos\theta+b_2\,(e^{2\alpha z}-e^{-2\alpha z})\cos 2\theta+\ldots\}$$

17. By multiplying the upper line of the first side of this equation by its factor, we obtain

$$(c-b_o)\,\alpha\,\Sigma\,r\,b_r\,(e^{r\alpha z}+e^{-r\alpha z})\cos r\,\theta$$

$$-\alpha^2\,\frac{a_,}{b_,}\,(c-b_o)\,\Sigma\,r\,b_r\,b_s\,(e^{r\alpha z}+e^{-r\alpha z})\,(e^{s\alpha z}+e^{-s\alpha z})\cos r\,\theta\sin s\,\theta:$$

the analogous term on the second side of the equation is

$$-\alpha^2\,(c-b_o)\,\frac{a_,}{b_,}\,\Sigma\,(e^{r\alpha z}-e^{-r\alpha z})\,(e^{s\alpha z}-e^{-s\alpha z})\,b_r\,b_s\,r\cos s\,\theta\sin r\,\theta\,;$$

if this result be brought to the first side of the equation, and combined with the former, we obtain

$$(c-b_0)\,\alpha\,\Sigma\, r\, b_r\,(e^{r\alpha z}+e^{-r\alpha z})\cos r\,\theta$$
$$+\alpha\,\Sigma\,(e^{\overline{r+s}\,\alpha z}+e^{-\overline{r+s}\,\alpha z})\, r\, b_r\, b_s \sin\overline{r-s}\,\theta$$
$$-\alpha\,\Sigma\,(e^{\overline{r-s}\,\alpha z}+e^{-\overline{r-s}\,\alpha z})\, r\, b_r\, b_s \sin\overline{r+s}\,\theta \quad (1)$$

it being observed that $\alpha\dfrac{a_{\prime}}{b_{\prime}}\,\overline{c-b_0}=1$ by (14).

18. The result of the factor unity is (omitting the part which has been already considered)

$$\frac{\alpha}{2}\,\Sigma\,(r-s)\, b_r\, b_s\,(e^{\overline{r+s}\,\alpha z}+e^{-\overline{r+s}\,\alpha z})\sin\overline{r-s}\,\theta$$
$$-\frac{\alpha}{2}\,\Sigma\,(r+s)\, b_r\, b_s\,(e^{\overline{r-s}\,\alpha z}+e^{-\overline{r-s}\,\alpha z})\sin\overline{r+s}\,\theta \quad (2).$$

The remainder of the upper side is

$$-\frac{\alpha^2 a_{\prime}}{2b_{\prime}}\,\Sigma\,(r-s)\, b_r\, b_s\, b_t\,(e^{\overline{r+s}\,\alpha z}+e^{-\overline{r+s}\,\alpha z})\,(e^{t\alpha z}+e^{-t\alpha z})\sin\overline{r-s}\,\theta\,.\sin t\,\theta$$
$$+\frac{\alpha^2 a_{\prime}}{2b_{\prime}}\,\Sigma\,(r+s)\, b_r\, b_s\, b_t\,(e^{\overline{r-s}\,\alpha z}+e^{-\overline{r-s}\,\alpha z})\,(e^{t\alpha z}+e^{-t\alpha z})\sin\overline{r+s}\,\theta\,\sin t\,\theta:$$

and the corresponding terms of the lower side are

$$\frac{\alpha^2 a_{\prime}}{2b_{\prime}}\,\Sigma\,(r+s)\, b_r\, b_s\, b_t\,(e^{\overline{r+s}\,\alpha z}-e^{-\overline{r+s}\,\alpha z})\,(e^{t\alpha z}-e^{-t\alpha z})\cos\overline{r-s}\,\theta\,\cos t\,\theta$$
$$-\frac{\alpha^2 a_{\prime}}{2b_{\prime}}\,\Sigma\,(r-s)\, b_r\, b_s\, b_t\,(e^{\overline{r-s}\,\alpha z}-e^{-\overline{r-s}\,\alpha z})\,(e^{t\alpha z}-e^{-t\alpha z})\cos\overline{r+s}\,\theta\,\cos t\,\theta\,;$$

which being brought over to the other side, the sum of the respective terms is

$$\begin{aligned}\frac{\alpha^2 a_{\prime}}{2b_{\prime}}\,\Sigma\, b_r\, b_s\, b_t\Big[&-(r-s)\,(e^{\overline{r+s+t}\,\alpha z}+e^{-\overline{r+s+t}\,\alpha z}+e^{\overline{r+s-t}\,\alpha z}+e^{-\overline{r+s-t}\,\alpha z})\sin(r-s)\,\theta\sin t\theta\\
&+(r+s)\,(e^{\overline{r-s+t}\,\alpha z}+e^{\overline{r-s-t}\,\alpha z}+e^{-\overline{r-s+t}\,\alpha z}+e^{-\overline{r-s-t}\,\alpha z})\sin(r+s)\,\theta\sin t\,\theta\\
&-(r+s)\,(e^{\overline{r+s+t}\,\alpha z}-e^{\overline{r+s-t}\,\alpha z}+e^{-\overline{r+s+t}\,\alpha z}-e^{-\overline{r+s-t}\,\alpha z})\cos(r-s)\,\theta\cos t\,\theta\\
&+(r-s)\,(e^{\overline{r-s+t}\,\alpha z}-e^{\overline{r-s-t}\,\alpha z}+e^{-\overline{r-s+t}\,\alpha z}-e^{-\overline{r-s-t}\,\alpha z})\cos(r+s)\,\theta\cos t\,\theta\Big]\end{aligned}$$

which becomes, by multiplying out, and adding together, such quantities as will make up cosines of sums or differences of the arcs which appear in the present form,

$$\begin{aligned}\frac{\alpha^2 a_{\prime}}{2b_{\prime}}\,\Sigma\, b_r\, b_s\, b_t\Big[&-r\,(e^{\overline{r+s+t}\,\alpha z}+e^{-\overline{r+s+t}\,\alpha z})\cos\overline{r-s-t}\,\theta\\
&-s\,(e^{\overline{r+s+t}\,\alpha z}+e^{-\overline{r+s+t}\,\alpha z})\cos\overline{r-s+t}\,\theta\\
&+r\,(e^{\overline{r+s-t}\,\alpha z}+e^{-\overline{r+s-t}\,\alpha z})\cos\overline{r-s+t}\,\theta\\
&+s\,(e^{\overline{r+s-t}\,\alpha z}+e^{-\overline{r+s-t}\,\alpha z})\cos\overline{r-s-t}\,\theta\end{aligned}$$

$$+r\left(e^{\overline{r-s+t}\,\alpha z}+e^{-\overline{r-s+t}\,\alpha z}\right)\cos\overline{r+s-t}\,\theta$$

$$-s\left(e^{\overline{r-s+t}\,\alpha z}+e^{-\overline{r-s+t}\,\alpha z}\right)\cos\overline{r+s+t}\,\theta$$

$$-r\left(e^{\overline{r-s-t}\,\alpha z}+e^{-\overline{r-s-t}\,\alpha z}\right)\cos\overline{r+s+t}\,\theta$$

$$+s\left(e^{\overline{r-s-t}\,\alpha z}+e^{-\overline{r-s-t}\,\alpha z}\right)\cos\overline{r+s-t}\,\theta\Big]$$

Or since

$$\Sigma r\cos\overline{r-s-t}\,\theta\, e^{\overline{r+s+t}\,\alpha z}$$

$$=\Sigma s\cos\overline{r-s+t}\,\theta\, e^{\overline{r+s+t}\,\alpha z}$$

&c. &c.

this term may be written

$$\frac{\alpha^2 a_{\prime}}{2b_{\prime}}\Sigma b_r b_s b_t\Big\{-2r\left(e^{\overline{r+s+t}\,\alpha z}+e^{-\overline{r+s+t}\,\alpha z}\right)\cos\overline{s+t-r}\,\theta$$

$$+2r\left(e^{\overline{r+s-t}\,\alpha z}+e^{-\overline{r+s-t}\,\alpha z}\right)\cos\overline{r+t-s}\,\theta$$

$$+2r\left(e^{\overline{r+t-s}\,\alpha z}+e^{-\overline{r+t-s}\,\alpha z}\right)\cos\overline{r+s-t}\,\theta$$

$$-2r\left(e^{\overline{s+t-r}\,\alpha z}+e^{-\overline{s+t-r}\,\alpha z}\right)\cos\overline{r+s+t}\,\theta\Big\}:$$

which may be written

$$=\frac{\alpha^2 a_{\prime}}{b_{\prime}}\Sigma r b_r b_s b_t\Big\{-\left(e^{f\alpha z}+e^{-f\alpha z}\right)\cos\overline{f-2r}\,\theta-\left(e^{\overline{f-2r}\,\alpha z}+e^{-\overline{f-2r}\,\alpha z}\right)\cos f\theta$$

$$+\left(e^{\overline{f-2t}\,\alpha z}+e^{-\overline{f-2t}\,\alpha z}\right)\cos(f-2s)\,.\,\theta+\left(e^{\overline{f-2s}\,\alpha z}+e^{-\overline{f-2s}\,\alpha z}\right)\cos(f-2t)\,\theta\Big\}\quad(3)$$

if $r+s+t$ be denoted by f.

19. The remainder of the lower side of our equation is

$$\frac{g\alpha a_{\prime}}{b_{\prime}}\Sigma b_r\left(e^{r\alpha z}-e^{-r\alpha z}\right)\cos r\theta.$$

By collecting these several terms our equation is reduced to the following very simple form:

$$(c-b_0)\,\alpha\,\Sigma r b_r\left(e^{r\alpha z}+e^{-r\alpha z}\right)\cos r\theta$$

$$+\alpha\Sigma\left(e^{\overline{r+s}\,\alpha z}+e^{-\overline{r+s}\,\alpha z}\right)r b_r b_s\sin\overline{r-s}\,\theta$$

$$-\alpha\Sigma\left(e^{\overline{r-s}\,\alpha z}+e^{-\overline{r-s}\,\alpha z}\right)r b_r b_s\sin\overline{r+s}\,\theta$$

$$+\frac{\alpha}{2}\Sigma(r-s)\,b_r b_s\left(e^{\overline{r+s}\,\alpha z}+e^{-\overline{r+s}\,\alpha z}\right)\sin\overline{r-s}\,\theta$$

$$-\frac{\alpha}{2}\Sigma\,\overline{r+s}\,b_r b_s\left(e^{\overline{r-s}\,\alpha z}+e^{-\overline{r-s}\,\alpha z}\right)\sin\overline{r+s}\,\theta$$

$$+\frac{\alpha^2 a_{\prime}}{b_{\prime}}\Sigma_r b_r b_s b_t \left\{-(e^{f\alpha z}+e^{-f\alpha z})\cos\overline{f-2r}.\theta-(e^{\overline{f-2r}.\alpha z}+e^{-\overline{f-2r}\alpha z})\cos f\theta\right.$$

$$\left.+(e^{\overline{f-2t}.\alpha z}+e^{-\overline{f-2t}.\alpha z})\cos\overline{f-2s}\,\theta+(e^{\overline{f-2s}\alpha z}+e^{-\overline{f-2s}.\alpha z})\cos\overline{f-2t}\,\theta\right\}$$

$$=\frac{g\,\alpha\,a_{\prime}}{b_{\prime}}\Sigma b_r(e^{r\alpha z}-e^{-r\alpha z})\cos r\theta:$$

or

$$(c-b_0)^2\alpha\Sigma r b_r(e^{r\alpha z}+e^{-r\alpha z})\cos r\theta$$

$$+2\alpha\Sigma(e^{\overline{r+s}\alpha z}+e^{-\overline{r+s}\alpha z})\,r b_r b_s\sin\overline{r-s}\,\theta\,(c-b_0)$$

$$-2\alpha\Sigma(e^{\overline{r-s}\alpha z}+e^{-\overline{r-s}\alpha z})\,r b_r b_s\sin\overline{r+s}\,\theta\,(c-b_0)$$

$$+\alpha\Sigma r b_r b_s b_t\left\{-(e^{f\alpha z}+e^{-f\alpha z})\cos\overline{f-2r}.\theta-(e^{\overline{f-2r}\alpha z}+e^{-\overline{f-2r}\alpha z})\cos f\theta\right.$$

$$\left.+(e^{\overline{f-2t}\alpha z}+e^{-\overline{f-2t}.\alpha z})\cos\overline{f-2s}\,\theta+(e^{\overline{f-2s}.\alpha z}+e^{-\overline{f-2s}.\alpha z})\cos\overline{f-2t}\,\theta\right\}$$

$$=g\Sigma b_r(e^{r\alpha z}-e^{-r\alpha z})\cos r\theta:$$

or finally

$$(c-b_0)^2\Sigma r b_r(e^{r\alpha z}+e^{-r\alpha z})\cos r\theta$$

$$+2(c-b_0)\Sigma r b_r b_s\left\{(e^{\overline{r+s}\alpha z}+e^{-\overline{r+s}\alpha z})\sin\overline{r-s}\,\theta-(e^{\overline{r-s}\alpha z}+e^{-\overline{r-s}\alpha z})\sin\overline{r+s}\,\theta\right\}$$

$$+\Sigma r b_r b_s b_t\left\{-(e^{f\alpha z}+e^{-f\alpha z})\cos\overline{f-2r}\,\theta-(e^{\overline{f-2r}\alpha z}+e^{-\overline{f-2r}\alpha z})\cos f\theta\right.$$

$$\left.+(e^{\overline{f-2t}\alpha z}+e^{-\overline{f-2t}\alpha z})\cos\overline{f-2s}\,\theta+(e^{\overline{f-2s}\alpha z}+e^{-\overline{f-2s}\alpha z})\cos\overline{f-2t}.\theta\right\}$$

$$=\frac{g}{\alpha}\Sigma b_r(e^{r\alpha z}-e^{-r\alpha z})\cos r\theta.$$

20. This expression is exceedingly simple and symmetrical, and might be very easily applied to any hypothesis respecting the coefficients $b_{\prime}$, b_2 ... It may be satisfactory, in the first place, to deduce from it the particular form already obtained, art. 10.

Let $b_{\prime}$ be the only value of b, then $r=s=t=1$, and we get

$$c^2 b(e^{\alpha z}+e^{-\alpha z})\cos\theta-2cb^2.2\sin 2\theta$$

$$+b^3\left(-(e^{+3\alpha z}+e^{-3\alpha z})\cos\theta-(e^{\alpha z}+e^{-\alpha z})\cos 3\theta+2(e^{\alpha z}+e^{-\alpha z})\cos\theta\right)$$

$$=\frac{g}{\alpha}b(e^{\alpha z}-e^{-\alpha z})\cos\theta$$

or

$$c^2 b(e^{\alpha z}+e^{-\alpha z})\cos\theta-8cb^2\sin\theta\cos\theta$$

$$-b^3\left\{(e^{3\alpha z}+e^{-3\alpha z})\cos\theta+\overline{e^{\alpha z}+e^{-\alpha z}}(\cos 3\theta+\cos\theta)-3\overline{e^{\alpha z}+e^{-\alpha z}}\cos\theta\right\}$$

$$=\frac{g}{\alpha}b(e^{\alpha z}-e^{-\alpha z})\cos\theta:$$

that is, if we divide by $b\cos\theta$

$$c^2(e^{\alpha z}+e^{-\alpha z})-8\ cb\sin\theta-b^2(e^{3\alpha z}+e^{-3\alpha z})$$
$$-2b^2(e^{\alpha z}+e^{-\alpha z})\cos 2\theta+3b^2(e^{\alpha z}+e^{-\alpha z})$$
$$=\frac{g}{\alpha}(e^{\alpha z}-e^{-\alpha z}).$$

Now the expression in art. 10. is this,

$$-8bc\sin\theta+c^2(e^{\alpha z}+e^{-\alpha z})+2b^2\overline{1-\cos 2\theta}\,(e^{\alpha z}+e^{-\alpha z})$$
$$-b^2(\overline{e^{3\alpha z}+e^{-3\alpha z}}-\overline{e^{\alpha z}+e^{-\alpha z}})=\frac{g}{\alpha}(e^{\alpha z}-e^{-\alpha z});$$

which is equivalent to

$$c^2(e^{\alpha z}+e^{-\alpha z})-8bc\sin\theta-b^2(e^{3\alpha z}+e^{-3\alpha z})+3b^2(e^{\alpha z}+e^{-\alpha z})$$
$$-2b^2(e^{\alpha z}+e^{-\alpha z})\cos 2\theta=\frac{g}{\alpha}(e^{\alpha z}-e^{-\alpha z})$$

an equation identical with the one above.

21. Let us now derive from the equation the value of c in terms of b, b_2 ... .

1. In the first place, since b_0 is in every place subtracted from c, never occurring in any other way, we derive the following important conclusion:

That a progressive motion of the fluid does not affect the velocity of transmission of the undulation relative to the position of corresponding particles; in other words, the velocity of transfer of the undulation is exactly equal to the sum of the velocity of progression, and that of undulation in a fluid at rest.

2. If we make $\theta=0$, we obtain

$$(c-b_0)^2\,\Sigma r\,b_r(e^{r\alpha h}+e^{-r\alpha h})$$
$$+\Sigma r\,b_r\,b_s\,b_t\left\{-(e^{f\alpha h}+e^{-f\alpha h})-(e^{(f-2r)\alpha h}+e^{-\overline{f-2r}\alpha h})\right.$$
$$\left.+(e^{\overline{f-2t}\alpha h}+e^{-\overline{f-2t}\alpha h})+(e^{\overline{f-2s}\alpha h}+e^{-\overline{f-2s}\alpha h})\right\}$$
$$=\frac{g}{\alpha}\Sigma b_r(e^{r\alpha h}-e^{-r\alpha h})$$

which equation gives $c-b_0$.

The two values so obtained will be equal, but will have opposite signs, since the term which involves the first power of $c-b_0$ does not contain any cosines.

This circumstance that the sines are combined with the odd powers, and cosines with even powers of $c-b_0$ is very remarkable, and it is probably connected with the relation existing between the quantities $b_{,}, b_2 \ldots$, but we shall not at present enter into a discussion of the subject.

If it be thought more simple to obtain $(c-b_0)^2$ in terms of $a_{,}, a_2 \ldots$, than in terms of $b_{,}, b_2 \ldots$, this can be at once effected by means of the equations in art. 14; the result being:

$$(c-b_0)^2\,\Sigma\, r^2 a_r\,(e^{r\alpha h}+e^{-r\alpha h})$$

$$+\alpha^2\,\Sigma\, r^2 s\,t\,(c-b_0)^2\,a_r\,a_s\,a_t\Big\{-(e^{f\alpha h}+e^{-f\alpha h})$$

$$-(e^{(f-2r)\alpha h}+e^{-(f-2r)\alpha h})+e^{(f-2t)\alpha h}+e^{-(f-2t)\alpha h}+e^{(f-2s)\alpha h}+e^{-(f-2s)\alpha h}\Big\}$$

$$=\frac{g}{\alpha}\,\Sigma\, r\,a_r\,(e^{r\alpha h}-e^{-r\alpha h}).$$

From this equation, we obtain

$$(c-b_0)^2=\frac{g}{\alpha}\,\Sigma\, r\,a_r\,(e^{r\alpha h}-e^{-r\alpha h})\div$$

$$\Big[\,\Sigma\, r^2 a_r\,(e^{r\alpha h}+e^{-r\alpha h})+\alpha^2\,\Sigma\, r^2 s\,t\,a_r\,a_s\,a_t\times$$

$$\Big\{-(e^{f\alpha h}+e^{-f\alpha h})-(e^{(f-2r)\alpha h}+e^{-(f-2r)\alpha h})+e^{(f-2s)\alpha h}+e^{-(f-2s)\alpha h}$$

$$+e^{(f-2t)\alpha h}+e^{-(f-2t)\alpha h}\Big\}\Big].$$

22. Since v must of necessity be zero at the points where z is a maximum and a minimum, we get for the values of θ at such points, first from the value of $\frac{dz}{dx}$ being 0 at such points,

$$0=a_1\,(e^{\alpha z}-e^{-\alpha z})\cos\theta+2\,a_2\,(e^{2\alpha z}-e^{-2\alpha z})\cos 2\,\theta+\ldots;$$

and from the circumstance that $v=0$,

$$0=b_1\,(e^{\alpha z}-e^{-\alpha z})\cos\theta+b_2\,(e^{2\alpha z}-e^{-2\alpha z})\cos 2\,\theta+\ldots$$

But by art. 14,

$$b_1=\alpha\,(c-b_0)\,a_1$$
$$b_2=\alpha\,(c-b_0)\,2\,a_2$$
$$..=\ldots\ldots$$

therefore, the second equation becomes

$$0=a_1\,(e^{\alpha z}-e^{-\alpha z})\cos\theta+2\,a_2\,(e^{2\alpha z}-e^{-2\alpha z})\cos 2\,\theta+\ldots$$

which is identical with the first. This is strongly confirmatory of the correctness of our operations.

If (as is probably always the case) the wave be symmetrical on both sides, we must have this zero occurring at points distant from each other by $\frac{\lambda}{2}$; hence

$$\theta=\frac{\pi}{2},\ \frac{3\pi}{2},\ \text{and } a_2,\ a_4,\ \&\text{c.}=0;\ b_2,\ b_4,\ \&\text{c.}=0.$$

The value of u at such points is

$$u=b_0+b_1\,e^{-\frac{2\pi}{\lambda}y}\sin\theta+b_3\,e^{-\frac{6\pi}{\lambda}y}\sin 3\,\theta+\ldots$$

$$\left.\begin{array}{ll}\theta=\dfrac{\pi}{2} & =b_0+b_1\,e^{-\frac{2\pi}{\lambda}y}-b_3\,e^{-\frac{6\pi}{\lambda}y}+b_5\,e^{-\frac{10\pi}{\lambda}y}-\&\text{c.}\\[2ex] \theta=\dfrac{3\pi}{2} & =b_0-b_1\,e^{-\frac{2\pi}{\lambda}y}+b_3\,e^{-\frac{6\pi}{\lambda}y}-\&\text{c.}\end{array}\right\}$$

2. In the case of common oscillatory waves, there ought one to be positive and the other negative, equal in magnitude,

$$\therefore \quad b_0 = 0:$$

from which it follows, that b_o is a quantity depending *entirely* on the progressive motion.

SECTION II.—VARIABLE WAVE MOTION.

23. Hitherto we have confined our attention to motion in two dimensions, limiting the channel to one of uniform breadth and depth.

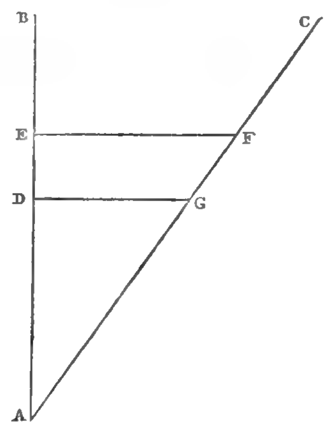
We proceed next to the more general case, where the breadth is variable but the depth constant.

To take the same order as before, we will commence with the hypothesis that all vertical sections have the same horizontal velocity at any time, the section being perpendicular to the direction of transmission.

By reference to arts. 2, 6, and 7, the process which follows will be perfectly intelligible, without any lengthened explanation.

We commence with that case wherein the section perpendicular to the direction of transmission is a triangle, one side of which we may suppose vertical, but it will not affect the calculation.

Let the axes be called those of x, y, and z; the introduction of a third being requisite in this case. In the former calculations, we adopted z as the representation of the depth: in the present case, we shall use the letter s for the same purpose.



The annexed figure is supposed to be a section of the fluid perpendicular to the direction of motion, and it is supposed that all sections so made are similar. Our object is to determine the motion of a portion of the fluid enclosed between two planes perpendicular to the direction of translation.

Let α be the distance between these planes at the time t,

$\alpha + \delta\alpha$ . . . . at the time $t + \delta t$;

s, the depth of the fluid at t,

$s + \delta s$ at $t + \delta t$;

r the breadth at the top at the time t,

$r + \delta r$ at $t + \delta t$;

then, from the property of the triangle, r and s bear a constant ratio: let $r = m\,s$.

Also, since the quantity of fluid between the planes is supposed to remain unchanged, we obtain

$$r\,s\,\alpha = (r + \delta r)\,(s + \delta s)\,(\alpha + \delta\alpha)$$

or
$$s^2\,\alpha=(s+\delta s)^2\,(\alpha+\delta\alpha)$$
$$=s^2\,\alpha+s^2\,\delta\alpha+2\,s\,\alpha\,\delta s \quad \text{nearly}$$
$$\therefore \quad \delta s=-\frac{s^2}{2\,s\,\alpha}\,\delta\alpha$$
$$=-\frac{s}{2\,\alpha}\,\delta\alpha$$
hence
$$\frac{d\,s}{d\,t}=-\frac{s}{2\,\alpha}\,\frac{d\,\alpha}{d\,t}.$$

24. Now, the moving force on this mass is
$$\textstyle\int\!\!\int dy\,dz\,p-\int\!\!\int dy\,dz\left(p+\frac{dp}{dx}\,\delta x\right)$$
$$=\int_0^s dy\int_0^{my} dz\,p-\int_0^{s+\frac{ds}{dx}\delta x} dy\int_0^{my} dz\left(p+\frac{dp}{dx}\,\delta x\right):$$
and
$$p=g\,\varrho\,.\,\overline{s-y}-\frac{1}{2}\,\varrho\,u^2,$$
$$\therefore \quad \frac{dp}{dx}=g\,\varrho\,\frac{d\,s}{d\,x}-\varrho\,u\,\frac{d\,u}{d\,x}.$$

Hence moving force $.\dfrac{1}{\varrho}$
$$=\int_0^s dy\int_0^{my} dz\left\{g\,\overline{s-y}-\frac{1}{2}\,u^2\right\}$$
$$-\int_0^{s+\frac{ds}{dx}\alpha} dy\int_0^{my} dz\left\{g\left(\overline{s+\frac{d\,s}{d\,x}\alpha-y}\right)-\frac{1}{2}\left(u^2+2\,u\,\frac{d\,u}{d\,x}\,\alpha\right)\right\}$$
$$=\int_0^s dy\left\{g\,\overline{s-y}\,my-\frac{1}{2}\,u^2\,my\right\}$$
$$-\int_0^{s+\frac{ds}{dx}\alpha} dy\left\{g\left(\overline{s+\frac{d\,s}{d\,x}\alpha-y}\right)my-\frac{1}{2}\,\overline{u^2+2\,u\,\frac{d\,u}{d\,x}\alpha}\,.\,my\right\}$$
$$=g\,m\,\frac{s^3}{2}-g\,m\,\frac{s^3}{3}-\frac{1}{4}\,u^2\,m\,s^2$$
$$-g\left.\overline{s+\frac{d\,s}{d\,x}\,\alpha}\right|^3\cdot\frac{m}{2}+g\,m\,\frac{\overline{s+\frac{d\,s}{d\,x}\,\alpha}^{\,3}}{3}+\frac{1}{4}\left(u^2+2\,u\,\frac{d\,u}{d\,x}\,\alpha\right)m\,\overline{s+\frac{d\,s}{d\,x}\,\alpha}^{\,2}$$
$$=-\frac{3}{2}\,g\,m\,s^2\,.\,\frac{d\,s}{d\,x}\,\alpha+g\,m\,s^2\,\frac{d\,s}{d\,x}\,\alpha+\frac{1}{2}\,m\left(u^2\,s\,\frac{d\,s}{d\,x}\,\alpha+s^2\,u\,\frac{d\,u}{d\,x}\,\alpha\right)$$
$$=-\frac{1}{2}\,m\,g\,\alpha\,s^2\,\frac{d\,s}{d\,x}+\frac{1}{2}\,m\,\alpha\left(u^2\,s\,\frac{d\,s}{d\,x}+s^2\,u\,\frac{d\,u}{d\,x}\right)$$

and mass moved $\quad =\dfrac{m}{2}\,s^2\,\alpha\,\varrho$

therefore, accelerating force $\quad =-g\,\dfrac{d\,s}{d\,x}+\dfrac{u^2\,\frac{d\,s}{d\,x}+s\,u\,\frac{d\,u}{d\,x}}{s}$

25. Also,

$$u=b\sin\frac{2\pi}{\lambda}(ct-x)$$

$$s=h+a\sin\frac{2\pi}{\lambda}(ct-x):$$

$$\therefore\quad \frac{du}{dx}=-b\frac{2\pi}{\lambda}\cos.\frac{2\pi}{\lambda}(ct-x),$$

$$\frac{ds}{dx}=-a.\frac{2\pi}{\lambda}\cos.\frac{2\pi}{\lambda}(ct-x),$$

$$\frac{du}{dt}=\frac{2\pi}{\lambda}bc\cos\theta-\frac{2\pi}{\lambda}b^2\sin\theta\cos\theta.$$

Consequently,

$$\frac{2\pi}{\lambda}bc.\cos\theta-\frac{2\pi}{\lambda}b^2\sin\theta\cos\theta=\frac{2\pi}{\lambda}ag\cos\theta$$

$$-\frac{2\pi}{\lambda}b^2.\frac{a\cos\theta\sin^2\theta+\sin\theta\cos\theta(h+a\sin\theta)}{h+a\sin\theta}$$

Multiplying out, this gives

$$(ag-bc+b^2\sin\theta)(h+a\sin\theta)=b^2a\sin^2\theta+b^2\sin\theta(h+a\sin\theta),$$

or

$$(ag-bc)(h+a\sin\theta)=b^2a\sin^2\theta.$$

Whence we get

$$hag-hbc=o;$$

$$\therefore\quad ag=bc.$$

But

$$\left(\frac{dx}{dt}\right)^2=u^2,$$

$$\therefore\quad \left(\frac{dx+d\alpha}{dt}\right)^2=\left(u+\frac{du}{dx}\alpha\right)^2$$

and

$$\frac{dx}{dt}.\frac{d\alpha}{dt}=u\frac{du}{dx}\alpha;$$

hence

$$\frac{d\alpha}{dt}=\alpha\frac{du}{dx},$$

$$\therefore\quad \frac{du}{dx}=\frac{1}{\alpha}\frac{d\alpha}{dt}=-\frac{2}{s}.\frac{ds}{dt};$$

whence, by substitution,

$$(h+a\sin\theta)b\cos\theta=2a\cos\theta(c-b\sin\theta)+\ldots$$

$$\therefore\quad hb=2ac$$

or if we put for b its value found above,

$$\frac{hag}{c}=2ac$$

$$hg=2c^2$$

$$c=\sqrt{\frac{gh}{2}}.$$

That is, the square of the velocity of transmission in a triangular channel, is half the square of the velocity in a rectangular channel of the same maximum depth.

It may be thought necessary to verify this result, although, from its simplicity, I suspect it must have been known to POISSON and others. As, however, I have never seen it, or met with any reference to such an investigation, I have added a few examples from Mr RUSSELL'S *Report on Waves*, in the last volume of the Reports of the British Association. The wave which Mr RUSSELL has examined, is different from that which has been assumed as the basis of calculation; but the difference makes no alteration in this result, as will be seen hereafter.

From the analysis which Mr RUSSELL has given of his experiments, I select the following results; being those for which the height of the fluid in the triangular tube most nearly coincides with that in the corresponding experiment in the quadrilateral one.

Depth of Fluid in inches.	Velocity observed in rectangular channel.	Velocity observed in triangular channel.	Do. computed from those in the second column.	Difference.
4.15	3.20	2.19	2.2	
4.23	3.35	2.42	2.4	
4.32	3.40	2.43	2.4	
4.38	3.40	2.46	2.4	
5.59	4.05	2.66	2.8	.1
6.26	4.08	2.88	2.9	
6.38	4.04	2.85	2.8	
6.52	4.05	3.02	2.8	.2
7.21	4.32	3.02	3.0	
7.36	4.39	3.02	3.1	

The only discrepancies are those which I have placed in the table; but it may be remarked that, in all cases, the circumstances are very different in the two experiments, depending partly on the height and partly on the length of the wave. That the two cases fixed on should be at variance, is not to be wondered at, when it is remarked that, in the rectangular channel, the velocity is the same for both cases, whilst the depths differ by about an inch; and it is by means of the depth alone, that we have compared the results in a triangular channel with those in a rectangular.

On the whole, I conceive the coincidence between the different results as a striking confirmation of the process which has been employed, approximative as that process confessedly is.

26. Let us now take the general case of any shape whatever to the vertical section.

Let K represent the area of the section of the fluid at the point x, y, z.

Then
$$\mathrm{K}\,\alpha=\left(\mathrm{K}+\frac{d\,\mathrm{K}}{d\,t}\,\delta\,t\right)\left(\alpha+\frac{d\,\alpha}{d\,t}\,\delta\,t\right),$$

$$\therefore\quad 0=\mathrm{K}\frac{d\,\alpha}{d\,t}+\alpha\frac{d\,\mathrm{K}}{d\,t}.$$

Now, K is a function of the height s; therefore, s is a function of K.

Let
$$s=f\mathrm{K}$$

$$\therefore\quad \frac{d\,s}{d\,t}=f'\,\mathrm{K}\,.\,\frac{d\,\mathrm{K}}{d\,t}$$

$$=-\frac{\mathrm{K}}{\alpha}\,f'\,\mathrm{K}\,.\,\frac{d\,\alpha}{d\,t}.$$

Now the moving force, on the mass of which the thickness is α

$$=\int_0^s dy\int_0^{\phi y}dz\,p-\int_0^{s+\frac{ds}{dx}\delta x}.\,dy\int_0^{\phi y}dz\left(p+\frac{d\,p}{d\,x}\,\delta\,x\right)$$

where $\phi\,y=z$ is the equation to the generating curve of the boundary of the section.

And
$$p=g\,\varrho\,(s-y)-\frac{1}{2}\varrho\,u^2,$$

$$\frac{d\,p}{d\,x}=g\,\varrho\,\frac{d\,s}{d\,x}-\varrho\,u\,\frac{d\,u}{d\,x}.$$

$$\therefore\quad \frac{\text{moving force}}{\varrho}=\int_0^s dy\int_0^{\phi y}dz\left(g\,\overline{s-y}-\frac{1}{2}u^2\right)$$

$$-\int_0^{s+\frac{ds}{dx}\alpha}dy\int_0^{\phi y}dz\left\{g\left(s+\frac{d\,s}{d\,x}\alpha-y\right)-\frac{1}{2}\left(u^2+2u\frac{d\,u}{d\,x}\alpha\right)\right\}$$

$$=\int_0^s dy\left(g\,\overline{s-y}-\frac{1}{2}u^2\right)\phi\,y$$

$$-\int_0^{s+\frac{ds}{dx}\alpha}dy\left(g\left(s+\frac{d\,s}{d\,x}\alpha-y\right)-\frac{1}{2}\overline{u^2+2\,u\frac{d\,u}{d\,x}\,\alpha}\right)\phi\,y$$

$$=\int_0^s dy\left\{-g\,\frac{d\,s}{d\,x}\,\alpha+u\,\frac{d\,u}{d\,x}\,\alpha\right\}\phi\,y$$

$$-\int_s^{s+\frac{ds}{dx}\alpha}dy\left\{g\,\overline{s-y}-\frac{1}{2}u^2\right\}\phi\,y.$$

Let
$$\int\phi\,y\,dy=\mathrm{F}\,y$$

$$\int y\,\phi\,y\,dy=f\,y$$

$$\therefore\quad \frac{\text{moving force}}{\varrho}=-\left(g\,\frac{d\,s}{d\,x}-u\,\frac{d\,u}{d\,x}\right)\alpha\,\mathrm{F}\,(s)$$

$$-\left(g\,s-\frac{1}{2}u^2\right)\alpha\,\frac{d\,s}{d\,x}\,.\,\mathrm{F}'\,s+\alpha\,\frac{d\,s}{d\,x}\,g\,f'\,s$$

$$=-\left(g\frac{ds}{dx}-u\frac{du}{dx}\right)\alpha\,\mathrm{F}\,s-\left(g s-\frac{1}{2}u^2\right)\alpha\frac{ds}{dx}\phi\,s+\alpha\,s\frac{ds}{dx}g\,\phi\,s$$

$$=-\left(g\frac{ds}{dx}-u\frac{du}{dx}\right)\alpha\,\mathrm{F}\,s+\frac{1}{2}\alpha\,u^2\frac{ds}{dx}\phi\,s$$

and mass moved $=\mathrm{K}\,\alpha\,\varrho$

$$=\alpha\,\varrho\,\mathrm{F}\,y=\alpha\,\varrho\,\mathrm{F}\,s$$

therefore moving force $=-g\dfrac{ds}{dx}+u\dfrac{du}{dx}+\dfrac{1}{2}u^2\cdot\dfrac{ds}{dx}\cdot\dfrac{\phi\,s}{\mathrm{F}\,s}$ (1)

This is an expression which can be applied to different shaped channels with great facility.

Also, as in other cases,

$$\left(\frac{dx+d\alpha}{dt}\right)^2=\left(u+\frac{du}{dx}\alpha\right)^2$$

$$\therefore\quad \frac{d\alpha}{dt}=\frac{du}{dx}\alpha$$

and $$\frac{du}{dx}=\frac{1}{\alpha}\frac{d\alpha}{dt}=-\frac{1}{\mathrm{K}}\frac{d\mathrm{K}}{dt}.$$

Now $$\mathrm{K}=\mathrm{F}\,s$$

$$\therefore\quad \frac{d\mathrm{K}}{dt}=\mathrm{F}'\,s\,.\frac{ds}{dt}$$

$$=\phi\,s\,.\frac{ds}{dt}$$

$$\therefore\quad \frac{du}{dx}=-\frac{\phi\,s}{\mathrm{F}\,s}\cdot\frac{ds}{dt}\quad(2)$$

By means of these equations the velocity and motion are discovered.

27. Ex. Let us take one example for the sake of illustration.

Suppose the canal to have the parabolic form, then

$$z=\sqrt{m\,y}=\phi\,y$$

and $$\mathrm{F}\,y=\frac{2}{3}\sqrt{m}\,.\,y^{\frac{3}{2}}\quad\text{and}\quad\frac{\phi\,s}{\mathrm{F}\,s}=\frac{3}{2}\,\frac{1}{s}.$$

By the substitution of this value in equation (1), we obtain

$$\frac{du}{dt}=-g\frac{ds}{dx}+u\frac{du}{dx}+\frac{1}{2}u^2\frac{ds}{dx}\frac{3}{2}\cdot\frac{1}{s}$$

or $$\frac{2\pi}{\lambda}b\,c\,.\cos\theta-\frac{2\pi}{\lambda}b^2\sin\theta\cos\theta=$$

$$\frac{2\pi}{\lambda}a\,g\cos\theta-\frac{2\pi}{\lambda}b^2\sin\theta\cos\theta-\frac{3}{4}\,\frac{b^2\sin^2\theta\frac{2\pi}{\lambda}a\cos\theta}{h+a\sin\theta}$$

therefore $a\,g-b\,c=0$ by equating the large terms as in other cases.

And from equation (2)

$$\frac{du}{dx} = -\frac{3}{2s}\frac{ds}{dt}$$

$$\therefore \quad (h+a\sin\theta)\,b\cos\theta = \frac{3}{2}a\cos\theta\,(c-b\sin\theta+\ldots$$

$$hb = \frac{3}{2}ac\,.$$

If we put for b this value in the former equation, we get

$$ag = \frac{3}{2}\frac{ac^2}{h}$$

$$c^2 = \frac{2}{3}gh$$

$$c = \sqrt{\frac{2}{3}gh}$$

or the square of the velocity in a parabolic vessel is $\frac{2}{3}$ of its value in a rectangular one.

28. We may readily deduce the more general results, viz.

$$ag - bc = 0 \quad (1)$$

$$b\cos\theta = a\cos\theta\,(c-b\sin\theta)\frac{\phi s}{\mathrm{F}s}$$

$$b = ac\frac{\phi h}{\mathrm{F}h} \quad (2)$$

$$\therefore \quad ag = ac^2\frac{\phi h}{\mathrm{F}h}$$

$$c^2 = g\frac{\mathrm{F}h}{\phi h}$$

$$= g\,.\,\frac{\text{area of vertical section}}{\text{breadth at surface}}\,.$$

29. The investigation supposes the curve a continuous curve, but, except in extreme cases, it will apply equally well to others. For instance, it will apply to the cases examined by Mr RUSSELL, viz. when the areas are trapeziums. We will apply the formula to one or two of these experiments of Mr RUSSELL, and then quit the subject.

In the channel M, the breadth is 12 inches, and the depth of the triangular part 4 inches: therefore area of triangular section equals 24 inches. And if we take the height to the centre of the wave as the height corresponding to h, we get for experiments XC, XCIII, mentioned p. 444;

	$h = 6.21$,
area of parallelogram	$= 26.52$,
therefore area of section	$= 50.52$

and $$c^2=g\,.\,\frac{50.52}{144}\text{ in feet}$$
$$=\frac{32.2\,.\,(50.52)}{144}=\frac{(32.2.)\,50.5}{144}\text{ suppose}$$
$$c=\frac{40.3}{12}=3.3\ \ldots$$

Experiment gives it 3.08.

The discrepancy is due entirely to our not knowing in *any* case the exact height (h) which gives $c^2=g\,h$ for a square formed canal.

This discrepancy is not much greater than that between two waves of the same height in a rectangular channel.

By computing the velocity of the next waves given by Mr RUSSELL, we obtain $$c=3.45.$$

By observation c is equal to 3.50.

For the next and last set we obtain 3.56, which by observation is 3.86.

30. I propose, in the last place, to deduce a first approximation to motion in a channel of variable breadth, taking only that case for which the channel diminishes very slowly and uniformly.

Let z be the breadth at the point whose other variables are x and t.

$$\therefore\quad z=m\,(l-x)$$

l being the whole length of the channel from the origin.

Adopting all the previous notation, we obtain

$$z\,s\,\alpha=\overline{z+\delta z}\ \overline{s+\delta s}\ \overline{\alpha+\delta\alpha}$$

or $$\alpha\,s\frac{d\,z}{d\,t}+\alpha\,z\frac{d\,s}{d\,t}+z\,s\frac{d\,\alpha}{d\,t}=0$$

or $$\frac{1}{\alpha}\frac{d\,\alpha}{d\,t}=-\frac{1}{z}\frac{d\,z}{d\,t}-\frac{1}{s}\frac{d\,s}{d\,t}$$

$$\therefore\quad \frac{d\,u}{d\,x}=\frac{1}{z}\,m\frac{d\,x}{d\,t}-\frac{1}{s}\frac{d\,s}{d\,t}\qquad(1)$$

31. Now, we may find the variation in the height of the wave, by supposing its length to remain constant, an hypothesis which must be considered as merely approximative.

The volume of the wave will vary as

$$\int_0^z\int_0^\pi a\sin\frac{2\pi}{\lambda}\,(c\,t-x)\,d\,x\,d\,z$$

or as $$a\,\lambda\,z+\mathrm{C}$$

hence, if a' be the value of a at the origin,

$$a'\lambda\,.\,l=a\,\lambda\,(l-x)$$
$$a=a'\,.\,\frac{l}{l-x}\,.$$
$$\therefore\quad s=h+\frac{a'\,l}{l-x}\sin\theta$$
$$u=b\sin\theta\,.$$

By substituting these values in equation (1), we obtain

$$-\frac{2\pi}{\lambda} b \,.\, \cos\theta + \sin\theta \frac{db}{dx} = \frac{m\,b\sin\theta}{m\,(l-x)}$$

$$-\frac{1}{h+\dfrac{a'\,l}{l-x}\sin\theta}\left\{\left.\frac{a'\,l}{(l-x)^2}\sin\theta - \frac{2\pi}{\lambda}\frac{a'\,l}{l-x}\cos\theta\right|\frac{dx}{dt} + \frac{2\pi}{\lambda}\frac{a'\,c\,l}{l-x}\cos\theta\right\}$$

Equating coefficients of like functions of x, we obtain approximately

$$b\,h = \frac{a'\,c\,l}{l-x}$$

and

$$\frac{db}{dx} = \frac{b}{l-x}$$

$$\therefore \qquad \log b = \log\frac{C}{l-x}$$

$$\frac{b}{b'} = \frac{l}{l-x}$$

$$b = \frac{b'\,l}{l-x}$$

b' being the value of b at the origin.

Again, $$\frac{du}{dt} = -g\frac{ds}{dx} + u\frac{du}{dx} + \&c.$$

$$\therefore \qquad -\frac{2\pi}{\lambda} b\,c\,.\cos\theta - \frac{2\pi}{\lambda} b^2 \sin\theta\cos\theta + \sin^2\theta\, b\frac{db}{dx}$$

$$= \frac{2\pi}{\lambda}\frac{a'\,g\,l}{l-x}\cos\theta - \frac{g\,a\,l}{(l-x)^2}\sin\theta + \&c.:$$

or equating only those coefficients which belong to the large terms,

$$b\,c = \frac{a'\,g\,l}{l-x}\,.$$

But we have shewn that

$$b\,h = \frac{a'\,c\,l}{l-x}$$

$$\therefore \qquad \frac{c}{h} = \frac{g}{c}$$

or $$c^2 = g\,h\,,$$

Thus it appears, that the velocity is not altered, whilst the height of the wave increases in harmonic progression. This result does not agree even roughly with Mr RUSSELL's experiments; the reason for which is, that his waves were of considerable length, so that the variation of the channel through the length of a single wave cannot be neglected.

To attempt the solution of the more general problem, would lead us into very complex analysis. It must consequently be reserved for another memoir.

SECTION III.—SOLITARY WAVE MOTION.

32. The subject for investigation in the ensuing section, is the transmission of a solitary wave. Waves of this kind are so generated, that, throughout the whole length of the wave, the velocity parallel to x is positive.

As to the vertical motion, it does not appear probable that any difference would be caused in it by the horizontal motion; we may then conceive all the circumstances to remain the same as before, with the exception that the whole wave has a transmission parallel to the axis of x. By art. 8, it appears that the only functions which will satisfy the conditions are,

$$u=b\,(e^{\alpha y}+e^{-\alpha y})\sin\theta+\phi\,y,$$

$$v=-b\,(e^{\alpha y}-e^{-\alpha y})\cos\theta,$$

θ being $=x-c\,t$.

We shall hereafter discuss the variation which these formulæ admit of, but there does not appear to be any other *form* capable of satisfying the necessary conditions.

We may remark that p is not now, as in the former case, a complete differential, except approximately; it becomes, then, a question in what manner to vary or increase the formulæ, &c., to render it so accurately.

This discussion will form a distinct branch of inquiry, into which I forbear to enter at present.

Let us return to our equations.

The condition to be satisfied is, that, when $t=0$ and $x=-\frac{\lambda}{4}$, u shall $=0$, and $v=0$: this gives

$$\phi\,y=e^{\alpha y}+e^{-\alpha y}$$

and we get

$$u=b\,(e^{\alpha y}+e^{-\alpha y})\,(1+\sin\theta)$$

$$v=-b\,(e^{\alpha y}-e^{-\alpha y})\cos\theta\,.$$

33. Lest it should be thought that, in the case before us, the assumption which we made in art. 8, that the form of the circular function is $\frac{\sin}{\cos}\frac{2\pi}{\lambda}\overline{x-c\,t}$, is inapplicable here, I offer the following demonstration of the point.

Take the most general form involving only one circular function:

$$u=(e^{\alpha y}+e^{-\alpha y})\,(\sin\alpha\,x\,f\,t-\cos\alpha\,x\,\phi\,t+\mathrm{C}+c)$$

$$v=-(e^{\alpha y}-e^{-\alpha y})\,(\cos\alpha\,x\,f\,t+\sin\alpha\,x\,\phi\,t+\mathrm{H})$$

where the circumstance that $v=0$ when $y=0$ for all values of x, gives the form of the exponentials; and the relation $\frac{d\,u}{d\,x}+\frac{d\,v}{d\,y}=0$, gives that of the circular functions. C and H are supposed to be functions of t.

Now, when $t=0$ and $\alpha x=-\frac{\pi}{2}$, $v=0$, and $u=0$,

$$\therefore \quad 0=-fo+C_o+c\,;\ \phi(o)=H_o$$

H_o being the value of H when $t=0$.

Let

$$z=h+(e^{\alpha z}-e^{-\alpha z})\,(D+\cos\alpha x\,F\,t+\sin\alpha x\,\psi\,t)$$

$$\therefore \quad \frac{dz}{dt}=\alpha\,(e^{\alpha z}+e^{-\alpha z})\,\frac{dz}{dt}\,(D+\cos\alpha x\,F\,t+\sin\alpha x\,\psi\,t)+(e^{\alpha z}-e^{-\alpha z})\left(\frac{dD}{dt}+\ldots\right)$$

$$-(e^{\alpha z}-e^{-\alpha z})\,(\alpha\sin\alpha x\,F\,t-\alpha\cos\alpha x\,\psi\,t)$$

$$\times(e^{\alpha z}+e^{-\alpha z})\,(\sin\alpha x f t-\cos\alpha x\,\phi\,t+C)$$

hence we obtain, by putting for $\frac{dz}{dt}$ its value $(v)_{y=z}$

$$(e^{\alpha z}-e^{-\alpha z})\,(\cos\alpha x f t+\sin\alpha x\,\phi\,t+H)$$

$$\times\{\alpha\,(e^{\alpha z}+e^{-\alpha z})\,(D+\cos\alpha x\,F\,t+\sin\alpha x\,\psi\,t)-1\}$$

$$=(e^{\alpha z}-e^{-\alpha z})\,\left(\frac{dD}{dt}+\cos\alpha x\,F'\,t+\sin\alpha x\,\psi'\,t\right)$$

$$-\alpha\,(e^{2\alpha z}-e^{-2\alpha z})\,(\sin\alpha x\,F\,t-\cos\alpha x\,\psi\,t)$$

$$\times(\sin\alpha x f t-\cos\alpha x\,\phi\,t+C).$$

Equate separately to zero the coefficients of $e^{\alpha z}-e^{-\alpha z}$, and of $e^{2\alpha z}-e^{-2\alpha z}$, and there results

$$(\cos\alpha x f t+\sin\alpha x\,\phi\,t+H)\,(D+\cos\alpha x\,F\,t+\sin\alpha x\,\psi\,t)$$

$$=-(\sin\alpha x\,F\,t-\cos\alpha x\,\psi\,t)\,(\sin\alpha x f t-\cos\alpha x\,\phi\,t+C)$$

and

$$\cos\alpha x f t+\sin\alpha x\,\phi\,t+H=-\frac{dD}{dt}-\cos\alpha x\,F'\,t-\sin\alpha x\,\psi'\,t.$$

From the former equation, we obtain

$$D\,(\cos\alpha x f t+\sin\alpha x\,\phi\,t)+\cos^2\alpha x f t\,F\,t+\sin^2\alpha x\,\phi\,t\,\psi\,t$$

$$+\sin\alpha x\cos\alpha x\,(f t\,\psi\,t+F\,t\,\phi\,t)+DH+H\cos\alpha x\,F\,t$$

$$+H\sin\alpha x\,\psi\,t=-C\,(\sin\alpha x\,F\,t-\cos\alpha x\,\psi\,t)$$

$$-\sin^2\alpha x\,F\,t f t-\cos^2\alpha x\,\phi\,t\,\psi\,t+\sin\alpha x\cos\alpha x\,(F\,t\,\phi\,t+f t\,\psi\,t):$$

in which, if we equate the coefficients of sines and cosines of αx, $2\alpha x$, &c., we get

$$D f t+H\,F\,t=C\,\psi\,t\ldots\ldots\ldots\ldots(1),$$

$$D\,\phi\,t+H\,\psi\,t=-C\,F\,t\ldots\ldots\ldots(2),$$

$$f t\,\psi\,t+F\,t\,\phi\,t=F\,t\,\phi\,t+f t\,\psi\,t \quad \text{an identity;}$$

$$2\,D\,H+f t\,F\,t+\phi\,t\,\psi\,t=-F\,t f t-\phi\,t\,\psi\,t$$

or

$$D\,H+f t\,F t+\phi\,t\,\psi\,t=0\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$f t\,F\,t-\phi\,t\,\psi\,t=F\,t f t-\phi\,t\,\psi\,t \quad \text{an identity.}$$

From the second equation, we obtain, in like manner,

$$\frac{dD}{dt}=-H\ldots\ldots\ldots\ldots\ldots(4)$$

$$f\,t = -\text{F}'\,t \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

$$\phi\,t = -\psi'\,t \ldots\ldots\ldots\ldots\ldots\ldots(6).$$

34. Let us next solve these six equations. To find C, we must combine (1), (2), and (3).

From (1) and (2),

$$\text{C}\,(\phi\,t\,\psi\,t + f\,t\,\text{F}\,t) = \text{H}\,(\phi\,t\,\text{F}\,t - \psi\,t\,f\,t) \qquad (a)$$

and from (3),

$$-\text{CDH} = \text{C}\,(f\,t\,\text{F}\,t + \phi\,t\,\psi\,t)$$

$$\therefore \qquad -\text{CD} = (\phi\,t\,\text{F}\,t - \psi\,t\,f\,t) \qquad (b)$$

To find H, we eliminate C by (1) and (2), and obtain

$$\text{H}\,\{(\text{F}\,t)^2 + (\psi\,t)^2\} + \text{D}\,(f\,t\,\text{F}\,t + \phi\,t\,\psi\,t) = 0 \qquad (c)$$

Putting for H its value from (3), we get

$$(\text{F}\,t)^2 + (\psi\,t)^2 = \text{D}^2 \qquad (d)$$

Also, by (5) and (6),

$$f\,t\,\text{F}\,t + \phi\,t\,\psi\,t = -(\text{F}\,t\,\text{F}'\,t + \psi\,t\,\psi'\,t) \qquad (e)$$

$$= -\frac{1}{2}\frac{d}{d\,t}\left(\overline{\text{F}\,t}^2 + \overline{\psi\,t}|^2\right)$$

$$= -\frac{1}{2}\frac{d}{d\,t}\,\text{D}^2 \text{ by } (d)$$

$$= -\text{D}\frac{d\,\text{D}}{d\,t} = \text{DH}$$

But by (3),

$$f\,t\,\text{F}\,t + \phi\,t\,\psi\,t = -\,\text{DH}$$

$$\therefore \qquad \text{DH} = 0,\; f\,t\,\text{F}\,t + \phi\,t\,\psi\,t = 0.$$

35. We may satisfy the equation DH=0 in two ways: 1. by making D=0, in which case, by virtue of equation (d), both Ft and ψt equal zero, or z is itself constant. This case corresponds to equilibrium, and we have nothing to do with it. 2. By making H=0; which hypothesis requires that D should be a constant, independent of the time, by equation (4). To prevent error, in consequence of a quantity equal to zero appearing in our equations, it will be desirable to write them down again, omitting H. We shall also omit t for the sake of brevity, and write f instead of ft, ϕ instead of ϕt, &c.

The equations are

$$\text{D}f = \text{C}\,\psi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\text{D}\,\phi = -\text{CF} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$f\text{F} + \phi\,\psi = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$f = -\text{F}' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

$$\phi = -\psi' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

The equation (3), combined with (4) and (5), gives

$$F^2+\psi^2=\text{a const},$$
$$=E^2 \text{ suppose}:$$

but $$F^2+\psi^2=\frac{D^2}{C^2}(f^2+\phi^2) \quad \text{by (1) and (2)};$$

$$\therefore \quad f^2+\phi^2=\frac{C^2 E^2}{D^2}$$

or $$F'^2+\psi'^2=\frac{C^2 E^2}{D^2} \quad \text{by (5) and (6)}.$$

If $F=E\sin\gamma$, ψ will $=E\cos\gamma$;

$$\therefore \quad \left(\frac{d\gamma}{dt}\right)^2=\frac{C^2}{D^2}.$$

36. Again, since $$\psi=\sqrt{E^2-F^2}$$
$$\psi'=\frac{-F F'}{\sqrt{E^2-F^2}}$$

by substituting which in the equation $F'^2+\psi'^2=\frac{C^2 E^2}{D^2}$ we get

$$F'^2+\frac{F^2 F'^2}{E^2-F^2}=\frac{C^2 E^2}{D^2}$$

or $$\frac{E^2 F'^2}{E^2-F^2}=\frac{C^2 E^2}{D^2}$$

$$\frac{F'^2}{E^2-F^2}=\frac{C^2}{D^2}$$

$$\frac{F'}{\sqrt{E^2-F^2}}=\frac{C}{D};$$

$$\therefore \quad \sin^{-1}\frac{F}{E}=\frac{1}{D}\int C\,dt,$$

and $$F=E\sin\left(\frac{1}{D}\int C\,dt\right).$$

Similar values may be obtained for the other quantities f, ϕ ..., and thus the number of arbitrary functions will be reduced to one, viz. C. This is the general solution of the problem.

For the present, however, we prefer the examination of the following particular case:

37. Let $$F\,t=e^{-mt}(f\cos\alpha c t+g\sin\alpha c t+k)$$
$$\psi\,t=e^{-mt}(p\cos\alpha c t+q\sin\alpha c t+r)$$
$$\therefore \quad f\,t=m e^{-mt}(f\cos.+g\sin.+k)+\alpha c e^{-mt}(f\sin.-g\cos.)$$
$$=e^{-mt}(\overline{mf-\alpha c g}.\cos.+\overline{mg+\alpha c f}.\sin.+mk)$$

and $$\phi\,t=m e^{-mt}(p\cos.+q\sin.+r)+\alpha c e^{-mt}(p\sin.-q\cos.)$$
$$=e^{-mt}\{\overline{mp-\alpha c q}\cos.+\overline{mq+\alpha c p}\sin.+mr\}:$$

by writing cos. for $\cos\alpha c t$ and so on for shortness.

And the equation $fF+\phi\psi=0$ gives us

$$(f\cos.+g\sin.+k)(\overline{mf-\alpha cg}\cos.+\overline{mg+\alpha cf}\sin.+mk)$$
$$+(p\cos.+q\sin.+r)(\overline{mp-\alpha cq}\cos.+\overline{mq+\alpha cp}\sin.+mr)=0.$$

Equating coefficients we get

$$f(mf-\alpha cg)+g(mg+\alpha cf)+p(mp-\alpha cq)$$
$$+q(mq+\alpha cp)+2mk^2+2mr^2=0 \quad \text{.......... (1)}$$
$$f(mf-\alpha cg)-g(mg+\alpha cf)+p(mp-\alpha cq)$$
$$-q(mq+\alpha cp)=0 \quad \text{.......... (2)}$$
$$f(mg+\alpha cf)+g(mf-\alpha cg)$$
$$+p(mq+\alpha cp)+q(mp-\alpha cq)=0 \quad \text{.......... (3)}$$
$$mkf+k(mf-\alpha cg)+mrp+r(mp-\alpha cq)=0 \quad \text{.......... (4)}$$
$$mkg+k(mg+\alpha cf)+mrq+r(mq+\alpha cp)=0 \quad \text{.......... (5)}$$

38. From combining (1) and (2), there arises the following equation:

$$f(mf-\alpha cg)+p(mp-\alpha cq)+m(k^2+r^2)=0 \quad \text{.......... (I.)}$$

From equation (2) alone we get

$$m(f^2+p^2-g^2-q^2)=2\alpha c(fg+pq) \quad \text{.......... (II.)}$$

From (3) $\quad 2m(fg+pq)+\alpha c(f^2+p^2-g^2-q^2)=0$ .... (III.)

From (4) $\quad 2m(kf+rp)=\alpha c(kg+rq)$ .... (IV.)

From (5) $\quad 2m(kg+rq)+\alpha c(kf+pr)=0$ .... (V.)

If it be allowable to eliminate $f^2+p^2-g^2-q^2$ between II. and III., we obtain

$$\frac{m}{2\alpha c}=-\frac{\alpha c}{2m}$$

or
$$m^2+\alpha^2c^2=0$$
$$\therefore \quad m=0,\ \alpha c=0;$$

and all the equations are satisfied without giving any other conditions. In fact, it is obvious this case is that in which Ft, ψt are constant.

Now, the only reason which can operate to prevent this elimination being effected, is the circumstance that one of the quantities $f^2+p^2-g^2-q^2$ or $fg+pq$ is equal to zero.

And, by means of the same equations, it appears that *both* the quantities must equal zero; or
$$f^2+p^2-g^2-q^2=0 \qquad (1')$$
$$fg+pq=0 \qquad (2')$$

But the first equation gives $m(f^2+p^2+k^2+r^2)=0$ by means of (2′); which again, combined with (1′), gives

$$m(g^2+q^2+k^2+r^2)=0 \qquad (3')$$

Either therefore $m=0$, or g, q, k, r, f, p, are all separately equal to 0; but the latter condition cannot be true,

$$\therefore \quad m=0;$$

also equation (IV.) gives

$$kg+rq=0 \qquad (4');$$

and equation (V)

$$kf+pr=0 \qquad (5');$$

If k and r are not each equal to zero, we obtain by equations (4′) and (5′)

$$\frac{f}{g}=\frac{p}{q}$$

and since by (2′)

$$fg=-pq,$$

$$\therefore \qquad \frac{f}{g}=-\frac{fg}{q^2} \text{ or } q^2=-g^2$$

$$\therefore \qquad q=0\ ,\ g=0\ .$$

But, in this case, by (1′),

$$f=0\ ,\ p=0\ .$$

Hence our functions are reduced to

$$\mathrm{F}=k\ ,\ \psi=r\ .$$

This solution is one with which we have no concern; it belongs to a state of rest, or of uniform motion. If, however, one of the quantities, as k, is equal to zero, then $rp=0$ and $rq=0$; either, therefore, $r=0$ or $p=0$, $q=0$. But if $p=0$, $q=0$ we have by (1′) $f^2=g^2$, and by (2′) $fg=0$:

$$\therefore \qquad f=0\ ,\ g=0;$$

and we are reduced to the same state as before.

But if $r=0$, the only equations to be satisfied are (1′) and (2′), which are

$$f^2+p^2-g^2-q^2=0$$

$$fg+pq=0;$$

and thus we reduce our equations to the form

$$\mathrm{F}\,t=f\cos\alpha c\,t+g\sin\alpha c\,t$$

$$\psi\,t=p\cos\alpha c\,t+q\sin\alpha c\,t$$

$$f t=-\alpha c\,(g\cos\alpha c\,t-f\sin\alpha c\,t)$$

$$\phi\,t=-\alpha c\,(q\cos\alpha c\,t-p\sin\alpha c\,t)$$

39. Now, when $\alpha x=-\frac{\pi}{2}$ and $t=0$, our assumed conditions are, that $u=0$, $v=0$; and since it has been proved that $\mathrm{H}=0$, it follows that $\phi\,t$ must be equal to 0 when $t=0$; hence we must have $q=0$: wherefore we require to have either $f=0$ or $g=0$, in order to satisfy equation (2′).

But

$$\mathrm{F}^2+\psi^2=\text{a constant}$$

hence

$$(f\cos\alpha c\,t+g\sin\alpha c\,t)^2+p^2\cos^2\alpha c\,t=\text{const}$$

$$\therefore \qquad \frac{f^2-g^2}{2}+\frac{p^2}{2}=0$$

$$fg=0$$

consequently

$$p^2=g^2-f^2$$

and p^2 is a positive quantity, therefore g^2 cannot be equal to 0, hence $f=0$; and

$$p=\pm g=-g \text{ suppose.}$$

We are therefore reduced to precisely the old form, viz.:

$$u=(e^{\alpha y}+e^{-\alpha y})\,(-g\,\alpha\,c\sin\alpha\,x\cos\alpha\,c\,t+g\,\alpha\,c\cos\alpha\,x\sin\alpha\,c\,t+\mathrm{C}+c)$$

$$=(e^{\alpha y}+e^{-\alpha y})\,(b\sin\alpha\,.\,\overline{x-c\,t}+\mathrm{C}+c)$$

if $\quad b=-g\,\alpha\,c$

$$v=-(e^{\alpha y}-e^{-\alpha y})\,(b\cos\alpha\,x\cos\alpha\,c\,t+b\sin\alpha\,x\sin\alpha\,c\,t)$$

$$=-b\,(e^{\alpha y}-e^{-\alpha y})\cos\alpha\,.\,\overline{x-c\,t}\,.$$

40. The corresponding value of z is

$$z=h+(e^{\alpha z}-e^{-\alpha z})\,(\mathrm{D}+g\sin\alpha\,c\,t\cos\alpha\,x-g\cos\alpha\,c\,t\sin\alpha\,x)$$

$$=h+(e^{\alpha z}-e^{-\alpha z})\left(\mathrm{D}+\frac{b}{\alpha\,c}\sin\alpha\,\overline{x-c\,t}\right).$$

Now D is independent of t (35); if, therefore, we put $\alpha\,x=-\frac{\pi}{2}$, $t=0$, we get by hypothesis $z=h$.

But we obtain $\quad z=h+(e^{\alpha h}-e^{-\alpha h})\left(\mathrm{D}-\frac{b}{\alpha\,c}\right)$

hence $\quad \frac{b}{\alpha\,c}=\mathrm{D}$

and $\quad z=h+(e^{\alpha z}-e^{-\alpha z})\left(\frac{b}{\alpha\,c}+\frac{b}{\alpha\,c}\sin\theta\right)$

$$=h+\frac{b}{\alpha\,c}(e^{\alpha z}-e^{-\alpha z})\,(1+\sin\theta)\,.$$

41. Lastly, $\quad \mathrm{C}^2\propto f^2+\phi^2$......(35)

$$\propto p^2\,\alpha^2\,c^2\sin^2\alpha\,c\,t+g^2\,\alpha^2\,c^2\cos^2\alpha\,c\,t$$

$$\propto p^2\,\alpha^2\,c^2 \qquad \because\ p^2=g^2$$

$\therefore$ C is independent of t.

And by making $\alpha\,x=-\frac{\pi}{2}$ and $t=0$, we obtain $u=0$,

$$\therefore\quad \mathrm{C}+c=b$$

and $\quad u=(e^{\alpha y}+e^{-\alpha y})\,b\,(1+\sin\theta)\,.$

42. Having thus obtained values of u, v, and z, it remains that we substitute them in the equations which determine the pressure. Now, in doing this, it must be borne in mind that the values of u, v, and z, are not correctly expressed by the above formulæ, and that consequently we must not expect accurately to satisfy the conditions of integrability of the function which expresses the differential of the pressure. Still whatever variation may be requisite in the above functions, to enable them accurately to satisfy all the conditions, it cannot be doubted that they hold true in the early part of the motion, as far as the large terms are concerned. If, then, in the process of finding the pressure, we take no notice of terms of the second order, our results will be a close approximation to the truth.

We proceed to the determination of the motion. All that remains for us to do, is to substitute the values of u, v, and z, in the equations of art. 8. We obtain

$$\frac{1}{\varrho}\frac{dp}{dx} = -b\,(e^{\alpha y}+e^{-\alpha y})\,(\alpha\cos\theta)\left(\frac{dx}{dt}-c\right) + \alpha b^2\,(e^{\alpha y}-e^{-\alpha y})^2\cos\theta\,(1+\sin\theta)$$

$$=-b\,\alpha\cos\theta\,(e^{\alpha y}+e^{-\alpha y})\,\{(b\overline{e^{\alpha y}+e^{-\alpha y}})\,(1+\sin\theta)-c\}$$

$$+\alpha b^2\,(e^{\alpha y}-e^{-\alpha y})^2\,(1+\sin\theta)\cos\theta$$

$$=-\alpha b^2\,(1+\sin\theta)\cos\theta\,.\,4+c\,\alpha\,b\cos\theta\,(e^{\alpha y}+e^{-\alpha y})$$

$$\frac{1}{\varrho}\frac{dp}{dy} = -g+\left\{-b\,\overline{e^{\alpha y}-e^{-\alpha y}}\,\sin\theta\,\alpha\left(\frac{dx}{dt}-c\right) + \alpha\,b\cos\theta\,(e^{\alpha y}+e^{-\alpha y})\left(\frac{dy}{dt}\right)\right\}$$

$$=-g-\alpha\,b\sin\theta\,\{b\,\overline{1+\sin\theta}\,(e^{2\alpha y}-e^{-2\alpha y})-c\,(e^{\alpha y}-e^{-\alpha y})\}$$

$$-\alpha b^2\cos^2\theta\,(e^{2\alpha y}-e^{-2\alpha y})$$

$$=-g-\alpha b^2\sin\theta\,(e^{2\alpha y}-e^{-2\alpha y})-\alpha b^2\,(e^{2\alpha y}-e^{-2\alpha y})$$

$$+\alpha\,b\,c\sin\theta\,(e^{\alpha y}-e^{-\alpha y})$$

$$\therefore\quad \frac{dp}{\varrho} = -\alpha b^2\,(4\cos\theta+2\sin 2\theta)\,dx+c\,\alpha\,b\cos\theta\,(e^{\alpha y}+e^{-\alpha y})\,dx$$

$$-g\,dy-\alpha b^2\,(1+\sin\theta)\,(e^{2\alpha y}-e^{-2\alpha y})\,dy+\alpha\,b\,c\,(e^{\alpha y}-e^{-\alpha y})\sin\theta\,dy$$

$$=c\,b\,d\,.\,(e^{\alpha y}+e^{-\alpha y})\sin\theta-g\,dy$$

$$\frac{p}{\varrho}=b\,c\,(e^{\alpha y}+e^{-\alpha y})\sin\theta-g\,y+\mathrm{P}$$

where P is a function of z.

Applying the formula $\frac{d\phi xz}{dx}+\frac{d\phi xz}{dz}\cdot\frac{dz}{dx}=0$, which we proved, art. 9, we obtain

$$0=b\,c\,\alpha\,(e^{\alpha z}+e^{-\alpha z})\cos\theta + (-g+b\,c\,\alpha\sin\theta\,(e^{\alpha z}-e^{-\alpha z}))\frac{dz}{dx}$$

and from the value

$$z=h+\frac{b}{\alpha c}(e^{\alpha z}-e^{-\alpha z})\,(1+\sin\theta),$$

we obtain

$$\frac{dz}{dx}=\frac{b}{c}(e^{\alpha z}-e^{-\alpha z})\cos\theta+\frac{b}{c}(e^{\alpha z}+e^{-\alpha z})\,(1+\sin\theta)$$

$$=\frac{\frac{b}{c}(e^{\alpha z}-e^{-\alpha z})\cos\theta}{1-\frac{b}{c}(e^{\alpha z}+e^{-\alpha z})\,(1+\sin\theta)}.$$

By substituting this value in the above equation, it becomes

$$b\,c\,\alpha\,(e^{\alpha z}+e^{-\alpha z})\cos\theta+\frac{\frac{b}{c}(e^{\alpha z}-e^{-\alpha z})\cos\theta}{1-\frac{b}{c}(e^{\alpha z}+e^{-\alpha z})\,(1+\sin\theta)}\,(-g+b\,c\,\alpha\sin\theta\,\overline{e^{\alpha z}-e^{-\alpha z}})=0$$

$$b\,c\,\alpha\cos\theta\left\{1-\frac{b}{c}\overline{1+\sin\theta}\,(e^{\alpha z}+e^{-\alpha z})\right\}(e^{\alpha z}+e^{-\alpha z})$$

$$=\frac{b}{c}(e^{\alpha z}-e^{-\alpha z})\cos\theta\left\{g-b\,c\,\alpha\sin\theta\,(e^{\alpha z}-e^{-\alpha z})\right\}$$

$$\therefore\quad c^2\alpha\,(e^{\alpha z}+e^{-\alpha z})-c\,b\,\alpha\,\overline{1+\sin\theta}\,(e^{\alpha z}+e^{-\alpha z})^2$$

$$=g\,(e^{\alpha z}-e^{-\alpha z})-b\,c\,\alpha\sin\theta\,(e^{\alpha z}-e^{-\alpha z})^2$$

$$c^2\alpha\,(e^{\alpha h}+e^{-\alpha h})=g\,(e^{\alpha h}-e^{-\alpha h})+b\,c\,\alpha\,(e^{\alpha h}-e^{-\alpha h})^2$$

$$c^2\alpha=g\,\frac{e^{\alpha h}-e^{-\alpha h}}{e^{\alpha h}+e^{-\alpha h}}+b\,c\,\alpha\cdot\frac{(e^{\alpha h}-e^{-\alpha h})^2}{e^{\alpha h}+e^{-\alpha h}}.$$

If e be the semi-elevation, $e=\frac{b}{c\,\alpha}(e^{\alpha h}-e^{-\alpha h})$

$$\therefore\quad c^2\alpha\left(1-e\,\alpha\,\frac{e^{\alpha h}-e^{-\alpha h}}{e^{\alpha h}+e^{-\alpha h}}\right)=g\,\frac{e^{\alpha h}-e^{-\alpha h}}{e^{\alpha h}+e^{-\alpha h}}$$

$$c^2=\frac{g}{\alpha}\cdot\frac{e^{\alpha h}-e^{-\alpha h}}{e^{\alpha h}+e^{-\alpha h}}\div\left(1-e\,\alpha\,\frac{e^{\alpha h}-e^{-\alpha h}}{e^{\alpha h}+e^{-\alpha h}}\right).$$

43. Thus we have obtained the velocity of transmission in a very simple form. As we have before pointed out, the function which properly represents the velocity of vibration is discontinuous; the *value* of this function at points of the fluid which are in actual motion cannot be correct, unless it lead to the conclusion that the motion ceases as soon as the wave has traversed a space equal to its length. This conclusion amounts to the fact which it is incumbent on us to prove, that all the superadded fluid, and no more, will pass onwards in the time occupied by the vibration.

Let Q be the volume of superadded fluid; R the volume of the portion which is carried forwards during the time of vibration.

$$z=h+\frac{b}{c\,\alpha}(e^{\alpha z}-e^{-\alpha z})(1+\sin\theta)$$

$$Q=\int_{-\frac{\lambda}{4}}^{\frac{3\lambda}{4}}dx\,\frac{b}{c\,\alpha}(e^{\alpha z}-e^{-\alpha z})(1+\sin\theta)$$

$$e^{\alpha z}=e^{\alpha h+\alpha m+\alpha m\sin\theta}$$

$$=e^{\alpha\,\overline{h+m}}\,\overline{1+\alpha\,m\sin\theta}$$

$$\therefore\quad Q=\int dx\,\frac{b}{c\,\alpha}\left\{e^{\alpha h}-e^{-\alpha h}+\alpha\,m\sin\theta\,(e^{\alpha h}+e^{-\alpha h})\right\}\overline{1+\sin\theta}$$

$$=\frac{b}{c\,\alpha}\int dx\left\{e^{\alpha h}-e^{-\alpha h}+\frac{\alpha\,m}{2}(e^{\alpha h}+e^{-\alpha h})+\ldots\right\}$$

$$=\frac{b\lambda}{c\,\alpha}\left\{e^{\alpha h}-e^{-\alpha h}+\frac{\alpha\,m}{2}(e^{\alpha h}+e^{-\alpha h})+\ldots\right\}$$

$$\mathrm{R} = b \int\int dy\, dt\, (e^{\alpha y} + e^{-\alpha y})(1 + \sin\theta)$$

$$= \frac{b}{\alpha} \int dt\, (e^{\alpha z} - e^{-\alpha z})(1 + \sin\theta)\,.$$

Now x and t enter always together, and the limits $x=p$ $x=q$ correspond to $t = \frac{q}{c} + r\,,\ \frac{p}{c} + r$; therefore the two are equal, *i. e.* $\mathrm{Q} = \mathrm{R}$ *accurately.*

Consequently we deduce the following important conclusion:

"That all the fluid which was elevated above the statical level has passed on with the wave, and no other fluid with it."

Combining this with the fact that the velocity of the particles in all vertical sections through points at the nearer extremity of the wave is zero, and that no particle is moving backwards, it follows *that a wave will not be followed by another resulting from the displacement which it has caused amongst the particles.* Hence the functions u, v, z are discontinuous ones, having the values assigned to them when the function $x - ct$ lies between $-\frac{\lambda}{4}$ and $+\frac{3\lambda}{4}$, but being zero in all other cases.

It is not necessary to determine what function this is: for, although the derived functions of certain orders might differ widely from those deduced from the equivalent formulæ, yet the first derived functions of which alone we make use, representing either a velocity or a force, cannot be very different in the wave as expressed by its *value* from that expressed by the form; they cannot, in fact, differ from those obtained in ordinary undulations, provided we conceive all the fluid in motion in the direction of the axis of x. We may conceive, however, a difference of pressure, and hence probably arises the fact that p is not in the present case a complete differential.

44. The formulæ we have deduced above will give us the velocity of transmission, provided we know the length of the wave.

Mr Russell has not yet published his results as to the length of the wave, nor have I been able at present to deduce it to my satisfaction in terms of the depth of the fluid and the height of the wave. It would appear at first sight probable that the higher the wave for a given depth of fluid, the longer it will be. Yet, from Mr Russell's conclusions, it appears the contrary to this is the case. I have appended a few results deduced from the value of λ, which Mr Russell has favoured me with as the result of his present researches. I must, however, take the liberty of observing, that, although the first part of it is just what I should have expected, the latter part is by no means so. Had I chosen a formula empirically, as that which the nature of the case seems to require, it would have differed from Russell's; thus, for $2\pi h - 2x$, I should have conceived $2\pi h + \pi x$, or $2\pi\left(h + \frac{x}{2}\right)$, as the probable form. But I forbear any further remarks until I shall have considered the subject with an especial view to the determination of the length of the wave.

45. The following results are all that I have *attempted* to obtain; consequently they may be considered as a fair specimen of the whole.

The account of the experiments will be found in the Report of the British Association, vol. vi.

Wave 1.

$$\left.\begin{aligned} h+2e&=4.50 \\ 2e&=\ .53 \end{aligned}\right\} \quad \therefore \quad h=3.97:$$

and

$$\frac{e^{\alpha h}-e^{-\alpha h}}{e^{\alpha h}+e^{-\alpha h}}=.78.$$

Hence

$$c^2=\frac{32.2\ (.78)}{2\pi}\ \frac{(2\pi h-4e)}{12}\div\{1-e\ (2\pi-4e)\ (.78)\}$$

$$=\frac{32.2}{12}h\ (.9575)\div\left\{1-\frac{e}{h}\ \frac{.78}{.9575}\right\}$$

$$=\frac{(32.2)\ (3.97)\ (.9575)}{12}\div\left\{1-\frac{(.53)\ (.39)}{(3.97)\ (.9575)}\right\}$$

$$=\frac{7.785}{1-\dfrac{(.53)\ (.39)}{3.97\ (.9575)}}$$

$$=\frac{7.785}{.94563}$$

$$c=2.8693=\frac{1}{.34853}$$

therefore, giving in twelve seconds 35 feet instead of 40.

Wave 27.

$$h=1,\ 2e=.30$$

$$\alpha=\frac{2\pi}{2\pi h-2e}=\frac{1}{h-\dfrac{.15}{3.1416}}$$

$$=\frac{3.1416}{3.1416-(.15)}=\frac{3.1416}{2.9916}$$

$$=1.05$$

$$\frac{e^{2\alpha h}-1}{e^{2\alpha h}+1}=\frac{7.1666}{9.1666}=.7818$$

$$e\,\alpha\,.\,\frac{e^{2\alpha h}-1}{e^{2\alpha h}+1}$$

$$=.15\ (1.05)\ \frac{7.1666}{9.1666}=.12314$$

$$\therefore \quad c^2 = \frac{\frac{32.2}{12}.7818}{.87686}$$

$$= \frac{32.2\,(.6515)}{8.7686} = 2.3924$$

$$\therefore \quad c = 1.547$$

hence, in eleven seconds, 17 feet ought to have been described instead of 20.

Wave 53.

$$h = 7.04$$

$$2e = .89$$

$$\alpha = \frac{1}{7.04}$$

$$\frac{c^2}{7.04}\left\{1 - \frac{.89}{14.08}\,.7616\right\} = \frac{32.2}{12}\,.7616$$

$$c^2 \cdot \frac{.952}{7.04} = \frac{32.2}{12}\,.7616$$

$$c^2 = \frac{(32.2)\,(.7616)\,(7.04)}{12\,(.952)}$$

$$= \frac{16.1)\,(.1904)\,(2.35)}{.476}$$

$$= \frac{(16.1)\,(.0476)\,(2.35)}{.119}$$

$$= 16 \text{ nearly}$$

$$c = 4$$

hence, in nine seconds, 36 feet ought to be described, whilst the result of observation is 40.

Wave in Mr Russell's memoir, entitled "Researches in Hydrodynamics," Trans. Royal Soc. Edin. vol. xiv.

$$h = 13, \quad 2e = 2.2$$

$$\lambda = 2\pi h - 4.4$$

$$c^2 = \frac{(32.2)\,.12}{2\pi}\,(2\pi h - 4.4)\,\mathrm{P} \div \left(1 - \frac{2\pi\,.\,(1.1)}{2\pi h - 4.4}\,\mathrm{P}\right)$$

$$2\alpha h = \frac{4\pi h}{2\pi h - 4.4} = \frac{2}{1 - \frac{2.2}{\pi h}}$$

$$= \frac{2}{1 - \frac{2.2}{(3.1416)\,(13)}} = 2.1061$$

$$\mathrm{P} = \frac{e^{2\alpha} - 1}{e^{2\alpha} + 1} = \frac{8.2155 - 1}{8.2155 + 1} = \frac{7.2155}{9.2155} = .78297$$

$$c^2 = \frac{(32.2)\,12\,(12.3032)\,(.78297)}{.928703}$$

in feet
$$c^2 = \frac{(32.2)\,(12.3032)\,(.78297)}{12\,(.928703)}$$

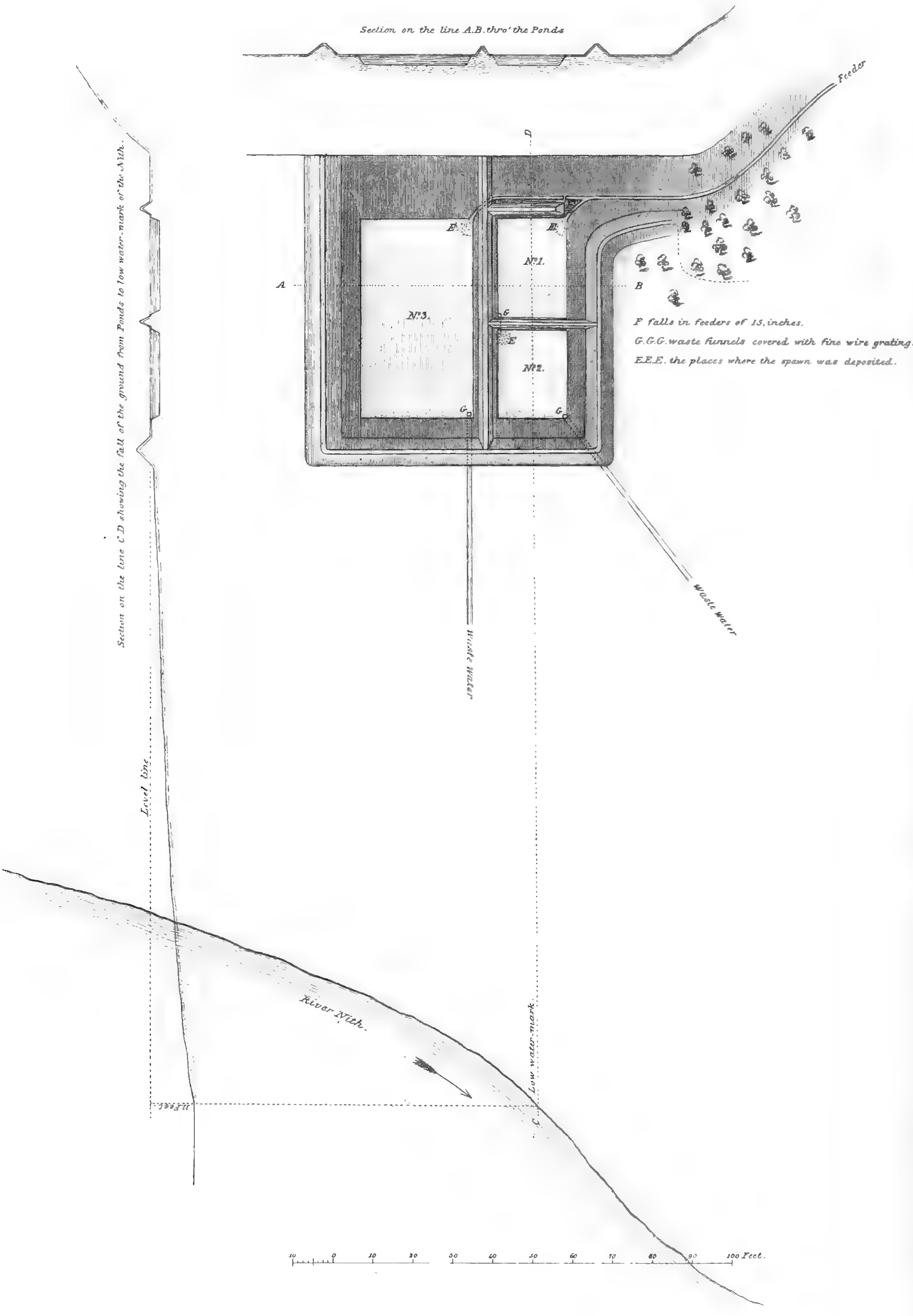
$$c = 5.275$$

remarkably near the truth.

These results, however, are of considerable importance, from the circumstance that they all err in defect; and it appears that the error is nearly of the same magnitude in all, as compared with the whole space, viz. about one-eighth. Whether this can be accounted for by any slight error in the formulæ, or in the value of λ, I leave it to subsequent investigations to determine.

EDINBURGH, *March* 12. 1839.

TO ILLUSTRATE M^R. SHAW'S EXPERIMENTS ON THE OVA OF THE SALMON.



1.

Day before hatching.

2.

One day old.

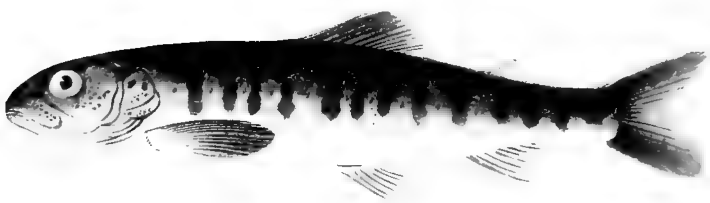
3.

Two months old.

4.

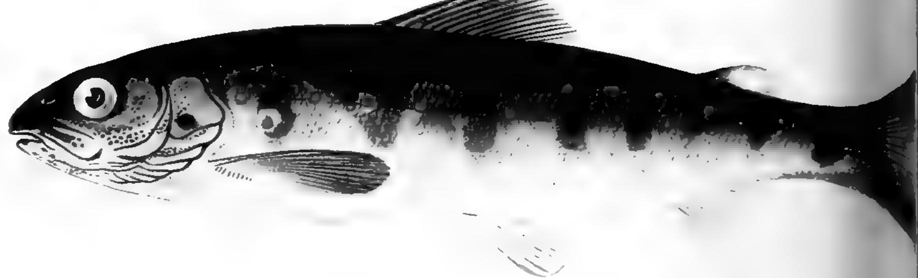
Four months old.

5.



Six months old.

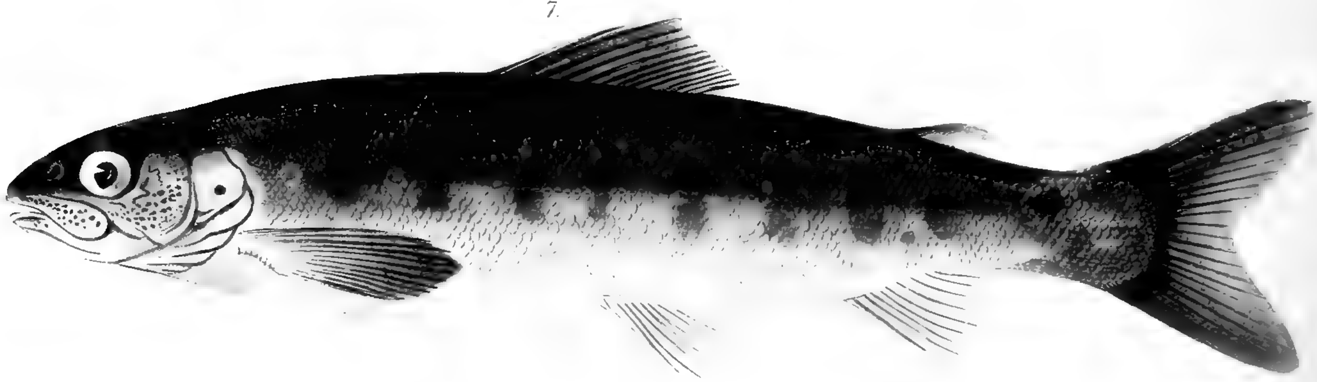
6.



Twelve months old.

The above are Parr produced from the ova of Salmon.

7.



Eighteen months old

Nos 7 & 8, bore the same aspect at the same age.

8.

Converted Parr or Smolt,— Two years old.

XXVII.—*Account of Experimental Observations on the Development and Growth of Salmon-Fry, from the exclusion of the Ova to the age of two years.* By Mr JOHN SHAW, *Drumlanrig.* Communicated by JAMES WILSON, Esq. F.R.S.E.

(Read 16th December 1839.)

"Experience once recognised as the fountain of all our knowledge of nature, it follows that in the study of nature and its laws, we ought at once to make up our minds to dismiss as idle prejudices, or at least suspend as premature, any preconceived notions of what might or ought to be the order of nature in any proposed case, and content ourselves with observing, as a plain matter of fact, what is."—Sir John Herschel's *Discourse on the Study of Natural Philosophy.*

That the facts which I communicate regarding the natural history of the salmon in its earlier stages, may not appear altogether undeserving of consideration, I may premise that my remarks have not proceeded from hasty or imperfect observation, but from the experience of many years sedulously devoted to the subject, the whole of my life, with the exception of a few seasons, having been spent on the banks of streams where salmon are in the habit of depositing their spawn, and where of course the parr is likewise abundant. My opportunities of observation have thus been as ample, as my efforts have been unremitting and laborious, to discover the true history of this invaluable species. I shall here present a brief abstract of my earlier proceedings in relation to the subject.*

I had long been of opinion, in opposition to the sentiments entertained by the majority of authors, that the fish commonly called *parr*, was the natural produce of the salmon, and that all recorded attempts to trace the history of the latter fish were fanciful in their nature, and delusive in their results. To enable me to watch the progressive growth of parr, I caught seven of these small fishes on the 11th of July 1833, and placed them in a pond supplied by a stream of

* My first paper, entitled "An Account of some Experiments and Observations on the Parr, and on the Ova of the Salmon, proving the Parr to be the young of the Salmon," was published in the *Edinburgh New Philosophical Journal* for July 1836, vol. xxi. p. 99. My second paper, under the title of "Experiments on the Development and Growth of the Fry of the Salmon, from the exclusion of the Ovum to the age of six months," was read before the Royal Society of Edinburgh on the 18th December 1837, and was published in the *Edinburgh New Philosophical Journal* for January 1838, vol. xxiv. p. 165. My third and concluding communication, which the Society now honours by its reception, contains an account of the continuance and confirmation of these experiments, with an introductory reference to the papers above named.

wholesome water. There they continued to thrive remarkably well, and were seen catching flies and other insects, and sporting on the surface in perfect health. In the month of April following (1834), they began to assume a different aspect from that which they exhibited when first put into the pond, and this change was evident enough even while they continued swimming at large in the water; but wishing to examine them more particularly, and at the same time to convince others of the fact of their having changed their external character, I caught them with a casting-net, on the 17th May 1834, and satisfied every individual present that they had assumed the usual appearance of what are called *salmon smolts* or *fry*. They were now of a fine deep blue upon the back, with a delicate silvery appearance on the sides, and the abdomen white; these silvery scales came easily off upon the hand. A circumstance occurred about the first week of May, which it may be proper to mention, as illustrating in some manner what may be deemed the migratory instinct of these fishes. They seemed to me at this time to be decreasing in numbers, and I found, on examination, that some had leapt altogether out of the pond, and were lying dead at a short distance from its edge.

In March 1835, I again took twelve parrs from the river of a larger size, that is, about six inches long; they then bore the perpendicular bars, and other usual characters of that fish. These I also transferred to a pond prepared for the purpose, and, by the end of April, they too assumed the characters of the salmon-fry,—the bars becoming overlayed by the new silvery scales, which parrs of two years old invariably assume before departing towards the sea. From these experiments I had no doubt that the larger parrs observable in rivers in autumn, winter, and early spring, were in reality the actual salmon-fry advancing to *the conclusion of their second year*, and that the smaller summer parrs (called in Dumfriesshire May parrs), were the same species, but younger as individuals, and *only entering upon* their second year. This, then, I conceived to be the detection of the main error of preceding observers, who had uniformly alleged that salmon-fry attain a size of six or eight inches in as many weeks, and after the lapse of this brief period take their departure to the sea. It is the rapidity with which the two year old parr assumes the aspect of the salmon-fry that has led to this false conclusion, and superficial or hasty observers, taking cognizance, 1*st*, of the hatching of the ova in early spring, and, 2*dly*, of the sea-ward migration of smolts soon afterwards, have imagined these two facts to take place in immediate or speedy succession. I may now mention what actually becomes of these young fishes for some weeks after they are hatched.

That the fish in question should not be found in the river in an earlier state than that in which it is named the May or summer parr, had long appeared to me to be an extraordinary and perplexing circumstance. I therefore made a minute examination of the streams where the old salmon had spawned the preceding winter, and I there found in vast numbers a very small but active fish, which

I concluded to be the young parr, or samlet of the season. To prove the fact, I scooped up with a gauze-net two or three dozen of them, on the 15th of May 1834. They measured about an inch in length; their heads were large in proportion to their bodies, and the latter tapered off toward the tail, in the form of a wedge. The small transverse bars, characteristic of the parr, were already distinctly marked. I placed them in two ponds, each provided with a run of water, where they throve well. In the course of the succeeding May (1835), that is, when they were more than a year old, and had been twelve months in my possession, I took a few of them from the pond for the purpose of examination. They had increased to the length of 3½ inches, on an average, and it is important to remark, that they corresponded in every respect with the parr of the same age which occurred in the river; but neither as yet indicated any approach to the silvery aspect of the smolt. Being satisfied, however, from the result of my former experiments on the parr, that they would ultimately assume that silvery aspect, I continued to detain them in the pond, and, accordingly, in May 1836, they were transmuted into smolts or salmon-fry, commonly so called. At this time they measured 6½ inches in length, their colour on the back a beautiful deep blue, the sides bright and silvery, the dorsal, caudal, and especially the pectoral fins, tipt with black, the abdomen, ventral, and anal fins, white. The undoubted smolts of *the river* were at this time descending sea-wards, and the most careful comparison of these with those in my possession did not elicit the slightest difference between the two. Mine had completed their second year, and is it likely that those in the river which so identically resembled them, were only a few weeks old?

The minute but active fish above alluded to, is at that early period to be no where found except in those streams (or their immediate vicinity) in which the old salmon had deposited their spawn during the preceding winter. Early in April 1835, I discovered them in one of these streams, but so young and weak, owing to their very recent emergence from the spawning-bed, as to be unable to struggle with the current where it flowed with any strength or rapidity. They therefore betook themselves to the gentler eddies, and frequently into the small hollows produced in the shingle by the hoofs of horses which had passed the ford. In these comparatively quiet places, and covered by a slight current of a few inches in depth, they continued with their little tails in constant motion, till such time as my near approach was perceived, when they immediately darted beneath the stones. They remain with these habits, and in the situations just mentioned, during the months of April, May, and even June; but as they increase in size and strength, they scatter themselves all over the shallower parts of the river, especially wherever the bottom is composed of fine gravel. They continue, in truth, comparatively unobserved throughout the whole of the first summer, being seldom taken by the angler during that season. But when the two-year-olds have disappeared (as smolts) in spring, these smaller fishes, now entering their second year, become bolder and more apparent, and now constitute the May and summer parr of anglers. But their timid habits during the first few months of their existence, and their consequent concealment in the shingle, greatly screen them from observation during that period, and have led to the erroneous belief, that the silvery smolts were the actual produce of the season, and were only a few weeks old. It certainly seems singular that it should never have occurred to any intelligent angler to inquire what had become of the older generation of parr, that is of the comparatively large individuals which he might have captured late in autumn and in earliest spring, but none of which he can detect after the departure of the so-called smolts. If the two are not identical, how does it happen that the one so constantly disappears simultaneously with the other? Yet no one alleges that he has ever seen parr, *as such*, performing their migration towards the sea. They cannot do so, because they have been previously converted into smolts.

I shall here allude briefly to three different occasions on which I have had an opportunity of witnessing the first migration of smolts or converted parr, that is, their descent in small shoals towards the sea. The first of these was in the first week of May 1831. I was able deliberately to inspect them as the several shoals arrived behind the sluices of a salmon cruive, and while they yet remained in the water, and were swimming in a particular direction, indistinct transverse lateral bars might still be seen, but as they changed their position, these became as it were lost in the silvery lustre. I also examined many of them in the hand, and could there also, by holding them at a certain angle in relation to the eye, produce the barred appearance, but when the fish were held with their broad side directly opposed to view, the character alluded to could not be seen. Its actual existence, however, could be easily proved by removing the deciduous silvery scales, when the barred markings became apparent, and, of course, continued so to whatever light exposed. My next opportunity occurred on the 3d of May 1833. The appearance was exactly the same as that which I have just described. They passed down the river in small family groups or shoals of from forty to sixty and upwards, their rate of progression being about two miles an hour. The caution which they exercised in descending the several rapids they met with in the course of their journey was very amusing. They no sooner came within the influence of any rapid current than they in an instant turned their heads up the stream, and would again and again permit themselves to be carried to the very brink, and as often retreat upwards, till at length one or two, bolder than the others, permitted themselves to be carried over the current, when the entire flock, one by one, disappeared, and then, so soon as they had reached comparatively still water, they again turned their heads towards the sea, and resumed their journey. The third opportunity to which I shall here refer occurred in May 1836, at which time, as I have stated, I compared a few of the descending smolts with those which (having been two years in my possession as parr) had, in the confine-ment of the pond, assumed the corresponding silvery aspect of the salmon-fry. The river during this month being remarkably low, I was thus enabled to ascertain more accurately the time during which they continued to migrate, which I found to be nearly throughout the whole of the month, but more especially in the course of the second week, in which the shoals were both larger, and more frequent in their successive arrivals. Their external aspect was the same as that of the former shoals, and the average length, as usual, from six to seven inches.

Having thus traced the progress of the parr from an inch in length, through its several stages up to the period of migration, I shall now detail my various experiments on the ova of the salmon, undertaken with a view to prove the identity of these two fish. On the 10th of January 1836, I observed a female salmon of considerable size (about 16 lb.), and two males, of at least 25 lb., engaged in depositing their spawn. The spot which they had selected for that purpose was a little apart from some other salmon which were engaged in the same process, and rather nearer the side, although still in pretty deep water. The two males kept up an incessant conflict during the whole of the day, for possession of the female, and, in the course of their struggles, frequently drove each other almost ashore, and were repeatedly on the surface displaying their dorsal fins, and lashing the water with their tails. Being satisfied that these were real salmon, there being at least ten brace of that fish engaged in the same process on the stream at the time, I took the opportunity of securing as much of the ova as I could possibly obtain. This I did three days after it was deposited, the males and female still occasionally frequenting the bed. The method by which I obtained the eggs was by using a thin canvass bag, stitched on a slight frame formed of small rod iron, in fashion of a large square landing-net, one person holding this bag a few inches farther down the stream than where the ova were deposited, and another with a spade digging up the gravel, the current carrying the eggs into the bag, while the greater portion of the gravel was left behind. Having thus obtained a sufficient quantity of the ova for my purpose, I placed them in gravel under a stream of water where I could have a convenient opportunity of watching their progress. The stream was pure spring water. On the 26th February, that is, forty-eight days after being deposited, I found on close inspection that they had some appearance of animation, from a very minute streak of blood which appeared to traverse for a short distance the interior of the egg, originating near two small dark spots not larger at that time than the point of a pin. These two dark spots, however, ultimately turned out to be the eyes of the embryo fish, which was distinctly seen resting against the interior surface of the egg a few days previous to its exclusion. On the 8th of April, which makes ninety days imbedded in the gravel, I found on examination that they were excluded from the egg, which was not the case a day or two previous. The temperature of the water at the time was 43°, the temperature of the water in the river 45°, and the temperature of the atmosphere 39°. On its first exclusion, the little fish has a very singular appearance. The head is large in proportion to the body, which is exceedingly small, and measures about *five-eighths of an inch* in length, of a pale blue or peach-blossom colour. But the most singular part of the fish is the conical bag-like appendage which adheres by its base to the abdomen. This bag is about two-eighths of an inch in length, of a beautiful transparent red, very much resembling a light red currant, and in consequence of its colour, may be seen at the bottom of the water when the fish itself can with difficulty be perceived. The body also presents another singular appearance, namely, a fin or fringe, resembling that of the tail of the tadpole, which runs from the dorsal and anal fins to the termination of the tail, and is slightly indented. This little fish does not leave the gravel immediately after its exclusion from the egg, but remains for several weeks beneath it with the bag attached, and containing a supply of nourishment, on the same principle, no doubt, as the umbilical vessel is known to nourish other embryo animals. By the end of fifty days, or the 30th May, the bag contracted and disappeared. The fin or tadpole-like fringe also disappeared by dividing itself into the dorsal, adipose, and anal fins, all of which then became perfectly developed. The little transverse bars, which for a period of two years (as I have already shewn) characterize it as the parr, also made their appearance. Thus, from the 10th January till the end of May, a period of upwards of 140 days was required to perfect this little fish, which even then measured little more than one inch in length, and corresponded in all respects with those on which I had formerly experimented, as well as with such as existed at that same time in great numbers in the natural streams.

Although I was myself satisfied by the preceding facts that parr and salmon fry were thus identical in kind, and differed only in respect to age, I was informed that my inferences were objected to, in as far as there was not sufficient evidence that the spawn experimented on was actually that of salmon, seeing that the same streams were accessible to other species of the genus. I therefore felt it incumbent on me to supply this desired link in the chain of evidence, and I accordingly repeated my experiments on ova which I *saw excluded*, which, in fact, I forced the salmon to exclude, in the manner after mentioned, preserving at the same time the skins of the parent fish, for the satisfaction of the curious or sceptical.

Before proceeding to make additional experiments, it was necessary to lay my experimental basins dry, not only for the purpose of removing the young salmon of the preceding season's produce, but also to enable me to fit them up on such a principle as would exclude any possibility of confusion either from the overflowing of the ponds themselves, or from the flooding of the river Nith, on the banks of which they are situate. The plan on which these ponds are constructed is shewn on Plate XXI. Every precaution was used not only to exclude error, but to place the young fry in circumstances as nearly resembling the state of nature as was consistent with their preservation.

The ponds, which are three in number, are two feet deep, and thickly embedded with gravel, while they are at the same time supplied with a small stream of spring-water in which the larvæ of insects abound. Pond No. 1 is 25 feet in length by 18 in breadth, and is fed by the stream, which debouches into it at the fall F. Pond No. 2 is 22 feet in length by 18 in breadth, and is fed from pond No. 1 at G, where the communication is carefully grated with wire. Pond No. 3 is 50 feet in length by 30 in breadth, and is fed by the stream at F, having no communication with either of the other ponds. The waste water from pond No. 1 is conducted into pond No. 2, through a square wooden pipe covered at the mouth with a wire-grating, the bars of which are about one-eighth of an inch apart. The waste water from pond No. 2 is conveyed under ground to the distance of 20 feet in a square wooden pipe grated in the same manner as the former. The waste water from pond No. 3 passes down a square wooden pipe 2 feet deep covered at the top with wire-gauze, and is conveyed under ground in a small covered drain to the distance of 20 feet from the pond. The water of the whole is then left to find its way to the river.

To prevent any communication arising from an accidental overflow of the ponds themselves, I raised embankments upon the intersecting walks of 2 feet in height, so that the several families of fish which the ponds contain can have no access, direct or indirect, to each other. Where the rivulet is divided for the purpose of supplying the several ponds, I have formed an artificial fall in each stream, of a construction to prevent the fish from ascending one stream and descending another. Finally, where the water discharges itself from the ponds, the channels are so secured by wire-grating that it is as impossible for the young fish to escape as for any other fish to have access to them. The whole occupies an area of nearly 80 feet square.

My experimental basins being thus prepared, my next object was to secure the fish, the progeny of which were to form the subject of experiment. With the view, therefore, of securing two salmon, male and female, while in the very act of continuing their kind, I provided myself with an iron hoop 5 feet in diameter, on which I fixed a net of a pretty large mesh, so constructed as to form a bag 9 feet in length by 5 in width. I then attached the hoop and net to the end of a pole 9 feet long, thus forming a landing-net on a large scale. The weight of the net with its iron hoop being upwards of 7 lb., it instantly sunk to the bottom on being thrown into the water.

Being thus prepared with all the means of carrying my experiment into practice, I proceeded to the river Nith on the 4th January 1837, and readily discovered a pair of adult salmon engaged in depositing their spawn. They were in a situation easily accessible, the water being of such a depth as to admit of my net being employed with certain success. Before proceeding to take the fish, I formed a small trench in the shingle by the edge of the stream, through which I directed a small stream of water from the river 2 inches deep. At the end of this trench, I placed an earthenware basin of considerable size, for the purpose of ultimately receiving the ova. I then, at one and the same instant, enclosed both the fish in the hoop, allowing them to find their way into the bag of the net by the aid of the stream. In capturing these fish, I considered myself fortunate in securing them by one cast of the net, for, in conducting the experiment of artificial impregnation, it appeared to me to be very desirable that the male should be taken, with the female of his own selection, at the very moment when they were mutually engaged in the continuance of their species. To take a female from one part of the stream and a male from another, might not have given the same chance of a successful issue to the experiment. Having drawn the fish ashore, I placed the female, while still alive, in the trench, and pressed from her body a quantity of ova. I then placed the male in the same situation, pressing from his body a quantity of milt, which, passing down the stream, thoroughly impregnated the ova. I then transferred the spawn to the basin, and deposited it in a stream connected with a pond previously formed for its reception, which, however, I have not considered it necessary to represent in the accompanying plan. The temperature of this stream was 39°, of the river from which the salmon were taken 33°, and of the atmosphere 36°. The skins of the parent salmon are now in my possession.

On examining the ova on the 23d of February (fifty days after impregnation), I found the embryo fish distinctly visible to the naked eye, and even exhibiting some symptoms of vitality by moving feebly in the egg. The temperature of the stream was at this time 36°, and of the atmosphere 38°. On the 28th of April (114 days after impregnation), I found the young salmon excluded from the egg, which was not the case when I visited them on the previous day. The temperature of the stream was then 44°. The ova, which for some time previous to being hatched, had been almost daily in my hand for inspection, did not appear to suffer at all from being handled. When I had occasion to inspect the ovum, I placed it in the hollow of my hand, covered with a few drops of water, where it frequently remained a considerable time without suffering any apparent injury. The embryo, however, while in this situation, shewed an increased degree of activity by repeatedly turning itself in the egg, an action probably produced by the increase of temperature arising from the warmth of the hand.

On the 24th of May (twenty-seven days after being hatched), the young fish had consumed the yolk, but in a few days afterwards the whole of this family, with the exception of one individual, were found dead at the bottom of the pond, a circumstance which has occurred more than once in the course of my experiments, arising, I apprehend, from a deposition of mud, the same result having previously taken place, when the pond had not been sufficiently imbedded with gravel.

To shew the effects of increased temperature in hastening the development of the infant fish, I may relate an experiment which I made upon a few of the same ova, from which this family proceeded. On the 20th of April (106 days after impregnation), finding the ova alluded to unhatched, and the temperature of the stream being 41°, I took four of them and placed them in a tumbler of water, covering the bottom with fine gravel, in which I imbedded the ova. I then suspended the tumbler from the top of my bed-room window, above which I placed a large earthenware jar, with a small spiggot inserted in its side, from which I easily directed a stream of pure spring water into the tumbler. The waste water was carried out at the window along a wooden channel fitted up for the purpose. As there was no fire in the bed-room, and the window facing the north, the temperature did not range very high, 47° being the average, while the average temperature of the water in the tumbler was 45°. During the night, however, the temperature would be very considerably increased, and the consequence was, the young fish in the tumbler were hatched in thirty-six hours, while those remaining in the stream did not hatch till the 28th of April, a difference of nearly seven days. At this stage the little fish are so very transparent, that their vital organs are distinctly visible, and, when placed immediately under the eye of the observer, they present a very interesting appearance. The pectoral fin is continually in rapid motion, even when the fish itself is otherwise in a state of perfect repose. They also begin to manifest an increasing desire to escape observation, a principle wisely implanted for their better security, during so feeble and helpless a condition. On the 24th of May (thirty-nine days after their birth), the fish in the tumbler were completely divested of the yolk, and the characteristic bars of the parr had become visible. At this time they measured nearly one inch in length, and appeared to be in perfect health; but fearing that after the yolk was consumed, I should be unable to supply them with appropriate food, I returned them to the pond from which I had taken them on the 20th of April, where they perished with the rest of the family.

This last experiment proves, that by placing the ova under a temporary stream of water in the house, the development of the young may be materially accelerated, while it also shews that they may be kept alive for a considerable time afterwards; at all events, until the yolk, which I presume to be their sole support at this period, is totally consumed.

The next experiment, the circumstances of which I have to relate, has been attended with more success than those which I had previously made. The process of taking the adult fish, and all the circumstances attending the impregnation, were entirely similar in this case to that already narrated.

That the pedigree of the young fish may not be called in question, I have preserved the skins of the parents. The weight of the male when taken was 16 lb., and of the female 8 lb.

The spawn was impregnated and deposited in the stream immediately below the fall, pond No. 1. E, on the 27th of January 1837; the temperature of the water in the stream being 40°, and that of the water in the river 36°. On the 21st of March (fifty-four days after impregnation), the embryo fish were visible to the naked eye. On the 7th May (101 days after impregnation), they had burst the envelope, and were to be found amongst the shingle of the stream. The temperature of the water was at this time 43°, and of the atmosphere 45°. It is this brood which I have now had an opportunity of watching continuously for a length of time, that is, for more than the entire period which was required to elapse from their exclusion from the egg, until their assumption of those characters which distinguish the undoubted salmon-fry. I therefore desire, even at the risk of repetition, to describe their progressive growth during these important and usually misconceived stages of existence.* But before doing so, I beg to be indulged in a few miscellaneous remarks.

It is indeed in no way surprising that any body of scientific men, before whom a portion of these observations on the growth of the salmon in fresh water may have been previously laid, should have been slow to express a decided opinion on the subject, more especially when the result of my experiments goes to prove facts so opposed to what has been the received opinion both of scientific and practical observers, ever since the natural history of the salmon became a subject of inquiry. I have no wish to attempt removing these opinions by the substitution of others which may be equally destitute of a correct foundation, but by the statement of facts resulting from the most careful and repeatedly verified experiments—experiments which, I believe, have been made by no other individual on the same principle for a similar purpose; for had they been so, I am persuaded the real history and economy of this valuable and interesting fish would long ere now have been more correctly understood by the community. However, should similar observations have been made, the results of which tend to support any material facts contradictory of those here stated, it would be most desirable that the scientific public should be immediately apprised of them.

It has been asserted, with some appearance of truth, in support of the old school theory, that owing to the comparatively limited range of my experimental ponds, the young salmon reared in them have not had a "supply of food sufficiently varied, or in sufficient quantity, to insure an equally rapid growth to those in the open river." This objection, I must repeat, is by no means tenable,

* I have transmitted a series of the specimens referred to, from the ovum to the smolt, and including the ordinary and transitionary state of the parr, to be exhibited when my paper is read.

as the streams and ponds in which they have existed from their birth abound with every species of insect food peculiar to the river, and, at the same time, the fishes themselves (which are certainly the best test), are in the highest possible health and condition, and correspond in every respect with those in the river. I have already stated that the young of the salmon remain in the river for the first two years after their birth, being then known under the various local denominations of parrs, pinks, fingerlings, &c. &c. However, in order to prevent any misconception of the terms employed in the course of these details, I shall adhere to the name *parr*, as being the designation by which this fish is most generally known in Scotland.

The early or late hatching of the salmon-spawn in the river is no doubt in a great measure regulated by the temperature which may prevail after its deposition. In severe winters, when the temperature of the river for many weeks barely exceeds the freezing point, the ova remain in the gravel at the bottom of the stream during that period with the living principle comparatively suspended, until the more genial temperature of the spring brings that principle into more active operation. In the course of experiments made in the beginning of 1838, I had an opportunity of observing the different effects of temperature in facilitating or retarding the development of the salmon-spawn. In ova placed in a stream of spring water, the average temperature of which was 40°, the embryo fish was visible to the naked eye by the end of the 60th day, and was hatched on the 108th day after impregnation. That which the same parent deposited the same day in the river, the average temperature of which during the eight following weeks did not exceed 33°, was not visible to the naked eye until the 90th day, and was not hatched until the 10th May, that is 131 days after impregnation. The temperature of the river, however, during the last forty days of that period, had considerably increased, and on the day on which the fishes were hatched, it had attained an elevation of 60°. Were it, then, the fact that the young salmon migrate to the sea the same season they are hatched, the effects of a mild or a rigid winter would alone regulate the period of their departure from the river. This, however, is not the fact, as the main body of the salmon-fry regularly quit our rivers about the first or second week in May, whatever may have been the temperature of the previous winter, and in this particular instance they were actually descending the river in shoals on the very day (10th May) on which that season's produce were only emerging from the ova.

Owing to the great family likeness which is known to exist amongst the young of the several species of the genus *Salmo* in their early stages, an idea has been entertained that unscientific observers are in the practice of confounding the progeny of the whole of the migratory species indiscriminately under the too general name of *Parr*. To obviate this inconvenience, and to mark the distinction of species in their earlier stages, recourse has been had to very fanciful and ill-defined attributes; and I am of opinion that in almost every instance these vague characters have been applied to individuals of the young of the real salmon, of which the characters had not been so fully developed as those of others, rather than to the young of any distinct species. With the view, therefore, of affording scientific men an opportunity of comparing the young of the *salmon trout* with that of the salmon, with which they are supposed to have been confounded, I have taken this opportunity of laying before the Royal Society a brood of the former produced by artificial impregnation, and exhibiting five successive stages, from the day on which they were hatched to the age of *nine months*, accompanied by the skins of the parent fishes. At the age of six months they bear no very marked resemblance to the young of the real salmon either in the parr or fry state, and as they advance in age and size, the resemblance becomes still slighter. However, on comparing them with the common trout, the resemblance is very striking, the general outline of the fish being much less elegant than that of the young salmon or parr, the external markings being also more peculiarly those of the trout species, so that, in the absence of the parent skins, it would be a matter of difficulty to determine to which kind of trout they actually belong. A specimen of the young common trout of this season's produce, taken from the *Clyde above the Falls*, is also exhibited; so that the young of the three species most common to this locality (and of corresponding age), viz. *Salmo salar*, *Salmo trutta*, and *Salmo fario*, may be carefully compared. The ova of the *Salmo eriox*, which is less common in these tributaries, I have not as yet had an opportunity of experimenting upon.

To resume my history of the so-called parr. Having brought the series of experiments on the ovum of the salmon, begun in January 1837, to a satisfactory conclusion, it may be gratifying to those who have taken an interest in this curious inquiry, to be put in possession of the results. I have already detailed the particulars regarding the mode practised in capturing the parent salmon, the process of fecundating the ovum artificially with the milt from the male, and the appearance it presents from that period up to the exclusion of the young fish from the capsule of the ovum, which took place on the 7th of May—101 days after impregnation. A complete series of specimens from the egg until the commencement of the third year, illustrates the following descriptive notes.

Specimens taken from the pond, when ten days old (16th May), had still a considerable portion of the vitelline bag attached to the abdomen. Specimens removed when forty-eight days old (24th June) had no recognisable bag, but the symmetry of the form was as yet but imperfectly developed. After the lapse of two months (7th July) the shape was found to be materially improved, and to exhibit in miniature much of the form and proportions of a mature fish. At the age of four months (7th September) the characteristic marks of the *parr* were clearly developed. Two months later (six months' old, 7th November) an accession both of size and strength was apparent, and on comparing the pond specimens with the parr of the river, no marked difference was perceptible. The average length at this time was three inches.

During the winter months, the general temperature of the rivers is so low, and the consequent deficiency of insect food so great, that the whole of the Scottish Salmonidæ which inhabit the fresh waters during that season, are well known to lose, rather than gain, in point of condition. The same rule holds in regard to the young salmon in the experimental ponds, although not to the same degree, they having maintained comparatively a superior condition throughout the winter to those found in the river of a corresponding age and size. The temperature of the ponds, averaging about 40° during the winter, not only keeps the young fishes which occupy them in a more active condition, but the insects themselves are also more abroad, and thus afford a convenient supply of food not to be obtained by those at that time in the river, the average temperature of which, in ordinary winters, barely exceeds 34°. I shall now refer more specially to the specimens before the Society.

No. 6 is a specimen from pond No. 1, of the age of nine months, taken in the middle of February 1838. It exhibits little or no particular accession of size or condition to that of No. 5, but may serve to shew the general appearance of the several broods of the young salmon in my possession at the age of nine months.

No. 7 is a specimen twelve months old, taken from pond No. 1, on the 10th May 1838. It is much improved in condition, as well as in external appearance, in comparison to that taken in February, and has exchanged its dusky antumnal and winter's coating for that which may be called its summer dress.* It measures about $3\frac{3}{4}$ inches in length, and is denominated, along with those of a corresponding age and size in the river, the "*May Parr*." Immediately after the migration of the two year-old parr (which the latter always affect about the beginning of May, under the name of salmon-fry), there is no other parr, besides such as have been recently hatched, to be found in the river, save those which correspond with this specimen, which is the Pink of the river Hodder, alluded to by Mr Yarrell.† As the summer advances they increase in size, and are actually the little fish which afford the angler in salmon rivers so much light amusement with the rod, during the months of August, September, and October. They remain over the

* On the approach of autumn, the whole of the Salmonidæ, resident as well as migratory, while in fresh water, acquire a dusky exterior, accompanied by a considerable increase of mucus or slime. The fins also become more muscular. However, on the return of spring, they resume their wonted beautiful colouring, and the fins, the cartilaginous portions of which are frequently damaged during the winter floods, grow up and acquire their former outline.

† "Pinks in the river Hodder, in the month of April, are rather more than three inches long, and

second winter in the river, during which period the males shed their milt, and are found continuing their kind along with the female adult salmon, although still bearing all the external markings of the parr, as I shall afterwards more particularly mention. No. 8 is a specimen eighteen months old, taken from pond No. 1, on the 14th November 1838. It measures 6 inches in length, and has now attained that stage when all the external characteristic markings of the parr are strikingly developed, and, in point of health and condition, cannot be exceeded by any taken from the river. All the males, at the age of eighteen months, of the several broods in my possession, last autumn (1838) attained a most important corroborative stage, viz. that of shewing a breeding state, by having matured the milt, which could be made to flow freely from their bodies, by the slightest pressure of the hand. The females of the same broods, however, although in equal health and condition, did not exhibit a corresponding appearance in regard to the maturing of roe. The male and female parrs in the river, of a similar age, are found respectively in precisely a corresponding state, which may surely be admitted as most important evidence in support of the fact, that all these individuals are, in truth, specifically the same.

No. 9 is a specimen two years old, taken from pond No. 1, on the 20th May 1839, *after having assumed the migratory dress.* The commencement of the change, which was perfected by the whole of the broods about the same time,* was first observable about the middle of the previous April, by the caudal, pectoral, and dorsal fins assuming a dusky margin, while, at the same time, the whole of the fish exhibited symptoms of a silvery exterior, as well as an increased elegance of form. The specimen in question, so recently a parr, exhibits a very perfect example of the salmon fry or smolt.

When the migratory change takes place in the young salmon in the ponds, a marked alteration also occurs in their habits. While in the parr state, they shew no disposition to congregate, but each individual occupies a particular station in the ponds, and should any one quit his place with the view of occupying the position already possessed by another, the intruder is at once expelled with an apparent degree of violence. But so soon as the whole brood has perfected the migratory dress, they immediately congregate into a shoal, and exhibit an anxious

are considered to be the fry of that year; at this time smolts of six inches and a half are also taken."—See Yarrell's *Supplement to British Fishes,* page 6. The *fry* of the same year, in mild winters, are only quitting the gravel in April, at which stage they measure not more than one inch.—J. S.

* One or two of each of the three broods assumed the migratory or smolt dress at the age of twelve months. This circumstance I am disposed to attribute to the high temperature of the spring-water ponds, which I have no doubt has hastened the change. I am greatly strengthened in this opinion by the fact of no instance of a similar change having occurred with individuals reared in similar ponds supplied with water from a rivulet, the temperature of which throughout the year ranges pretty nearly with that of the River Nith.

desire to effect their escape, by scouring all over the ponds, leaping and sporting, and altogether displaying a vastly increased degree of activity.

No. 10 is a specimen twenty months old, taken from pond No. 3, on the 5th January 1839. It measures 6 inches in length, and still displays all the characteristic markings of the parr.

No. 11 is a specimen two years old, also taken from pond No. 3, on the 24th May 1839. After assuming the migratory dress, it measures about $6\frac{1}{2}$ inches in length, being about the average size of the brood. I have elsewhere stated that "the circumstances attending the development and growth of the brood in pond No. 3, so exactly correspond with those of the preceding brood in pond No. 1, that their history would only be a repetition of the former. I may, however, state, that the individuals in pond No. 3 are considerably larger than those in pond No. 1, the difference, at the age of six months, amounting to an inch."* This superiority in point of size, for the first six months, of those in pond No. 3, over those reared in pond No. 1, was not, however, maintained, with the exception of two individuals, much beyond the first six months, as by the period at which they assumed the migratory dress (two years), no difference existed in regard either to size or condition.

In order to be more distinctly understood regarding the specimen next in order (No. 12), the history of which is most interesting, and highly important in establishing the identity of the parr and salmon, it will be necessary here to recur to a passage in my former communication on this subject, where I stated that "pond No. 2, was occupied by a brood of young salmon also produced by artificial impregnation, the history of which should form the subject of another paper, after I had an opportunity of verifying the experiment by repetition."† I have now repeatedly verified the experiment alluded to, and take this opportunity of giving publicity to the very extraordinary nature of the results.

The circumstance of the male parrs with the milt matured, and flowing in profusion from their bodies, being at all times found in company with the adult female salmon while depositing her spawn in the river, and the female parrs being in every instance absent, suggested the idea that the males were probably present with the female salmon at such seasons for a sexual purpose. And to demonstrate the fact, I, in January 1837, took a female salmon weighing 14 lb. from the spawning bed, from whence I also took a male parr weighing $1\frac{1}{2}$ oz., with the milt of which I impregnated a quantity of her ova, and placed it in the stream E, pond No. 2 (See Plate XXI), where, to my great astonishment, the process succeeded in every respect as it had done with that which had been impregnated by the adult male salmon, and exhibited, from the first visible appearance of the embryo fish up to their assuming the migratory dress, the utmost health and vigour. The very ex-

* *Edinburgh New Phil. Journ.* for January 1838 (vol. xxiv. p. 172, note).

† *Ibid.* same page.

traordinary results of these experiments, although made with the utmost possible care, induced me to defer giving them publicity until I had repeatedly verified the fact. I, therefore, removed this brood to another pond, apart from all other fish, where they had an abundant supply of insect food and wholesome water; and again, early in the following January (1838), I repeated the experiment by taking another female salmon, weighing 14 lb., and two male parrs from the same spawning bed (See parent specimens marked A), and impregnated two lots of her ova with the milt from the two parrs, and afterwards placed them in two different streams, inclosed in boxes open at the top, temperature 45°. The extreme severity of the weather which succeeded had, however, nearly proved fatal to the whole. On the evening of the 8th January, the day on which I took the parents from the river, the frost set in, and continued with such intensity for a succession of many weeks, that the wild fowl generally, and the wild ducks in particular, suffered severe privations, and in the course of their wanderings in search of food they unfortunately stumbled on my boxes of ova, one lot of which they wholly devoured, to the amount at least of 500. My feelings of mortification and disappointment on the discovery of this unforeseen disaster may readily be conceived. However, on examining my other box, I found there were still a few remaining, which I carefully collected, and put into a place of greater safety. The progressive growth of these, from the impregnation of the ova up to the age of eighteen months (See specimens A), has also been uniformly the same as those produced by male and female adult parents, and reared under similar circumstances.

As a further illustration of the singular economy of the salmon in their native streams, I have yet to detail another experiment or two, not less interesting than conclusive. In December last (1838) I took a female salmon from the river weighing 11 lb., and four male parrs from the same spawning bed. After impregnating four different lots of her ova, one lot to each individual parr, I placed the four parrs in a pond, where they remained until the following May, at which period they assumed the migratory dress. (See specimen No. 13.) The ova were placed in streams to which no other fish had access, and where they became mature in a similarly progressive manner to those already detailed, thus clearly demonstrating that the young salmon of eighteen months old, while yet in the parr or early state, actually perform the duties of a male parent before quitting the river.*

* As I believe it has been objected to my views, or rather practice, regarding this mode of impregnation, that the generative influence may have been in some other way effected than through the medium of the parr, I therefore took every means to prove the truthful results of my experiments by varying in some measure their conditions. Thus, in two instances, I took a portion of the ova from a female salmon, and placed them, *without impregnation*, in a stream of pure water. The result was as I anticipated:—up to the termination of the general hatching season they exhibited no appearance of vitality. The female from which one lot of ova was taken, and placed in water without impregnation, was the

While the males of the three several broods which occupy ponds No. 1, 2, and 3, continued in a breeding state, which lasted throughout the whole of the winter of 1838–39, I impregnated the ova of three adult female salmon from the river with the milt of a male taken from each of the three ponds, the whole of which ova matured. This at once removes any doubt which may have been entertained regarding the constitutional strength of individuals reared under such circumstances.

Specimen No. 12, is one of the males used in the above experiments, and is itself the produce between a male parr and female adult salmon taken from the river on the 4th January 1837, and reared in pond No. 2, as already mentioned. The result of the experiment practised with this specimen and the female salmon from the river, being of the utmost importance in establishing the identity of the species (on a principle recognised by physiologists as a law of nature), every necessary precaution to avoid error or confusion was observed. It was taken from pond No. 2 on the 5th January 1839, being then twenty months old, with the milt flowing from its body. A female adult salmon weighing twelve lb. (see parent specimen, B.) was taken at the same time from the river, in the act of spawning in the absence of the male. A quantity of her ova was impregnated in the same manner in every respect as practised in the preceding experiments, and, for the better security of the lot, the whole was placed in a wooden trough, over which a sheet of fine copper-wire gauze was fixed. The trough was then placed in a stream of water previously prepared for its reception, and the results were precisely of a corresponding nature to those already detailed, the embryo fish becoming visible after fifty-five days, and being excluded from the egg at the end of 109 days after impregnation, under a temperature of 40°.

It has been maintained by individuals whose opinions are opposed to mine on this question, that the parr is a distinct species, and that, by a forced connection between it and the female salmon, I was producing a *hybrid*. This idea at once brings the importance of the last experiment more immediately into view, from the circumstance of the male parent of specimen No. 12 being actually a parr, while No. 12 itself, the alleged hybrid, in its turn became the parent of a numerous brood. (See specimens, B.)

Were these two species, then, really distinct, it would follow that the produce would be *hybrids*, and " nature herself has provided against the confusion of different species by a conservative law, according to which all hybrids are

female with which the four parrs above alluded to were spawned. They were placed in the same stream but in a separate vessel from the four lots impregnated. The other lot was taken from the female with which the male from pond No. 3. was spawned. The unimpregnated lot was placed in the same stream with the former. The impregnated lot was placed in the stream of pond No. 3. To avoid contact the unimpregnated lots were in each case taken first, and removed to a distance.

barren:" consequently, upon this principle—a law in the economy of nature—the *parr* and salmon are really identical in species, as proved by the fact now narrated, of the young produced between them having actually the power of reproducing their kind.

Apart from these experiments, it was at one time held, that the parrs found in their native streams were hybrids, from the anomalous circumstance of the males being always found in the autumn with the milt matured, while females, of a corresponding size, could at no season be found exhibiting the least approximation to a breeding state.* However, this idea, if it ever was seriously entertained by scientific men, has now given way to the opinion " that they are a distinct species, and have no connection whatever with the migratory salmon."† Were the parr a distinct species, the result of their attendance on the female salmon would have the effect of producing universal confusion among the migratory inhabitants of rivers, from the circumstance of the male parrs in a breeding state occupying in great numbers the very centre of the salmon spawning bed, while the female salmon herself is at the same instant pouring thousands of her ova into the very spot where they are thus genially congregated.

Had these extraordinary results proceeded from a solitary experiment, there might have been some ground for believing that I was probably deceiving myself, and, consequently, misleading others,—a fear I myself at first entertained. But after such a series of experiments, made with all possible care, and uniformly ending in the same results, the fact can no longer, I conceive, admit of doubt. Having altogether within these last two years, made eight distinct experiments by artificially impregnating the ova of the salmon with the milt of a corresponding number of male parrs from the river, besides three experiments with those of eighteen-month-old parrs from the pond—each with perfect success—I trust that I have thrown some interesting light on *the breeding of parrs*,—a subject which has hitherto defeated all inquiry when sought after on the principle of their breeding among themselves as a distinct species.

* Solitary instances have occurred of large female parrs having been found in salmon rivers with the roe considerably developed, and I find, by detaining the female smolts in fresh water until the end of the third winter, that individuals are found in this comparatively mature condition. From this fact, therefore, it may be inferred, that the large parr, either male or female, of nine and ten inches in length, which are occasionally found in rivers, are the young of the salmon, which, for some natural reason, had not been prepared to migrate at the ordinary period, and had, therefore, remained for another year in the fresh water.

† Recent experiments having been made on the young of the salmon by very competent individuals, it is now admitted that they " remain *one year* in the river before they go to the sea as smolts." However, owing to these fishes having escaped the observation of those individuals during the intermediate stage, that is, from the ovum up to the length of three inches, they were actually twelve months old at the commencement of the experiments referred to by Mr Yarrell, in place of being the " fry of that year."—See Mr Yarrell's *Supplement to British Fishes*.

The fact of the young salmon propagating its kind while it is yet itself in other respects in an immature condition, is certainly an extraordinary departure from the ordinary laws of nature, so far, at least, as land animals are concerned. From certain observed facts, however, there is reason to believe that the economy of the class of fishes differs in this respect from that of land animals—a disparity which, in consequence of the medium they inhabit, has hitherto escaped the observation of the naturalist. As the young of the other migratory species do not quit the river during the first year, it is probable that they also observe a similar economy to that of their more valuable congener.

It has been generally supposed that the male salmon, during the spawning season, assists the female in forming the spawning bed. This idea is, I think, founded in error, as, during the whole course of my experience, I have never been able to detect the male taking any share whatever in the more laborious portion of these parental duties. The only part he performs, beyond the mere sexual function, consists in the unwearied vigilance which he exhibits in protecting the spawning-bed from the intrusion of rival males, all of which he assiduously endeavours to expel. The female, regardless of the occasional absence of the males during these contests, and probably satisfied with the presence of the male parrs, proceeds with her operations by throwing herself at intervals of a few minutes upon her side, and while in that position, by the rapid action of her *tail*,* she digs a receptacle in the gravel for her ova, a portion of which she deposits, and, again turning upon her side, she covers it up by a renewed action of the tail,—thus alternately digging, depositing, and covering ova, until the process is completed by the laying of the whole mass, an operation which generally occupies three or four days. In the course of these experiments, it has been ascertained that the milt of a single male parr, whose entire weight may not exceed one and a-half ounce, is capable, when confined in a small stream, of effectually impregnating all the ova of a very large female salmon. On the spawn first quitting the body of the female, it is found to be enveloped in a thin coating of viscous matter, which the action of the water does not immediately destroy, but which continues to admit of a partial adherence to the gravel at the bottom of the spawning-bed, where the ova receive the necessary fecundation of the milt, and are afterwards covered with gravel by the instinctive efforts of the female parent, in the manner above described.

How long these ova will remain excluded from the body of the female, and yet continue capable of receiving with effect the fecundating action of the milt, I have not hitherto ascertained. I have, however, made several experiments on

* I am aware it has been a matter of dispute amongst observers as to which of the two extremities of the fish is employed in the formation of the spawning-bed. However, from late opportunities of observation, which rarely occur, owing to the turbid state of the river in the spawning season, I am now satisfied that it is by the action of the caudal extremity alone that the gravel is removed.

the ova after the parent had been a considerable time dead, and removed from the river. In one particular instance, the female had been dead for nearly two hours without the vital principle of the spawn being in the least degree affected,—as, on being afterwards placed in water, and the milt of a living male poured upon it, it exhibited within the usual period the same healthy and progressing vivification, under a similar temperature, as that taken and impregnated the moment it quitted the body of the living parent. I have merely stated this fact as being in part corroborative, so far as relates to the salmon, of similar experiments made by M. Jacobi on individuals of the same genus.

The extraordinary nature of the experiments made with the parr and salmon, I have no doubt will tend to stagger the belief of many who may be disposed to admit the truth of the facts resulting from the experiments upon the adult fishes. Nevertheless, they are strictly true; and I would strongly recommend that all those interested should immediately turn their attention to a subject so curious in a zoological point of view, and so important in its bearings on the history of the most highly prized of all the species which ever sojourn in our river waters.

ILLUSTRATIVE PLATES.

Plate XXI. exhibits a plan of the Experimental Ponds, as constructed by Mr Shaw, and described at p. 553.

Plate XXII. contains representations of Parr or Salmon-Fry, in various stages from the ovum to the age of two years,—by which period the characteristic aspect of the Smolt (commonly so called) has been assumed. This Plate is lettered in such a manner as to explain itself, and therefore need not here be more particularly described.

XXVIII.—*On General Differentiation. Part I.* *By* The Rev. P. KELLAND, *M.A., F.R.SS. L. & E., F.C.P.S., late Fellow of Queens' College, Cambridge; Professor of Mathematics, &c. in the University of Edinburgh.*

(Read 2d December 1839.)

We owe to LEIBNITZ the first suggestion of Differentiation, with fractional and negative indices, but no definite notion of the theory was attained until EULER expounded it in the Petersburgh Commentaries for 1731. Still EULER wrote only a few pages on the subject, so that the theory could scarcely be said to have come into existence, until LAPLACE, in his *Théorie des Probabilités*, and FOURIER, in his *Théorie de la Propagation de la Chaleur*, shewed how general differential coefficients might be deduced by means of definite integrals, provided we assume or prove, by means of some elementary definition, that the differential coefficient of a circular or of an exponential function has a certain form. The formula given by M. FOURIER is a very simple one; and our astonishment is great, when we reflect on the time which elapsed from its announcement to the first application that was made of it. This took place in 1832, in a memoir by M. LIOUVILLE, entitled *Questions of Geometry and Mechanics resolved by a new analysis*, which memoir is followed by two others on the more immediate theory of the analysis itself. Although M. LIOUVILLE regards his analysis as a new invention, we have no doubt that the idea is due to FOURIER; but still to M. LIOUVILLE belongs the honour of moulding it into a shape capable of being made use of in the solution of problems.

M. LIOUVILLE, in this memoir, adopts a different line of proceeding, in order to deduce the differential coefficients of positive powers of x from that by which he obtains the differential coefficients of the negative powers. It is true he shews, in one or two cases, the possibility of deducing the one *directly* from the other, by means of complementary functions or constant functions of differentiation. We think, however, with Mr PEACOCK, who reviewed this memoir in the *Reports of the British Association*, that the process is far from satisfactory. Indeed, M. LIOUVILLE appears to have entertained some suspicion that it would be thought so; for we find, in the *eleventh volume of* CRELLE'S *Journal for* 1834, a short memoir by him on the theory of complementary functions, in which he corrects his previous memoir, by adopting a more enlarged definition of Γn (LEGENDRE'S or EULER'S Function). But this correction does not appear by any means perfectly to answer the desired end. We cannot find that M. LIOUVILLE has attempted to obtain the differentials of quantities without direct recourse to his complementary function; and consequently, although we acknowledge a degree of improvement on his first essay, we are far from thinking the subject free from objection. We are not aware that M. LIOUVILLE has done any thing further in the establishing of the first principles of the science. We have three other memoirs by him, two on formulæ, by which the subject may be applied, and the third on the mode of effecting a transformation of the independent variable, in none of which does he say a word about the principles.

About the end of 1837, appeared a paper in the Cambridge Mathematical Magazine, the author of which is ignorant of the greater portion of what has been written on the subject: indeed, he must be supposed to have read M. LIOUVILLE's first series of memoirs, and those only. The author professes to have translated a part of these memoirs, altering the parts against which objections had been raised. Independently of the fact, that there are proofs of the limited extent of the author's reading, the mode in which the subject is treated is such as to deserve a high degree of praise. M. LIOUVILLE had left the matter very vague as regards the determination of all differential coefficients, except those which come under a particular *form*. This vagueness is done away with by the author of this paper, who shews how to proceed in all cases of powers of the independent variable. The whole subject, too, is arranged in a logical form, commencing with a generalization of the fundamental formulæ of the science. This, as far as we know, is all that has hitherto been written on the subject, if we may except generalizations of M. FOURIER's theorems, and an arrangement of EULER's notion. The subject is indeed in its infancy, but it is to be hoped that it will rapidly grow to a full stature. We venture to express our belief, that the excessive rage for elliptic functions, which has engrossed analysts for the last ten years, will be turned, partially at least, into the channel of general analysis. That we may contribute our part towards effecting this desirable object, it is our intention to present to this Society two or three memoirs on the subject, endeavouring to place it in so simple a light, that no greater difficulty shall be experienced in appreciating the evidence on which it rests, than is attached to common algebra. We hope, too, we shall be enabled to remove the barrier, which, doubtless, has been the real obstacle to its reception, viz. the *extreme multiplication of cases*, which Mr PEACOCK complains of with justice. To explain what we mean, we may be allowed to observe that, in the present state of the science, the expression for $\frac{d^m . x^n}{dx^m}$ has four different and very dissimilar forms, depending on the signs and the relative magnitude of m and n. We shall shew, in the sequel, that one form comprehends them all; and shall thus be enabled to give to this science its proper dignity, by relieving it from the imputation of being subject to a tentative process of examining the results of each particular form, and retaining only that one which gives no absurdity as its result.

We are aware that the whole substance of an Academic memoir is usually, as far as is known, *original*, either in its results or in its mode of arriving at them. We trust, however, that we shall offer a sufficient reason for deviating from the usual custom to a slight extent, when we state that the analysis which we treat of is hardly known by name even in this country; and even where it is known, from M. GREATHEED'S paper, its value is far from appreciated. The latter circumstance is undoubtedly owing to its want of generality; for we confess that, before we were so fortunate as to discover a *general* form for all the differential coefficients, we subscribed to Mr PEACOCK'S views of the memoir in question, and agreed with him in treating it as almost or altogether erroneous.

It is right at the same time, that we should state our conviction that M. LIOUVILLE himself clearly saw, in his second memoir, the unity of his calculus; for although he appears to rest his conclusions, even in his very last memoir, on the complementary function, yet, from one phrase in the 15th volume of the *Polytechnique Journal*, we are induced to infer that he considers the fundamental form to be not merely generally true as a definition, but also as a means of operation. We had not seen either this memoir, or those in CRELLE'S Journal, until long after our own views had been settled. It is strange, however, that M. LIOUVILLE does not appear to be able to *apply* his ideas in the establishment of the theorems requisite for the foundation of an analytical calculus.

We trust that no apology will be requisite for introducing into the present memoir a considerable portion of analysis, derived from M. LIOUVILLE'S memoir, as we fear there is no other way of making the subject interesting, if indeed we can otherwise make it intelligible. We trust, too, that the nature of our additions to the theory will plead our apology, and we hope that the importance of our generalization will suffice to make some amends for the introduction of borrowed materials. Not to spend more time, then, on apology, we proceed to the definitions and first principles of the science, leaving it to the reader to discover, in many instances, what is original and what not.

SECTION I.—FUNDAMENTAL PRINCIPLES.

1. There are two ways of arriving at the fundamental formula: we shall exhibit them both; the latter is, however, merely the converse process to that contained in the former.

Since $$\frac{d \,.\, e^{mx}}{dx} = m\, e^{mx}$$

$$\frac{d^2 . e^{mx}}{dx^2} = m^2 e^{mx}$$

$$\frac{d^{-1} . e^{mx}}{dx^{-1}} = \int e^{mx} dx = m^{-1} e^{mx}$$

&c. &c.

$$\therefore \quad \frac{d^\mu e^{mx}}{dx^\mu} = m^\mu e^{mx}$$

whenever μ is a positive or negative integer. Let us retain the general form, viz. that $\frac{d^\mu e^{mx}}{dx^\mu} = f(m) e^{mx}$; then, if μ be a fraction $\frac{p}{q}$, we must have $\frac{d^{\frac{p}{q}}}{dx^{\frac{p}{q}}} \cdot \frac{d^{\frac{p}{q}}}{dx^{\frac{p}{q}}} \cdot \frac{\cdots}{\cdots} e^{mx}$, the symbol of differentiation being repeated q times, equal to $\overline{f(m)}|^q e^{mx}$.

But the result is also $m^p e^{mx}$, since, by repeating $\frac{d^{\frac{p}{q}}}{dx^{\frac{p}{q}}}$ q times, we get $\frac{d^p}{dx^p}$:

hence
$$\overline{f(m)}|^q = m^p$$

$$\therefore \quad \overline{f(m)}| = m^{\frac{p}{q}}$$

$$\frac{d^{\frac{p}{q}} . e^{mx}}{dx^{\frac{p}{q}}} = m^{\frac{p}{q}} . e^{mx};$$

and by the usual extension, we obtain $\frac{d^\mu . e^{mx}}{dx^\mu} = m^\mu . e^{mx}$ whatever μ may be.

Hence, if any function of x can be expanded in terms of e^{mx} &c., we can find its general differential coefficient.

2. From the above proposition, we deduce the two following:

$$\frac{d^\mu}{dx^\mu} \cdot \frac{d^\nu}{dx^\nu} . u = \frac{d^{\mu+\nu}}{dx^{\mu+\nu}} . u$$

$$\frac{d^\mu}{dx^\mu} . (u+v) = \frac{d^\mu u}{dx^\mu} + \frac{d^\mu v}{dx^\mu}.$$

The former of these propositions was requisite for the demonstration above: we may, however, assume the result of the last article as the definition. In this case we can prove the formula before us thus:

Let
$$u = \Sigma A e^{mx}$$

$$\therefore \quad \frac{d^\nu u}{dx^\nu} = \Sigma m^\nu A e^{mx}$$

and
$$\frac{d^\mu}{dx^\mu} \cdot \frac{d^\nu}{dx^\nu} . u = \Sigma m^{\mu+\nu} A e^{mx}$$

$$= \frac{d^{\mu+\nu} . u}{dx^{\mu+\nu}}.$$

For the second proposition, let $\quad v=\Sigma\,B\,e^{n'x}$

$$\therefore \quad u+v=\Sigma\,A\,e^{mx}+\Sigma\,B\,e^{nx}$$

and
$$\frac{d^\mu(u+v)}{dx^\mu}=\Sigma\,m^\mu\,A\,e^{mx}+\Sigma\,n^\mu\,B\,e^{nx}$$

$$=\frac{d^\mu u}{dx^\mu}+\frac{d^\mu v}{dx^\mu}.$$

It must be observed that all functions are supposed to be susceptible of expansion in the form $\Sigma\,A\,e^{mx}$: with the correctness of this assumption we have no concern, provided we limit our results if the assumption is not correct.

3. The demonstrations above exhibited are due to M. LIOUVILLE. The following, which are deduced by reversing M. LIOUVILLE'S process, are Mr GREATHEED'S.

A general differential coefficient is defined to be such a function that the following equations are satisfied by it:

$$\frac{d^\mu(u+v)}{dx^\mu}=\frac{d^\mu u}{dx^\mu}+\frac{d^\mu v}{dx^\mu}\;\ldots\;(1)$$

$$\frac{d^\mu}{dx^\mu}\cdot\frac{d^\nu}{dx^\nu}\,u=\frac{d^{\mu+\nu}u}{dx^{\mu+\nu}}\;\ldots\ldots\ldots\;(2)$$

Now let
$$y=e^{mx},$$

$$\therefore \quad \frac{dy}{dx}=m\,e^{mx}$$

and
$$\frac{d^\mu}{dx^\mu}\cdot\frac{dy}{dx}=m\frac{d^\mu e^{mx}}{dx^\mu}$$

or
$$\frac{d^{\mu+1}y}{dx^{\mu+1}}=m\,\frac{d^\mu y}{dx^\mu}\qquad\text{(by 2)}$$

or
$$\frac{d}{dx}\cdot\frac{d^\mu y}{dx^\mu}=m\,\frac{d^\mu y}{dx^\mu}\qquad\text{(do.)}$$

hence, by integration,

$$\frac{d^\mu y}{dx^\mu}=C\,e^{mx}.$$

Now if
$$\mu=\frac{p}{q},\quad \frac{d^{\frac{p}{q}}\,d^{\frac{p}{q}}\ldots\text{ to } q \text{ terms}}{dx^{\frac{p}{q}}\,dx^{\frac{p}{q}}\ldots\text{ to } q\text{ terms}}\cdot y=C^q\,e^{mx}$$

or
$$\cdot\;m^p=C^q,\;C=m^{\frac{p}{q}}.$$

$$\therefore \quad \frac{d^\mu y}{dx^\mu}=m^{\frac{p}{q}}\,e^{mx}.$$

If there be any doubt about the correctness of the assumption that

$$\frac{d^\mu \, m y}{d x^\mu} = \frac{m \, d^\mu y}{d x^\mu},$$

it may be removed by means of equation (1), from which this equation is deducible, by putting u, $2u$ ... successively for v.

By means of this fundamental formula, or definition (as it may be termed) of the calculus, we are enabled to obtain the differential coefficients of different functions of x.

4. *To find the differential coefficient of* $\frac{1}{x}$.

Since
$$\frac{1}{x} = \int_0^\infty \epsilon^{-\alpha x} \, d\alpha$$

$$\frac{d^\mu \cdot \frac{1}{x}}{d x^\mu} = \int_0^\infty \frac{d^\mu}{d x^\mu} \epsilon^{-\alpha x} \, d\alpha,$$

$$= \int_0^\infty (-\alpha)^\mu \, \epsilon^{-\alpha x} \, d\alpha;$$

a form which is readily put into a numerical shape, when x is positive, in the following manner.

Let
$$\alpha x = \theta$$

$$\therefore \quad d\alpha = \frac{d\theta}{x};$$

and
$$\frac{d^\mu \cdot \frac{1}{x}}{d x^\mu} = \int_0^\infty \left(-\frac{\theta}{x}\right)^\mu \epsilon^{-\theta} \, \frac{d\theta}{x}$$

$$= \frac{(-1)^\mu}{x^{1+\mu}} \int_0^\infty \theta^\mu \, \epsilon^{-\theta} \, d\theta$$

$$= \frac{(-1)^\mu}{x^{1+\mu}} \, \Gamma(1+\mu)$$

since $\Gamma(1+\mu) = \int_0^\infty \theta^\mu \epsilon^{-\theta} \, d\theta$, where Γ is LEGENDRE's function gamma.

But if x be negative, we have

$$\frac{1}{x} = -\int_0^\infty \epsilon^{\alpha x} \, d\alpha$$

$$\therefore \quad \frac{d^\mu \frac{1}{x}}{d x^\mu} = -\int_0^\infty \alpha^\mu \, \epsilon^{\alpha x} \, d\alpha$$

Let
$$\alpha x = -\theta$$

$$\therefore \quad d\alpha = -\frac{d\theta}{x}$$

and $$\frac{d^{\mu}\frac{1}{x}}{dx^{\mu}}=-\int_{0}^{\infty}\left(-\frac{\theta}{x}\right)^{\mu}e^{-\theta}\left(-\frac{d\theta}{x}\right)$$

$$=\frac{(-1)^{\mu}}{x^{1+\mu}}\int_{0}^{\infty}\theta^{\mu}e^{-\theta}d\theta$$

$$=\frac{(-1)^{\mu}\Gamma(1+\mu)}{x^{1+\mu}}$$

the same form as before.

5. *To find the differential coefficient of* $\frac{1}{x^{n}}$.

If x be positive, $\int_{0}^{\infty}e^{-\alpha x}\alpha^{n-1}d\alpha$ becomes, by the substitution of θ for αx,

$$\int_{0}^{\infty}e^{-\theta}\left(\frac{\theta}{x}\right)^{n-1}\frac{d\theta}{x}=\frac{1}{x^{n}}\Gamma n:$$

if x be negative, $\int_{0}^{\infty}e^{+\alpha x}\alpha^{n-1}d\alpha$ becomes

$$\int_{0}^{\infty}e^{-\theta}\left(\frac{\theta}{x}\right)^{n-1}(-1)^{n}\frac{d\theta}{x}=\frac{(-1)^{n}}{x^{n}}\Gamma n.$$

In the first case, if we differentiate with respect to x, to the index μ we get

$$\frac{d^{\mu}.\frac{1}{x^{n}}}{dx^{\mu}}=\frac{1}{\Gamma n}\int_{0}^{\infty}(-\alpha)^{\mu}e^{-\alpha x}\alpha^{n-1}d\alpha$$

$$=\frac{1}{\Gamma n}(-1)^{\mu}\int_{0}^{\infty}e^{-\alpha x}\alpha^{n+\mu-1}d\alpha$$

$$=\frac{1}{\Gamma n}(-1)^{\mu}\cdot\frac{\Gamma(n+\mu)}{x^{n+\mu}}.$$

In the second case, if we differentiate with respect to x as before, we get

$$\frac{d^{\mu}.\frac{1}{x^{n}}}{dx^{\mu}}=\frac{1}{(-1)^{n}\Gamma n}\int_{0}^{\infty}\alpha^{n+\mu-1}e^{\alpha x}d\alpha$$

$$=\frac{(-1)^{n+\mu}}{(-1)^{n}\Gamma n}\frac{\Gamma(n+\mu)}{x^{n+\mu}}$$

$$=\frac{(-1)^{\mu}\Gamma(n+\mu)}{\Gamma n\ x^{n+\mu}},\text{ as before.}$$

Now, LEGENDRE considered the function Γ as restricted to positive quantities; consequently when either $n+\mu$ is negative, or n negative, the above expression appears to fail, and others quite different have been shewn to apply to these cases. If we have no means of remedying this defect, the system is utterly useless as a branch of analysis, and we should do well to attempt to establish another in its place; but, fortunately, there is no occasion for this, as we shall shew that the above form is applicable in all cases, and may therefore be used as the *general* form, without any reference whatever to the specific values of n and μ. As this proposition is of the utmost importance, we shall give more than one proof of it. We must, however, make a slight change in the notation, in order to avoid the charge of misapplying LEGENDRE's functions. We shall, therefore, write $\overline{|\,p}$ instead of $\Gamma(p)$ or Γp, lengthening the top of the Γ, and assuming the function $\overline{|\,p}$ to be such, that, like LEGENDRE's function, it satisfies the condition $\overline{|\,1+p}=p\,\overline{|\,p}$. By this hypothesis, Γ and $\overline{|\quad}$ become coincident, whenever the quantities under them are positive.

We suppose, then, that the general expression for

$$\frac{d^{\mu}\frac{1}{x^{n}}}{dx^{\mu}} \text{ is } \frac{(-1)^{\mu}\,\overline{|\,n+\mu}}{\overline{|\,n}\;x^{n+\mu}}$$

whether n or $n+\mu$ be positive or negative.

This we call formula (I).

6. II. *To find the differential coefficient of x^{n} where n is greater than μ.*

The result will be

$$\frac{d^{\mu}.\,x^{n}}{dx^{\mu}}=\frac{1}{\overline{|\,-n}}\,(-1)^{\mu}\,\frac{\overline{|\,\mu-n}}{x^{-n+\mu}}.$$

Now,

$$\overline{|\,1-(n-\mu)}=-(n-\mu)\,\overline{|\,-(n-\mu)}$$

$$\overline{|\,2-(n-\mu)}=+(1-n-\mu)\,\overline{|\,1-(n-\mu)}$$

$$\dots\dots=\dots\dots$$

$$\overline{|\,r-(n-\mu)}=+(r-1-n-\mu)\,\overline{|\,(r-1)-(n-\mu)}$$

therefore, by multiplication,

$$\overline{|\,r-(n-\mu)}=(-1)^{r}(n-\mu)(n-\mu-1)\dots(n-\mu-\overline{r-1})\times\overline{|\,-(n-\mu)}$$

r being the integer next greater than $n-\mu$.

In the same manner, $\overline{|\,-n}$ may be reduced by the formula

$$\overline{|\,s-n}=(-1)^{s}\,n\,(n-1)\dots(n-\overline{s-1})\,\overline{|\,-n}$$

where s is the integer next greater than n.

Hence, by division,

$$\overline{|\,-(n-\mu)}=(-1)^{r}\,\frac{\overline{|\,r-(n-\mu)}}{(n-\mu)\,(n-\mu-1)\dots(n-\mu-r+1)}$$

$$\overline{|\,-n}=(-1)^{s}\,\frac{\overline{|\,s-n}}{n\,(n-1)\dots(n-s+1)}$$

$$\therefore\quad\frac{\overline{|\,-(n-\mu)}}{\overline{|\,-n}}=(-1)^{r-s}\,\frac{\overline{|\,r-(n-\mu)}}{\overline{|\,s-n}}\cdot\frac{n\,(n-1)\dots(n-s+1)}{(n-\mu)\dots(n-\mu-r+1)}$$

$$=(-1)^{r-s}\,\frac{\overline{|\,r-(n-\mu)}}{\overline{|\,s-n}}\cdot\frac{\overline{|\,n+1}}{\overline{|\,n-s+1}}\cdot\frac{\overline{|\,n-\mu-r+1}}{\overline{|\,n-\mu+1}}.$$

Now, $\quad \Gamma(p)\,.\,\Gamma(1-p)=\dfrac{\pi}{\sin p\pi}$ $\quad$ when p is a fraction less than 1.

$$\therefore \qquad \Gamma(r-(n-\mu))\;\Gamma(1+n-\mu-r)=\frac{\pi}{\sin(r-n+\mu)\pi}$$

$$\Gamma(s-n)\;\Gamma(1-s+n)=\frac{\pi}{\sin(s-n)\pi}$$

hence
$$\frac{\Gamma(-(n-\mu))}{\Gamma(-n)}=(-1)^{r-s}\,.\,\frac{\Gamma(n+1)}{\Gamma(n-\mu+1)}\,.\,\frac{\sin(s-n)\pi}{\sin(r-n+\mu)\pi}$$

$$=\frac{\Gamma(n+1)}{\Gamma(n-\mu+1)}\,.\,\frac{(-1)^{-s+1}\,.\cos s\pi\sin n\pi}{(-1)^{-r+1}\,.\cos r\pi\sin(n-\mu)\pi}$$

$$=\frac{\Gamma(n+1)}{\Gamma(n-\mu+1)}\,.\,\frac{\sin n\pi}{\sin(n-\mu)\pi}$$

provided that n and $n-\mu$ are both fractions.

In this case, then, $\quad \dfrac{d^\mu x^n}{dx^\mu}=(-1)^\mu\,\dfrac{\Gamma(n+1)}{\Gamma(n-\mu+1)}\,.\,\dfrac{\sin n\pi}{\sin(n-\mu)\pi}\,x^{n-\mu}$.

But if one of the quantities be an integer, we must proceed differently.

(*a*) Let n be an integer:

then
$$\Gamma(n-n)=(-1)^n\,n(n-1)\ldots1\,.\,\Gamma(-n)$$
$$=(-1)^n\,\Gamma(n+1)\,\Gamma(-n)$$

$$\therefore \qquad \frac{\Gamma(\mu-n)}{\Gamma(-n)}=\frac{(-1)^r\,\dfrac{\Gamma(r-(n-\mu))\,.\,\Gamma(n-\mu-r+1)}{\Gamma(n-\mu+1)}}{(-1)^n\,\dfrac{\Gamma(n-n)}{\Gamma(n+1)}}$$

$$=\frac{(-1)^r}{(-1)^n}\,.\,\frac{\pi}{\sin(r-n+\mu)\pi}\;\frac{\Gamma(n+1)}{\Gamma(n-\mu+1)}\,.\,\frac{1}{\Gamma(n-n)}$$

$$=(-1)^{n+1}\,.\,\frac{\pi}{\sin(n-\mu)\pi}\,.\,\frac{\Gamma(n+1)}{\Gamma(n-\mu+1)}\,.\,\frac{1}{\Gamma(n-n)}$$

$$=0,\;\because\;\Gamma(n-n)=\infty$$

hence the formula above will give the result, whether it be correct in form or not.

(*b*) Let $n-\mu$ be an integer; then it follows from the last case that $\dfrac{\Gamma(\mu-n)}{\Gamma(-n)}=\infty$,

so that the formula above gives the correct result in this case also.

(*c*) Let both n and $n-\mu$ be integers;

then
$$\Gamma(n-n)=(-1)^n\,\Gamma(n+1)\,\Gamma(-n)$$
$$\Gamma(n-\mu-(n-\mu))=(-1)^{n-\mu}\,\Gamma(n-\mu+1)\,\Gamma(\mu-n)$$

hence
$$\frac{\overline{/\mu-n}}{\overline{/-n}}=\frac{\overline{/n-\mu-(n-\mu)}}{\overline{/n-n}}(-1)^{\mu}\frac{\overline{/n+1}}{\overline{/n-\mu+1}}$$

and
$$\frac{d^{\mu}x^{n}}{dx^{\mu}}=\frac{\overline{/n+1}}{\overline{/n-\mu+1}}\cdot x^{n-\mu}$$

and the above formula is not correct.

We can, however, render it applicable by adopting the following mode of proceeding, viz. by treating the formula $(-1)^{\mu}\cdot\frac{\sin n\pi}{\sin(n-\mu)\pi}$ as *general* with respect to n, but not general with respect to μ; and consequently, writing $\sin n\pi\cos\mu\pi$ for $\sin\overline{n-\mu}\,.\,\pi$, when μ is an integer as well as n. It must be borne in mind that our present transformations are not intended for the purpose of obtaining formulæ general in their nature and form, but are merely formulæ of calculation. We wish, in fact, to shew that the fundamental formula itself includes all others, and consequently that, in all cases of general operation, it can be adopted without error.

Bearing in mind the restriction imposed by the last case, we are able to make use of the following formula in our calculations whether n and $n-\mu$ are integers or fractions.

$$\therefore\quad\frac{d^{\mu}x^{n}}{dx^{\mu}}=(-1)^{\mu}\cdot\frac{\overline{/n+1}}{\overline{/n-\mu+1}}\cdot\frac{\sin n\pi}{\sin(n-\mu)\pi}\cdot x^{n-\mu}$$

7. III. Next let n be negative, but less than μ; then shall we obtain from the fundamental formula

$$\frac{d^{\mu}x^{n}}{dx^{\mu}}=\frac{(-1)^{\mu}}{\overline{/-n}}\frac{\overline{/\mu-n}}{x^{\mu-n}}.$$

Now we have already shewn that

$$\overline{/-n}=\frac{(-1)^{s}\,\overline{/s-n}}{n(n-1)\ldots(n-s+1)}$$

$$=(-1)^{s}\frac{\overline{/s-n}\;\overline{/1+n-s}}{\overline{/n+1}}$$

$$=(-1)^{s}\cdot\frac{\pi}{\sin(s-n)\pi}\cdot\frac{1}{\overline{/n+1}}$$

$$=-\frac{\pi}{\sin n\pi}\cdot\frac{1}{\overline{/n+1}}$$

$$\therefore\quad\frac{d^{\mu}x^{n}}{dx^{\mu}}=(-1)^{\mu+1}\cdot\frac{\sin n\pi}{\pi}\cdot\frac{\overline{/n+1}\;\overline{/\mu-n}}{x^{\mu-n}}.$$

8. IV. Lastly, if the form be

$$\frac{d^{\mu}x^{-n}}{dx^{\mu}}=\mathrm{P}x^{-(n+\mu)}$$

where $n+\mu$ is a negative quantity,

$$\frac{d^{\mu} x^{-n}}{d x^{\mu}}=(-1)^{\mu} \cdot \frac{\overline{/n+\mu}}{/n} x^{-(n+\mu)}$$

and since $n+\mu$ is negative, let it equal $-m$;

$$\therefore \quad \overline{/-m}=(-1)^{t} \frac{\overline{/t-m}}{m(m-1)\ldots(m-t+1)}$$

(t being the integer next greater than m)

$$=(-1)^{t} \frac{\overline{/t-m}}{\overline{/m+1}} \cdot \overline{/m-t+1}$$

$$=(-1)^{t} \cdot \frac{1}{\overline{/m+1}} \frac{\pi}{\sin(t-m)\pi}$$

$$=-\frac{\pi}{\sin m\pi} \frac{1}{\overline{/m+1}}$$

or

$$\frac{d^{\mu} x^{-n}}{d x^{\mu}}=(-1)^{\mu+1} \frac{\pi}{\sin m\pi} \frac{1}{\overline{/n}\ \overline{/m+1}} x^{-(n+\mu)}$$

$$=(-1)^{\mu+1} \frac{\pi}{\sin(-\overline{n+\mu}.\pi)} \frac{1}{\overline{/n}\ \overline{/1-\overline{n+\mu}}} x^{-\overline{n+\mu}}$$

If $n+\mu=0$, the result is infinite; but it is *constant;* consequently we may suppose some arbitrary constant to have been omitted in the differentiation. In other words, when μ is negative, the fundamental formula does not give the complete result. It must therefore be rectified, as in the case of ordinary integration, by the introduction of an arbitrary constant *of the form* of the integral. The complete result, then, we shall assume to be

$$\frac{d^{\mu} x^{-n}}{d x^{\mu}}=(-1)^{\mu+1} \frac{\pi}{\sin(-\overline{n+\mu}\,\pi)} \cdot \frac{1}{\overline{/n}\ \overline{/1-(n+\mu)}} \cdot \left(x^{-(n+\mu)}-a^{-(n+\mu)}\right).$$

Now $$\frac{x^{-(n+\mu)}-a^{-(n+\mu)}}{\sin(-\overline{n+\mu}\,\pi)}=\frac{0}{0} \text{ when } n+\mu=0;$$

hence we must find its value by the usual method of differentiations, and we obtain

$$\frac{-\log \frac{x}{a}}{-(\pi)\cos(n+\mu)\pi}=\frac{1}{\pi} \log . \frac{x}{a}$$

$$\therefore \quad \frac{d^{-n} x^{-n}}{d x^{-n}}=(-1)^{-\mu+1} \cdot \frac{1}{/n} \log . \frac{x}{a}.$$

We shall recur to this process in the sequel.

9. We have thus deduced from a single formula results which are applicable to any case, and we may consequently adopt this formula as our standard, and

use it without any restriction as to the value or sign of the symbols which it includes. That our formulæ are correct, we shall give abundant evidence as we proceed. At present we shall be occupied in the demonstration of two or three propositions which will be useful hereafter, and will also serve to verify our results.

Mr PEACOCK suggests the adoption of the following form of the differential coefficient.

$$\frac{d^\mu x^n}{d x^\mu} = \frac{/\overline{n+1}}{/\overline{n-\mu+1}} \cdot x^{n-\mu} .$$

In fact, if μ be a positive integer, we get

$$\frac{d^\mu x^n}{d x^\mu} = n(n-1) \ldots (n-\mu+1)\, x^{n-\mu}$$

$$= \frac{/\overline{n+1}}{/\overline{n-\mu+1}}\, x^{n-\mu} ;$$

and are thus in possession of a form very convenient to be adopted as a definition. We can easily shew that it is a correct *form* by the following process:

Since $\frac{d^\mu x^n}{d x^\mu} = \mathrm{A}\, x^{n-\mu}$ where A is a function of n and μ, if we take the $(n-\mu)$th differential coefficient of each side, we obtain

$$\frac{d^n x^n}{d x^n} = \mathrm{A} \frac{d^{n-\mu} x^{n-\mu}}{d x^{n-\mu}}$$

$$\therefore \qquad \mathrm{A} = \frac{d^n x^n}{d x^n} \div \frac{d^{n-\mu} x^{n-\mu}}{d x^{n-\mu}}$$

if therefore $\frac{d^n x^n}{d x^n}$ be abbreviated by $f(n+1)$, we obtain

$$\mathrm{A} = \frac{f(n+1)}{f(n-\mu+1)}$$

$$\therefore \qquad \frac{d^\mu x^n}{d x^\mu} = \frac{f(n+1)}{f(n--\mu+1)}\, x^{n-\mu}$$

the *form* required.

10. Mr PEACOCK is, however, not justified in assuming that $f(n+1)$ coincides with $/\overline{(n+1)}$. If we examine equation (2), we shall find that, as far as that formula is concerned, we may write

$$(-1)^n \sin n\pi\, /\overline{n+1} \quad \text{for} \quad f(n+1)$$

Taking this as true, we obtain

$$(-1)^{n-1} \sin (n-1)\pi\, /\overline{n} = f(n)$$

or

$$(-1)^n \sin n\pi\, /\overline{n} = fn$$

$$\therefore \qquad f(n+1) = (-1)^n \sin n\pi\, /\overline{n+1}$$

$$= (-1)^n \sin n\pi\, n\, /\overline{n}$$

$$= n f(n).$$

This function will consequently satisfy the requisite condition, as well as $\overline{/n+1}$.

In order to assure ourselves whether our induction is true or not, we must inquire whether it will satisfy the fundamental formula.

By adopting it, we obtain

$$\frac{d^{\mu} \cdot \frac{1}{x^n}}{dx^{\mu}} = \frac{f(-n+1)}{f(-n-\mu+1)} \, x^{\overline{-n+\mu}}$$

$$= \frac{(-1)^{-n} \sin(-n\pi) \, \overline{/-n+1}}{(-1)^{-(n+\mu)} \sin(-n-\mu)\pi \, \overline{/-n-\mu+1}} \, x^{\overline{-n+\mu}}$$

$$= (-1)^{\mu} \frac{\sin n\pi}{\sin(n+\mu)\pi} \cdot \frac{\overline{/-n+1}}{\overline{/-n-\mu+1}} \cdot \frac{1}{x^{n+\mu}}$$

Now we have shewn under formula (2) that

$$\frac{\overline{/-n'}}{\overline{/-(n'-\mu)}} = \frac{\overline{/n'-\mu+1}}{\overline{/n'+1}} \cdot \frac{\sin(n'-\mu)\pi}{\sin(n'\pi)} \; .$$

Let $n'=n-1$, and write $-\mu$ for μ;

$$\therefore \qquad \frac{\overline{/-n+1}}{\overline{/-n-\mu+1}} = \frac{\overline{/n+\mu}}{\overline{/n}} \cdot \frac{\sin \overline{n+\mu-1}\,\pi}{\sin \overline{n-1}\,\pi}$$

$$= \frac{\overline{/n+\mu}}{\overline{/n}} \cdot \frac{\sin(n+\mu)\pi}{\sin n\pi}$$

therefore, by substitution,

$$\frac{d^{\mu} \frac{1}{x^n}}{dx^{\mu}} = (-1)^{\mu} \frac{\overline{/n+\mu}}{\overline{/n}} \cdot \frac{1}{x^{n+\mu}}$$

This value of $f(n+1)$, therefore, completely verifies the general formula. We must not at the same time conclude that it is *complete*, although we may safely trust it as the *variable factor* of the complete form. It will be seen in the sequel that the other factor is infinite; but, as each function has the same factor, this produces no effect on the result.

We see in this circumstance a remarkable instance of the failure of *the principle of the permanence of equivalent forms*, as it is called. According to that principle, $f(n+1)$ should have been equal to $\overline{/n+1}$, and not to $(-1)^n \sin n\pi \, \overline{/n+1}$. There can be no doubt that such a principle has no real existence, sanctioned as it is by the names of the greatest analysts. But we forbear discussion of this matter.

11. Our next proposition is the following, analogous to that in the theory of *whole* differentials.

Let u, v be two functions of x capable of expansion in the form of exponentials; then

$$\frac{d^\mu (u\,v)}{d\,x^\mu} = v\,\frac{d^\mu u}{d\,x^\mu} + \mu\,.\,\frac{d\,v}{d\,x}\,.\,\frac{d^{\mu-1}\,u}{d\,x^{\mu-1}} + \frac{\mu\,(\mu-1)}{1\,.\,2} \times$$

$$\frac{d^2\,v}{d\,x^2}\,.\,\frac{d^{\mu-2}\,v}{d\,x^{\mu-2}} + \&c.\,\ldots$$

the order of the Binomial Theorem being observed.

Let $$u = \Sigma\,\mathrm{A}\,e^{mx}\,,\ v = \Sigma\,\mathrm{B}\,e^{nx}$$

$$\therefore \quad u\,v = \Sigma\,\mathrm{A}\,e^{mx}\,.\,\Sigma\,\mathrm{B}\,e^{nx}$$

$$= \Sigma\,\mathrm{A}\,\mathrm{B}\,e^{\overline{m+n\,x}}$$

$$\therefore \quad \frac{d^\mu (u\,v)}{d\,x^\mu} = \Sigma\,\mathrm{A}\,\mathrm{B}\,(m+n)^\mu\,e^{\overline{m+n\,x}}$$

$$= \Sigma\,(m+n)^\mu\,\mathrm{A}\,e^{mx}\,.\,\mathrm{B}\,e^{nx}$$

$$= \Sigma\,\left(m^\mu + \mu\,n\,m^{\mu-1} + \frac{\mu\,(\mu-1)}{1\,.\,2}\,n^2\,m^{\mu-2} + \ldots\right)\mathrm{A}\,e^{mx}\,\mathrm{B}\,e^{nx}$$

$$= v\,\frac{d^\mu\,u}{d\,x^\mu} + \mu\,\frac{d\,v}{d\,x}\,.\,\frac{d^{\mu-2}\,u}{d\,x^{\mu-2}} + \frac{\mu\,(\mu-1)}{1\,.\,2}\,.\,\frac{d^2\,v}{d\,x^2}\,\frac{d^{\mu-2}\,u}{d\,x^{\mu-2}} + \ldots$$

12. The application which we purpose to make of this theorem at present is the following: to deduce the formulæ (2), (3), and (4), from the fundamental formula.

Let r be an integer such that $n=r-m$, where m is a fraction less than unity;

then $$\frac{d^\mu\,x^n}{d\,x^\mu} = \frac{d^\mu\left(x^r\,.\,\dfrac{1}{x^m}\right)}{d\,x^\mu}\,.$$

Let, therefore, $u=\dfrac{1}{x^m}$, $v=x^r$; and we get from the theorem

$$\frac{d^\mu\,x^n}{d\,x^\mu} = x^r\,(-1)^\mu\,\frac{\overline{|m+\mu}}{\overline{|m}}\,\frac{1}{x^{m+\mu}} + \mu\,r\,x^{r-1}\,(-1)^{\mu-1}\,\frac{\overline{|m+\mu-1}}{\overline{|m}}\,\frac{1}{x^{m+\mu-1}} + \&c.$$

$$+ \frac{\mu\,(\mu-1)\ldots\overline{\mu-r+1}}{1\,.\,2\ \ldots\ r}\,r\,(r-1)\ldots 2\,.\,1\,(-1)^{\mu-r}\,\frac{\overline{|m+\mu-r}}{\overline{|m}}\,.\,\frac{1}{x^{m+\mu-r}}$$

$$= (-1)^\mu\,\frac{x^n}{x^\mu\,\overline{|m}}\left\{\overline{|m+\mu} - \mu\,r\,\overline{|m+\mu-1} + \ldots\right.$$

$$\left. + \mu\,(\mu-1)\,\ldots\,(\mu-r+1)\,\frac{\overline{|m+\mu-r}}{(-1)^r}\right\}.$$

$$= \frac{(-1)^\mu\,x^{n-\mu}}{\overline{|m}}\,.\,\Big\{\,(m+\mu-1)\,\ldots\,(m+\mu-r)\,\overline{|m+\mu-r}$$

$$- \frac{\mu\,r}{1}\,(m+\mu-2)\,\ldots\,(m+\mu-r)\,\overline{|m+\mu-r}$$

$$+\frac{r(r-1)}{1.2}\mu(\mu-1)(m+\mu-3)\dots(m+\mu-r)\ \Gamma\overline{m+\mu-r}+\dots\Big\}$$

$$=(-1)^\mu\frac{x^{n-\mu}}{\Gamma m}\ \Gamma\overline{m+\mu-r}.\Big\{(m+\mu-1)\dots(m+\mu-r)$$

$$-\frac{\mu r}{1}(m+\mu-2)\dots(m+\mu-r)+\dots$$

$$+(-1)^{r-1}\mu(\mu-1)\dots(\mu-r+2).r(m+\mu-r)$$

$$+(-1)^r\mu(\mu-1)\dots(\mu-r+1)\Big\}$$

$$=(-1)^\mu\frac{x^{n-\mu}}{\Gamma m}\ \Gamma\overline{\mu-n}\Big\{(\mu-n)(\mu-n+1)\dots(\mu-n+\overline{r-1})$$

$$-\frac{\mu.r}{1}(\mu-n)\dots(\mu-n+\overline{r-2})+\frac{\mu(\mu-1)}{1.2}r(r-1)\times(\mu-n)\dots(\mu-n)+\overline{r-3})-\dots\Big\}$$

$$=(-1)^\mu\frac{x^{n-\mu}}{\Gamma m}\ \Gamma\overline{\mu-n}\Big\{(-1)^r\frac{d^r}{dx^r}.\frac{1}{x^{\mu-n}}.x^{\mu-n+r}$$

$$-\frac{r}{1}\frac{d}{dx}x^\mu.\frac{d^{r-1}}{dx^{r-1}}.\frac{(-1)^{r-1}}{x^{\mu-n}}.x^{\mu-n+r-1-\overline{\mu-1}}$$

$$+\frac{r(r-1)}{1.2}\frac{d^2}{dx^2}.x^\mu\frac{d^{r-2}}{dx^{r-2}}\frac{(-1)^{r-2}}{x^{\mu-n}}.x^{\mu-n+r-2-\overline{\mu-2}}-\dots\Big\}$$

$$=(-1)^\mu.\frac{x^{n-\mu}}{\Gamma m}\ \Gamma\overline{\mu-n}\Big\{(-1)^r x^{r-n}\times$$

$$\Big[x^\mu\frac{d^r}{dx^r}.\frac{1}{x^{\mu-n}}+\frac{r}{1}\frac{d}{dx}x^\mu.\frac{d^{r-1}}{dx^{r-1}}\frac{1}{x^{\mu-n}}+\dots\Big]\Big\}$$

$$=(-1)^\mu.\frac{x^{n-\mu}}{\Gamma m}\ \Gamma\overline{\mu-n}(-1)^r x^{r-n}\frac{d^r}{dx^r}\Big(x^\mu\frac{1}{x^{\mu-n}}\Big)$$

$$=(-1)^\mu\frac{x^{n-\mu}}{\Gamma m}\ \Gamma\overline{\mu-n}(-1)^r x^{r-n}\frac{d^r.x^n}{dx^r}$$

$$=(-1)^\mu\frac{x^{n-\mu}}{\Gamma m}\ \Gamma\overline{\mu-n}(-1)^r x^{r-n}\frac{n(n-1)\dots(n-r+1)}{x^{r-n}}$$

$$=(-1)^\mu\frac{x^{n-\mu}}{\Gamma m}\ \Gamma\overline{\mu-n}(-1)^r\frac{\Gamma\overline{n+1}}{\Gamma\overline{n-r+1}}$$

$$=(-1)^\mu\frac{x^{n-\mu}}{\Gamma m}\ \frac{\Gamma\overline{\mu-n}\ \Gamma\overline{n+1}(-1)^r}{\pi}.\sin(r-n)\pi$$

$$=(-1)^{\mu+1}\frac{\Gamma\overline{\mu-n}\ \Gamma\overline{n+1}}{x^{\mu-n}}.\frac{\sin n\pi}{\pi}$$

which coincides with (3).

We cannot help remarking, that the process adopted here is most satisfactory, as we do not take for granted even that the function $\overline{|n+1}$ satisfies the condition $\overline{|n+1} = n\,\overline{|n}$ except in those cases in which n is a positive quantity, where, of course, we have no difficulty. It, therefore, confirms our hypothesis, if indeed confirmation is requisite.

13. We have still another mode of verifying our formulæ from the theorem before us. We have seen that

$$\frac{d^{\mu} x^{n}}{d x^{\mu}} = \frac{f(n+1)}{f(n-\mu+1)} x^{n-\mu};$$

where $$f(n+1) = \frac{d^{n} x^{n}}{d x^{n}}.$$

But the theorem gives, by writing n for μ,

$$\frac{d^{n} x^{n}}{d x^{n}} = \frac{(-1)^{n}}{\overline{|m}} \left\{ \overline{|m+n} - n \,.\, r\, \overline{|m+n-1} + \ldots + (-1)^{r} n \ldots (n-r+1)\, \overline{|m+n-r} \right\}$$

but $n=r-m$, therefore $n-r+m=0$, so that the last term of this expansion is $\overline{|0}$ $\times$ a quantity. Also, $\overline{|0} \times 0 = \overline{|1}$; therefore $\overline{|0} = \infty$, consequently the last term is infinite; we may therefore neglect all the other terms compared with it, and we get

$$f(n+1) = \frac{(-1)^{n} \,.\, (-1)^{r}\, \overline{|n+1}\ \overline{|0}}{\overline{|r-n}\quad \overline{|n-r+1}}$$

$$= (-1)^{n} (-1)^{r} \frac{\sin (r-n) \pi}{\pi}\, \overline{|n+1} \,.\, \overline{|0}$$

$$= (-1)^{n+1} \frac{\sin n \pi}{\pi} \,.\, \overline{|n+1} \,.\, \overline{|0}.$$

Hence the constant factor which we omitted in $f(n+1)$ is $\dfrac{-\overline{|0}}{\pi}$, and we have now the complete value of that factor. The result completely verifies the formula (2).

14. But its effect is not confined to this particular formula.

We have generally

$$\frac{d^{n} x^{n}}{d x^{n}} = \frac{(-1)^{n+r}}{\overline{|r-n}} n (n-1) \ldots (n-r+1)\, \overline{|0}$$

$$\therefore \quad \frac{d^{n-\mu} \,.\, x^{n-\mu}}{d x^{n-\mu}} = \frac{(-1)^{n-\mu+s}}{\overline{|s-(n-\mu)}} (n-\mu) \ldots (n-\mu-s+1)\, \overline{|0}$$

if $n-\mu$ be positive;

and $$\frac{d^{n-\mu} \,.\, \dfrac{1}{x^{\mu-n}}}{d x^{n-\mu}} = \frac{(-1)^{n-\mu}\, \overline{|0}}{\overline{|\mu-n}}$$

from the fundamental formula, if $n-\mu$ be negative.

Therefore, in the latter case,

$$\frac{d^\mu x^n}{d x^\mu} = (-1)^\mu \, \overline{/\mu-n} \, . \, \frac{n\,(n-1)\ldots(n-r+1)}{\overline{/r-n}} \, \frac{(-1)^r}{x^{\mu-n}}$$

$$= (-1)^\mu \, \frac{\overline{/\mu-n} \; \overline{/n+1}}{\overline{/r-n} \; \overline{/n-r+1}} \, \frac{(-1)^r}{x^{\mu-n}}$$

$$= (-1)^{\mu+1} \, \frac{\sin n\,\pi}{\pi} \, \frac{\overline{/\mu-n} \; \overline{/n+1}}{x^{\mu-n}}$$

which coincides with formula (3).

If n and $n+\mu$ are both negative, we can obtain the results by means of the fundamental formula alone, without having recourse to the theorem before us.

15. We subjoin the fundamental formula, with its three modifications, in order to apply it to a few examples.

(1.) $$\frac{d^\mu \frac{1}{x^n}}{d x^\mu} = \frac{1}{\overline{/n}} \, (-1)^\mu \, . \, \frac{\overline{/(n+\mu)}}{x^{n+\mu}}$$ the fundamental formula.

(2.) $$\frac{d^\mu x^n}{d x^\mu} = (-1)^\mu \, . \, \frac{\overline{/n+1}}{\overline{/n-\mu+1}} \, . \, \frac{\sin n\,\pi}{\sin (n-\mu)\,\pi} \, . \, x^{n-\mu}$$

(3.) $$\frac{d^\mu x^n}{d x^\mu} = (-1)^{\mu+1} \, . \, \frac{\sin n\,\pi}{\pi} \, . \, \frac{\overline{/n+1} \, . \, \overline{/\mu-n}}{x^{\mu-n}}$$

(4.) $$\frac{d^\mu \frac{1}{x^n}}{d x^\mu} = (-1)^{\mu+1} \, . \, \frac{\pi}{\sin (-n+\mu\,\pi)} \, . \, \frac{1}{\overline{/n} \; \overline{/1-(n+\mu)}} \; x^{-(n+\mu)} \, .$$

Ex. 1. Find the differential coefficient of x^n to index 1.

Here, if we adopt the fundamental formula, we obtain

$$\frac{d\,x^{-m}}{d x} = \frac{1}{\overline{/m}} (-1) . \frac{\overline{/m+1}}{x^{1+m}}$$

and by supposing $m = -n$, this gives

$$\frac{d\,x^n}{d x} = \frac{-1}{\overline{/-n}} \, . \, \frac{\overline{/1-n}}{x^{1-n}} \, .$$

But $$\overline{/1-n} = -n \, \overline{/-n},$$

$$\therefore \quad \frac{\overline{/1-n}}{\overline{/-n}} = -n$$

and $$\frac{d\,x^n}{d x} = \frac{+n}{x^{1-n}} = n\,x^{n-1} \, .$$

Ex. 2. Find the μ^{th} differential coefficient of x^n where μ is an integer.

1. If μ be less than n, we may use formula (2), and we obtain

$$\frac{d^{\mu}x^{n}}{dx^{\mu}} = (-1)^{\mu}\frac{\overline{|n+1}}{\overline{|n-\mu+1}}\,\frac{\sin n\pi}{\sin(n-\mu)\pi}\,x^{n-\mu}.$$

But
$$\overline{|n+1} = n\,\overline{|n}$$
$$\overline{|n} = (n-1)\,\overline{|n-1}$$
$$\ldots = \ldots$$
$$\overline{|n-\mu+2} = (n-\mu+1)\,\overline{|n-\mu+1}$$
$$\therefore \quad \overline{|n+1} = n(n-1)\ldots(n-\mu+1)\,\overline{|n-\mu+1}$$

and
$$\frac{d^{\mu}x^{n}}{dx^{\mu}} = (-1)^{\mu}\,n(n-1)\ldots(n-\mu+1)\,\frac{\sin n\pi}{\sin(n-\mu)\pi}\,x^{n-\mu}$$

Now, if n be a fraction, $\dfrac{\sin n\pi}{\sin(n-\mu)\pi} = \dfrac{1}{\cos\mu\pi}$; and $\dfrac{(-1)^{\mu}}{\cos\mu\pi} = 1$:

hence
$$\frac{d^{\mu}x^{n}}{dx^{\mu}} = n(n-1)\ldots(n-\mu+1)\,x^{n-\mu}.$$

But if n be an integer,

$\because \quad \dfrac{\sin n\pi}{\sin(n-\mu)\pi} = \dfrac{0}{0}$ we obtain by the observations in art. 6.

$\dfrac{\sin n\pi}{\sin(n-\mu)\pi}$ with the limitations imposed on it equivalent to

$$\frac{\sin n\pi}{\sin n\pi\cos\mu\pi} = \frac{1}{\cos\mu\pi} \quad \text{as before;}$$

hence the result is true in all cases.

2. If μ be greater than n, we may use formula (3).
If n be a fraction,

$$\frac{d^{\mu}x^{n}}{dx^{\mu}} = (-1)^{\mu+1}\cdot\frac{\sin n\pi}{\pi}\cdot\frac{\overline{|n+1}\,\overline{|\mu-n}}{x^{\mu-n}}.$$

If n be an integer, $\sin n\pi = 0$, and $\dfrac{d^{\mu}x^{n}}{dx^{n}} = 0$.

3. If $\mu = n$, $\dfrac{d^{\mu}x^{n}}{dx^{\mu}} = n(n-1)\ldots 2.1.$

Ex. 3. To find the differential of x^{n} when n is a positive quantity and μ a negative integer.

Here we may adopt our formula (2).

$$\therefore \quad \frac{d^{-m}x^{n}}{dx^{-m}} = (-1)^{-m}\,\frac{\overline{|n+1}}{\overline{|n+m+1}}\cdot\frac{\sin n\pi}{\sin(n+m)\pi}\cdot x^{n+m}$$

and by the restrictions, if n be an integer, or actually if it be not

$$(-1)^m \frac{\sin n\,\pi}{\sin (n+m)\,\pi} = 1$$

also $$\overline{/n+m+1} = (n+m)(n+m-1)\ldots(n+1)\,\overline{/n+1}$$

$$\therefore \qquad \frac{d^{-m}\,x^n}{d\,x^{-m}} = \frac{x^{n+m}}{(n+1)(n+2)\ldots(n+m)}\,.$$

Obs. The introduction of arbitrary functions (of integration) is of course requisite to render this complete; but we shall defer the discussion of the nature of these functions to a separate section.

Ex. 4. Find the value of $\frac{d^{\frac{1}{2}}\,x^{\frac{1}{2}}}{d\,x^{\frac{1}{2}}}$.

Here we must use the second formula; and we get

$$\frac{d^{\frac{1}{2}}\,x^{\frac{1}{2}}}{d\,x^{\frac{1}{2}}} = (-1)^{\frac{1}{2}}\,\frac{\overline{/\tfrac{3}{2}}}{\overline{/1}}\;\frac{\sin \frac{1}{2}\,\pi}{\sin (n-\mu)\,\pi}\;(x^{n-\mu} - a^{n-\mu})$$

if we suppose the differential to vanish when $x=a$.

Now, $$\frac{x^{n-\mu} - a^{n-\mu}}{\sin (n-\mu)\,\pi} = \frac{1}{\pi}\log\frac{x}{a}\,.$$

$$\therefore \qquad \frac{d^{\frac{1}{2}}\,x^{\frac{1}{2}}}{d\,x^{\frac{1}{2}}} = (-1)^{\frac{1}{2}}\,\frac{\overline{/\tfrac{3}{2}}}{\pi}\log\frac{x}{a}$$

$$= \sqrt{-1}\;\tfrac{1}{2}\,\frac{\overline{/\tfrac{1}{2}}}{\pi}\log\frac{x}{a}\,.$$

But $\overline{/\tfrac{1}{2}} = \sqrt{\pi}$ by the well known formula $\overline{/r}\;\overline{/1-r} = \frac{\pi}{\sin r\,\pi}$; putting $r=\frac{1}{2}$

$$\therefore \qquad \frac{d^{\frac{1}{2}}\,x^{\frac{1}{2}}}{d\,x^{\frac{1}{2}}} = \frac{\sqrt{-1}}{2}\,\frac{1}{\sqrt{\pi}}\log\frac{x}{a}\,.$$

or $$= \frac{\sqrt{-1}}{2}\cdot\frac{1}{\sqrt{\pi}}\log x,$$

if we omit the constant.

Ex. 5. Find the value of $\frac{d^{\frac{1}{2}}\;x^{\frac{3}{2}}}{d\,x^{\frac{1}{2}}}$.

By formula (2) $$\frac{d^{\frac{1}{2}}\,x^{\frac{3}{2}}}{d\,x^{\frac{1}{2}}} = (-1)^{\frac{1}{2}}\,\frac{\overline{/\tfrac{5}{2}}}{\overline{/2}}\cdot\frac{\sin\frac{3}{2}\,\pi}{\sin\pi}\cdot x$$

$$= \infty\,.$$

Ex. 6. Find the value of $\frac{d^{\frac{3}{2}}\,x^{\frac{1}{2}}}{d\,x^{\frac{3}{2}}}$.

By formula (3) $$\frac{d^{\frac{3}{2}} x^{\frac{1}{2}}}{dx^{\frac{3}{2}}} = (-1)^{\frac{5}{2}} \frac{1}{\pi} \frac{\Gamma\frac{3}{2}\ \Gamma 1}{x}$$

$$= \frac{\sqrt{-1}}{2\sqrt{\pi}} \cdot \frac{1}{x}.$$

This example will serve us to verify the very singular and unexpected result which we obtained in Ex. 4.

For $$\frac{d^{\frac{3}{2}} x^{\frac{1}{2}}}{dx^{\frac{3}{2}}} = \frac{d\, d^{\frac{1}{2}} x^{\frac{1}{2}}}{dx\, dx^{\frac{1}{2}}} = \frac{d}{dx} \cdot \frac{\sqrt{-1}}{2\sqrt{\pi}} \log x \qquad \text{(by Ex. 4.)}$$

$$= \frac{\sqrt{-1}}{2\sqrt{\pi}} \cdot \frac{1}{x}$$ which coincides with the result above.

The verification here obtained is stronger than at first sight may appear, inasmuch as the result of Ex. 6 was obtained without having recourse to the introduction of an arbitrary constant, whereas that in Ex. 4 depended entirely on the arbitrary constant.

Ex. 7. Find the value of $\frac{d^n x^n}{dx^n}$ when n is a fraction.

By the second formula,

$$\frac{d^{\mu} x^n}{dx^{\mu}} = (-1)^n \frac{\Gamma n+1}{\Gamma 1} \cdot \frac{\sin n\pi}{\sin (n-\mu)\pi} (x^{n-\mu} - a^{n-\mu})$$

$$= (-1)^n \Gamma n+1 \cdot \frac{\sin n\pi}{\pi} \log . \frac{x}{a} .$$

Ex. 8. Find the value of $\frac{d^{\frac{1}{2}} x}{dx^{\frac{1}{2}}}$.

By the second formula,

$$\frac{d^{\frac{1}{2}} x}{dx^{\frac{1}{2}}} = (-1)^{\frac{1}{2}} \frac{\Gamma 2}{\Gamma\frac{3}{2}} \cdot \frac{\sin . \pi}{\sin \frac{1}{2}\pi} . x^{\frac{1}{2}} = 0.$$

16. This example appears to have been the first to induce men to think on the subject.

EULER, in the Petersburgh Commentaries, vol. v. for 1730, gives the following result as the basis of general differentiation.

$$d^n z^e = z^{e-n}\, dz^n \frac{\int dx\, (-\log x)^e}{\int dx\, (-\log x)^{e-n}} . :$$

and obtains from it, by putting $e=1$, $n=\frac{1}{2}$, $d^{\frac{1}{2}} z = \sqrt{\frac{z\, dz}{A}}$ . . , where A is a constant.

If A $= \infty$, this result coincides with that which we have deduced.

M. LIOUVILLE, on the other hand, makes $\frac{d^{\frac{1}{2}}x}{dx^{\frac{1}{2}}} = \mathrm{A} + \mathrm{B}x + \ldots + \mathrm{C}x^m$; by adding *any* rational integral function of x, which he calls the complementary function. In applying the analysis to differential formula, this result, if admissible, would totally stultify all our processes. We should prefer writing the result in the form which EULER has given; for then we could proceed to differentiate a second time, and obtain the differential coefficient of x, *without reference* to a complementary function, which would be desirable, otherwise the complementary function could not depend in any determinate way on n and μ. We conceive that, whatever may be the *value* of the differential coefficient, its *form* ought to be such as to resolve itself into a known function when subjected to known operations. On this account, we should think it advisable to write

$$\frac{d^{\frac{1}{2}}.\,x}{dx^{\frac{1}{2}}} = (-1)^{\frac{1}{2}} \frac{\Gamma 2}{\Gamma \frac{3}{2}} \frac{\sin \pi}{\sin \frac{1}{2}\pi} x^{\frac{1}{2}}.$$

Now, differentiating this function to the index $\frac{1}{2}$, the result is

$$\frac{dx}{dx} = (-1)^{\frac{1}{2}} (-1)^{\frac{1}{2}} \frac{\Gamma 2}{\Gamma \frac{3}{2}} \cdot \frac{\Gamma \frac{3}{2}}{\Gamma 1} \cdot \frac{\sin \frac{1}{2}\pi}{\sin 0\pi} \frac{\sin \pi}{\sin \frac{1}{2}\pi} x^{0}$$

$$= -\frac{\sin \pi}{\sin 0\pi} x^{0} = x^{0} = 1.$$

The logarithmic form of Ex. 4. cannot be here introduced, on account of the process in the first part not being a *final* process; the introduction of a constant at all going on the supposition that the differential shall vanish for some value of x, which, in the case before us, it cannot do.

17. Ex. 9. To find $\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}$. ($a$ const.)

By form (3) $$\frac{d^{\frac{1}{2}}x^n}{dx^{\frac{1}{2}}} = (-1)^{\frac{3}{2}} \frac{\sin n\pi}{\pi} \frac{\Gamma (n+1) \,.\, \Gamma (\mu - n)}{x^{\mu - n}}$$

Let $n = 0$

$$\therefore \quad \frac{d^{\frac{1}{2}}a}{dx^{\frac{1}{2}}} = a(-1)^{\frac{3}{2}} \frac{\sin 0}{\pi} \frac{\sqrt{\pi}}{x^{\frac{1}{2}}}$$

$$= \frac{a \sin 0}{\sqrt{\pi}} (-1)^{\frac{3}{2}} \frac{1}{x^{\frac{1}{2}}}.$$

SECTION II.—DIFFERENTIAL COEFFICIENTS OF FUNCTIONS OF x.

Logarithmic Functions.

18. Our first proposition must be the differential coefficients of $\log x$. Now we have already shewn in Ex. 7 that, when n is a fraction,

$$\frac{d^n x^n}{dx^n}=(-1)^n \overline{/n+1}\,.\,\frac{\sin n\pi}{\pi}\,.\log x$$

omitting the constant.

Hence, taking the differential of the $-n$th order, we obtain

$$x^n=(-1)^n \overline{/n+1}\,.\,\frac{\sin n\pi}{\pi}\;\frac{d^{-n}\log x}{dx^{-n}}$$

$$\therefore\quad \frac{d^{-n}\log x}{dx^{-n}}=\frac{(-1)^{-n}}{\overline{/n+1}}\,.\,\frac{\pi}{\sin n\pi}\,.\,x^n.$$

From the nature of the process, all the functions corresponding with constants of integration are necessarily omitted, and these may, as we shall see presently, embrace the most important part of the result.

Again, we saw in Art. 8. that

$$\frac{d^{-n}x^{-n}}{dx^{-n}}=(-1)^{-n+1}\frac{1}{\overline{/n}}\log x\,;$$

therefore, taking the nth differential, we obtain

$$\frac{d^n\log x}{dx^n}=(-1)^{(n-1)}\overline{/n}\,.\,x^{-n}.$$

The last formula includes the former; for

$$\overline{/-n}=-\frac{\pi}{\sin n\pi}\;\frac{1}{\overline{/n+1}}$$

$$\therefore\quad \frac{d^{-n}\log x}{dx^{-n}}=(-1)^{-n}\,.\,\frac{\pi}{\sin n\pi}\,.\,\frac{x^n}{\overline{/n+1}}$$

which is the result above.

19. To obtain a more general form in each case, if there be one, we must proceed in the following manner.

To find $\dfrac{d^n\log x}{dx^n}$ *generally.*

Since
$$\frac{d^n\log x}{dx^n}=\frac{d^{n+1}}{dx^{n+1}}\;\frac{d^{-1}}{dx^{-1}}\log x.$$

$$=\frac{d^{n+1}}{dx^{n+1}}\{x\log x-x\}$$

and that generally $\dfrac{d^r x}{dx^r}=\dfrac{1}{\overline{/-1}}\,(-1)^r\,.\,\dfrac{\overline{/r-1}}{x^{r-1}}$ from the fundamental formula;

we obtain $$\frac{d^{n+1} . x \log x}{d x^{n+1}} = \log x \frac{d^{n+1} . x}{d x^{n+1}} + \frac{(n+1)}{x} . \frac{d^n x}{d x^n}$$

$$- \frac{(n+1) n}{1.2} \frac{1}{x^2} \frac{d^{n-1} x}{d x^{n-1}} + \frac{(n+1) n (n-1)}{1.2.3} \frac{2}{x^3} \frac{d^{n-2} x}{d x^{n-1}}$$

$$- \text{\&c.}$$

$$= \log x \frac{1}{|\overline{-1}} (-1)^{n+1} \frac{|\overline{n}}{x^n} + \frac{n+1}{x} . \frac{1}{|\overline{-1}} (-1)^n \frac{|\overline{n-1}}{x^{n-1}}$$

$$- \frac{(n+1) n}{1.2} . \frac{1}{x^2} . \frac{1}{|\overline{-1}} (-1)^{n-1} \frac{|\overline{n-2}}{x^{n-2}} + \ldots\ldots$$

$$= \frac{(-1)^{n+1}}{|\overline{-1} . x^n} \left\{ \log x |\overline{n} - |\overline{n-1} . (n+1) \right.$$

$$\left. - \frac{(n+1) . n}{1.2} |\overline{n-2} - 1.2 . \frac{(n+1) n (n-1)}{1.2.3} |\overline{n-3} - \text{\&c.} \right\}$$

$$= \frac{(-1)^{n+1}}{|\overline{-1}\, x^n} \left\{ \log x |\overline{n} - |\overline{n} - (n+1) |\overline{n-1} - \frac{(n+1)(n)}{2} |\overline{n-2} \right.$$

$$\left. - \frac{(n+1) n (n-1)}{3} |\overline{n-3} - \frac{(n+1)(n)(n-1)(n-2)}{4} |\overline{n-4} - \text{\&c.} \ldots \right\}$$

$$\therefore \quad \frac{d^n \log x}{d x^n} = \text{above series} - \frac{(-1)^{n+1}}{|\overline{-1}} . \frac{|\overline{n}}{x^n}$$

$$= \frac{(-1)^{n+1}}{|\overline{-1}\, x^n} \left\{ (\log x - 1) |\overline{n} - (n+1) |\overline{n-1} \right.$$

$$\left. - \frac{(n+1) n}{2} |\overline{n-2} . - \frac{(n+1) n (n-1)}{3} |\overline{n-3} - \text{\&c.} \right\}$$

Now the series may be put under forms as follows:

$$|\overline{n-1} = (n-2) |\overline{n-2}$$

$$\text{\&c.} = \text{\&c.}$$

$$\therefore \quad (n+1) |\overline{n-1} + \frac{(n+1) n}{2} |\overline{n-2} + \ldots$$

$$= |\overline{n} \left(\frac{n+1}{n-1} + \frac{(n+1) n}{(n-1)(n-2)} \frac{1}{2} + \frac{(n+1) n (n-1)}{(n-1)(n-2)(n-3)} . \frac{1}{3} + \ldots \right) \text{\&c.} \ldots$$

This series is *divergent*, except when n is negative or fractional.

Now, $$\frac{d^{-r} (u v)}{d x^{-r}} = \frac{u\, d^{-r} v}{d x^{-r}} - r \frac{d u}{d x} \frac{d^{\overline{-r+1}} v}{d x^{\overline{-r+1}}} + \ldots$$

if, therefore, $$u = x^{n+1}, \quad v = x^{-(n)}$$

$$\frac{d^{-r} (u v)}{d x^{-r}} = \frac{x^{n+1} (-1)^r x^{-(n-r)}}{(n-1)(n-2) \ldots (n-r)}$$

$$- \frac{r (n+1) x^n . (-1)^{r+1} x^{-(n-r-1)}}{(n-1) \ldots (n-r-1)}$$

$$+ \frac{r(r+1) x^{n-1}}{1.2} \frac{(-1)^{r+2} x^{-(n-r-2)} (n+1) n}{(n-1) \quad \ldots \quad (n-r-2)} - \text{\&c.}$$

$$= x^{r+1} (-1)^r \left\{ \frac{1}{(n-1)\ldots(n-r)} + r \frac{(n+1)}{(n-1)\ldots(n-r-1)} \right.$$

$$\left. + \frac{r(r+1)}{1.2} \frac{(n+1)\,n}{(n-1)\ldots(n-r-2)} + \ldots \right\}$$

which gives the above series if $r=0$

$$\therefore \quad \frac{n+1}{n-1} + \frac{1}{2} \frac{(n+1)\,n}{(n-1)(n-2)} + \ldots$$

$$= \frac{1}{r} \left(\frac{d^{-r} x}{d x^{-r}} . x^{\overline{-r+1}} (-1)^r - \frac{1}{(n-1)\ldots(n-r)} \right), r=0$$

$$= \frac{1}{r} \left\{ (-1)^r x^{\overline{-r+1}} \frac{d^{-r} x}{d x^{-r}} - \frac{\sqrt{n-r}}{\sqrt{n}} \right\}, r=0$$

$$= \frac{1}{r} \left\{ \frac{\sqrt{-(1+r)}}{\sqrt{-1}} - \frac{\sqrt{n-r}}{\sqrt{n}} \right\}, r=0$$

$$= \frac{0}{0}, \text{ an illusory expression.}$$

The fact is, that the function admits of different values dependent on the value of n. We shall proceed by a process of comparison to exhibit these values.

1. If n be a positive integer, the series must equal $-1\sqrt{-1}$ in order to make the result coincide with the above particular case, and

$$-1\sqrt{-1} = \sqrt{0} = \frac{\sqrt{1}}{0} = \left(\frac{1}{r}\right), r=0:$$

hence the above numerator is unity. The same is true if n is a positive or negative fraction, in all which cases $\sqrt{-1}$ in the denominator causes the first term to vanish, and the result is

$$\frac{d^n \log x}{d x^n} = (-1)^{n-1} \sqrt{n}\; x^{-n}$$

But if n be a negative whole number (and in no other case), $\frac{\sqrt{n}}{\sqrt{-1}}$ is finite; so that in this case the first term does not vanish, and we get

$$\frac{\sqrt{n}}{\sqrt{-1}} = \frac{(-1)^{n-1}}{(-n)(-n-1)\ldots(2)}$$

$$\therefore \quad \frac{d^n \log x}{d x^n} = \frac{1}{x^n} \frac{(\log x - 1)}{(-n)(-n-1)\ldots(2)} - \frac{(-1)^{n+1}}{\sqrt{-1}\, x^n} \sqrt{n}\,.\,\mathrm{P}$$

or writing $-m$ for n:

$$\frac{d^{-m} \log x}{d x^{-m}} = x^m \frac{\log x - 1}{m(m-1)\ldots 2} + (-1)^{-m} \frac{\sqrt{-m}}{\sqrt{-1}} x^m\, \mathrm{P}$$

$$= x^m \frac{\log x - 1}{m(m-1)\ldots 2} - \frac{x^m\, \mathrm{P}}{m(m-1)\ldots 2}$$

consequently we know that $\mathrm{P} = \frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{m-1}$ in this case.

20. We may exhibit the value of P as a definite integral thus:

$$P=\frac{n+1}{n-1}+\frac{1}{2}\,\frac{(n+1)\,n}{(n-1)\,(n-2)}+\frac{1}{3}\,\frac{(n+1)\,(n)}{(n-2)\,(n-3)}+\dots$$

$$=(n+1)\,n\left\{\frac{1}{n\,(n-1)}+\frac{1}{2}\,\frac{1}{(n-1)\,(n-2)}+\dots\right\}$$

Let therefore, $$S=\frac{x^n}{n\,(n-1)}+\frac{1}{2}\,\frac{x^{n-1}}{(n-1)\,(n-2)}+\dots$$

then $$\frac{d^2\,S}{d\,x^2}=x^{n-2}+\frac{1}{2}\,x^{n-3}+\frac{1}{3}\,x^{n-4}+\dots$$

$$=x^{n-1}\left\{\frac{1}{x}+\frac{1}{2}\cdot\frac{1}{x^2}+\frac{1}{3}\cdot\frac{1}{x^3}+\dots\right\}$$

$$=x^{n-1}\log\left(1-\frac{1}{x}\right)$$

$$\therefore\qquad S=-\int^x\!\!\int^x dx\,dx\,x^{n-1}\log\left(1-\frac{1}{x}\right)$$

and $$P=-\,(n+1)\,.\,n\int^1\!\!\int^x dx\,dx\,x^{n-1}\log\left(1-\frac{1}{x}\right).$$

The limits of the integral in this case are not determined. In fact, the inferior limit will depend on the value of n. If n be negative, this limit is ∞.

21. We shall then, adopt the following value for $\frac{d^n\log x}{d\,x^n}$, viz.:

$$\frac{/\overline{n}\,.\,(-1)^{n+1}}{/\overline{-1}\,.\,x^n}\left\{\log x-(1+P)\right\};\quad\text{where}\quad P=\frac{n+1}{n-1}+\frac{(n+1)\,n}{(n-1)\,(n-2)}\cdot\frac{1}{2}+\&\text{c.}$$

Ex. 1. To find the differential to the index $\frac{1}{2}$ of $\log x$.

$$\frac{d^{\frac{1}{2}}\log x}{d\,x^{\frac{1}{2}}}=\frac{(-1)^{\frac{3}{2}}\,/\overline{\tfrac{1}{2}}}{/\overline{-1}}\,\frac{\log x}{\sqrt{x}}-(-1)^{\frac{3}{2}}\,\frac{/\overline{\tfrac{1}{2}}}{/\overline{-1}}\,\frac{(1+P)}{\sqrt{x}}.$$

Now, by what we have seen, $$\frac{1+P}{/\overline{-1}}=-1$$

$$\therefore\qquad\frac{d^{\frac{1}{2}}\log x}{d\,x^{\frac{1}{2}}}=/\overline{\tfrac{1}{2}}\,.\,(-1)^{\frac{3}{2}}\,x^{-\frac{1}{2}}$$

$$=-\frac{\sqrt{-\pi}}{\sqrt{x}}.$$

Ex. 2. To find $\frac{d^{\frac{3}{2}}\log x}{d\,x^{\frac{3}{2}}}$.

We have the choice of three methods, which we give as follows:

(1) $$\frac{d^{\frac{3}{2}}\log x}{d\,x^{\frac{3}{2}}}=(-1)^{\frac{1}{2}}\,/\overline{\tfrac{3}{2}}\,.\,x^{-\frac{3}{2}}$$

$$=\frac{1}{2}\sqrt{-\pi}\;x^{-\frac{3}{2}}\qquad\text{from the formula.}$$

(2) $$\frac{d^{\frac{3}{2}}\log x}{dx^{\frac{3}{2}}} = \frac{d}{dx}\cdot\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\log x$$

$$=\frac{1}{2}\cdot\frac{\sqrt{-\pi}}{x^{\frac{3}{2}}}$$ by differentiating the result in the last example.

(3) $$\ldots \quad = \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\cdot\frac{d}{dx}\log x$$

$$=\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\cdot\frac{1}{x}$$

$$=(-1)^{\frac{1}{2}}\frac{\Gamma\frac{3}{2}}{\Gamma 1}\frac{1}{x^{\frac{3}{2}}}$$ by the fundamental formula

$$=\sqrt{-1}\,\frac{1}{2}\sqrt{\pi}\,\frac{1}{x^{\frac{3}{2}}}$$

$$=\frac{1}{2}\sqrt{-\pi}\,\frac{1}{x^{\frac{3}{2}}}.$$

Hence all the results coincide.

Ex. 3. To find $\frac{d^{\frac{3}{2}}\,x\log x}{dx^{\frac{3}{2}}}$.

$$\frac{d^{\frac{3}{2}}\,x\,.\log x}{dx^{\frac{3}{2}}} = x\,.\,\frac{d^{\frac{3}{2}}\log x}{dx^{\frac{3}{2}}} + \frac{3}{2}\,\frac{d^{\frac{1}{2}}\log x}{dx^{\frac{1}{2}}}$$

$$=\frac{x}{2}\sqrt{-\pi}\,\frac{1}{x^{\frac{3}{2}}} - \frac{3}{2}\,\frac{\sqrt{-\pi}}{\sqrt{x}}$$

$$=-\frac{\sqrt{-\pi}}{\sqrt{x}}$$

$$=\frac{d^{\frac{1}{2}}\log x}{dx^{\frac{1}{2}}}.$$

If we desire to obtain the differential coefficients of powers or other functions of $\log x$, we have, in general, no other way of proceeding than to adopt the series for the differential coefficient of a product.

Ex. 4. To find $\frac{d^n(\log x)^2}{dx^n}$.

Here $$\frac{d^n(\log x)^2}{dx^n} = \frac{d^n\log x\,.\log x}{dx^n}$$

$$=\log x\,\frac{d^n\log x}{dx^n} + \frac{n}{x}\,\frac{d^{n-1}\log x}{dx^{n-1}} - \frac{n(n-1)}{1.2}\cdot\frac{1}{x^2}\,\frac{d^{n-2}\log x}{dx^{n-2}}$$

$$+\frac{n(n-1)(n-2)}{1.2.3}\cdot\frac{1.2}{x^3}\cdot\frac{d^{n-3}\log x}{dx^{n-3}} - \&c.$$

Let $1+P=Q_n$, and write Q_{n-1} &c. for the functions corresponding to $\frac{d^{n-1}\log x}{dx^{n-1}}$:

then $$\frac{d^n (\log x)^2}{dx^n} = \log x \, . \, (-1)^{n+1} \frac{\lceil n}{\lceil -1} \, . \, \frac{1}{x^n} (\log x - Q_n)$$

$$+ \frac{n}{x} \, . \, (-1)^n \, . \frac{\lceil n-1}{\lceil -1} \, . \, \frac{1}{x^{n-1}} (\log x - Q_{n-1})$$

$$- \frac{n(n-1)}{1.2} \, . \, \frac{1}{x^2} (-1)^{n-1} \frac{\lceil n-2}{\lceil -1} \, . \, \frac{1}{x^{n-2}} (\log x - Q_{n-2})$$

$+$ &c.

$$= (-1)^{n+1} \frac{\lceil n}{\lceil -1} \, . \, \frac{\log x}{x^n} (\log x - Q_n)$$

$$+ (-1)^n \frac{\lceil n-1}{\lceil -1} \frac{n}{x^n} (\log x - Q_{n-1})$$

$$+ (-1)^n \frac{\lceil n-2}{\lceil -1} \, . \, \frac{1}{x^n} \, \frac{n(n-1)}{1.2} (\log x - Q_{n-2})$$

$$+ (-1)^n \frac{\lceil n-3}{\lceil -1} \, \frac{1.2}{x^n} \, . \, \frac{n(n-1)(n-2)}{1.2.3} (\log x - Q_{n-3})$$

$+$ &c.

$$= (-1)^{n+1} \frac{\lceil n}{\lceil -1} \, . \, \frac{\log x}{x^n} \, . \, (\log x - Q_n)$$

$$+ (-1)^n \frac{\lceil n-1}{\lceil -1} \, . \, \frac{1}{x^n} \, . \left\{ n (\log x - Q_{n-1}) + \frac{n(n-1)}{1.2(n-2)} (\log x - Q_{n-2}) \right.$$

$$\left. + \frac{n(n-1)(n-2)}{1.2.3(n-2)(n-3)} 1.2 (\log x - Q_{n-3}) + \text{\&c.} \right\}$$

$$= (-1)^{n+1} \frac{\lceil n}{\lceil -1} \frac{\log x}{x^n} (\log x - Q_n)$$

$$+ (-1)^n \frac{\lceil n-1}{\lceil -1} \frac{1}{x^n} \cdot \left\{ \left(n + \frac{n(n-1)}{n-2} \cdot \frac{1}{2} + \frac{n(n-1)(n-2)}{(n-2)(n-3)} \cdot \frac{1}{3} + \ldots \right) \log x \right.$$

$$\left. - \left(n Q_{n-1} + \frac{n(n-1)}{n-2} \cdot \frac{1}{2} Q_{n-2} + \frac{n(n-1)(n-2)}{(n-2)(n-3)} \cdot \frac{1}{3} Q_{n-3} + \ldots \right) \right\}$$

SECTION III.—CIRCULAR FUNCTIONS.

22. *To find the differential coefficient of* $\cos mx$ *to any index* n.

Since $$\cos mx = \frac{1}{2} \left(e^{mx\sqrt{-1}} + e^{-mx\sqrt{-1}} \right)$$

$$\frac{d^n \cos mx}{dx^n} = \frac{1}{2} m^n \left\{ (\sqrt{-1})^n e^{mx\sqrt{-1}} + (-\sqrt{-1})^n e^{-mx\sqrt{-1}} \right\}$$

$$=\frac{1}{2}m^n\left\{\left(\cos\frac{4r+1}{2}n\pi+\sqrt{-1}\sin\frac{4r+1}{2}n\pi\right)e^{mx\sqrt{-1}}\right.$$

$$\left.+\left[\cos\left(2r'-\frac{1}{2}\right)n\pi+\sqrt{-1}\sin\left(2r'-\frac{1}{2}\right)n\pi\right]e^{-mx\sqrt{-1}}\right\}$$

$$=\frac{1}{2}m^n\left\{\cos\left(\overline{2r+\frac{1}{2}}\,.\,n\pi+mx\right)+\sqrt{-1}\sin\left(\overline{2r+\frac{1}{2}}\,n\pi+mx\right)\right.$$

$$\left.+\cos\left(\overline{2r'-\frac{1}{2}}\,n\pi-mx\right)+\sqrt{-1}\sin\left(\overline{2r'-\frac{1}{2}}\,n\pi-mx\right)\right\}$$

$$=m^n\left\{\cos\overline{r+r'}\,n\pi\,.\cos.\left(\overline{r-r'+\frac{1}{2}}\,n\pi+mx\right)\right.$$

$$\left.+\sqrt{-1}\sin\overline{r+r'}\,n\pi\cos\left(\overline{r-r'+\frac{1}{2}}\,.\,n\pi+mx\right)\right\}$$

$$=m^n\left\{\cos\overline{r+r'}\,n\pi+\sqrt{-1}\sin\overline{r+r'}\,.\,n\pi\right\}\cos\left(\overline{r-r'+\frac{1}{2}}\,.\,n\pi+mx\right)$$

23. *To find* $\dfrac{d^n\sin mx}{dx^n}$.

$$d^n\sin mx=\frac{m^n}{2\sqrt{-1}}\left\{(\sqrt{-1})^n e^{mx\sqrt{-1}}-(-\sqrt{-1})^n e^{-mx\sqrt{-1}}\right\}$$

$$=\frac{m^n}{2}\left\{(\sqrt{-1})^{n-1}e^{mx\sqrt{-1}}+(\sqrt{-1})^{n-1}e^{-mx\sqrt{-1}}\right\}$$

so that the quantity under the bracket differs from that in the expression, for $\dfrac{d^n\cos mx}{dx^n}$ only in having $(n-1)$ in the place of n.

Hence $$\frac{d^n\sin mx}{dx^n}=m^n\,.\left\{\cos\overline{r+r'}\,.\,\overline{n-1}\,\pi+\sqrt{-1}\sin\overline{r+r'}\,.\,\overline{n-1}\,\pi\right\}$$

$$\times\cos\left(\overline{r-r'+\frac{1}{2}}\,.\,\overline{n-1}\,\pi+mx\right)$$

Cor. $$\frac{d^{n+1}\,.\sin mx}{dx^{n+1}}=m\,.\,\frac{d^n\cos mx}{dx^n}.$$

Ex. If $m=1$, $n=\dfrac{1}{2}$, we get

$$\frac{d^{\frac{1}{2}}\cos x}{dx^{\frac{1}{2}}}=\left\{\cos\overline{r+r'}\,.\,\frac{\pi}{2}+\sqrt{-1}\sin\overline{r+r'}\,\frac{\pi}{2}\right\}\cos\left(\overline{r-r'+\frac{1}{2}}\,\frac{\pi}{2}+x\right)$$

Now we may give to r and r' any integral positive values we please :

Let $r'=0$, $\begin{cases} r=0 \text{ gives } \cos\left(\dfrac{\pi}{4}+x\right) \\ r=1 \;\ldots\; \sqrt{-1}\cos\left(\dfrac{3\pi}{4}+x\right)\end{cases}$

$$r'=1,\ \begin{cases} r=0 & \ldots\ \sqrt{-1}\cos\left(-\frac{\pi}{4}+x\right) \\ r=1 & \ldots\ -\cos\left(\frac{\pi}{4}+x\right) \end{cases}$$

Also $$\cos\left(\frac{3\pi}{4}+x\right) = -\cos\left(-\frac{\pi}{4}+x\right)$$

therefore the four values of $\dfrac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\cos x$ are

$$\pm\cos\left(\frac{\pi}{4}+x\right),\ \pm\sqrt{-1}\cos\left(-\frac{\pi}{4}+x\right)$$

and a trial of any other values of r and r' will shew that these are the only results.

Again $$\frac{d^{\frac{1}{3}}\cos x}{dx^{\frac{1}{3}}} = \left\{\cos\overline{r+r'}\,.\,\frac{\pi}{3}+\sqrt{-1}\sin\overline{r+r'}\frac{\pi}{3}\right\}\cos\overline{\left(r-r'+\frac{1}{2}\right)\frac{\pi}{3}+x}$$

Let
$$r'=0\begin{cases} r=0 \text{ gives } \cos\left(\frac{\pi}{6}+x\right) \\ r=1 \ \ldots\ \left(\cos\frac{\pi}{3}+\sqrt{-1}\sin\frac{\pi}{3}\right)\cos\left(\frac{\pi}{2}+x\right) \\ r=2 \ \ldots\ \left(\cos\frac{2\pi}{3}+\sqrt{-1}\sin\frac{2\pi}{3}\right)\cos\left(\frac{5\pi}{6}+x\right) \end{cases}$$

$$r'=1\begin{cases} r=0 \ \ldots\ \left(\cos\frac{\pi}{3}+\sqrt{-1}\sin\frac{\pi}{3}\right)\cos\left(-\frac{\pi}{6}+x\right) \\ r=1 \ \ldots\ \left(\cos\frac{2\pi}{3}+\sqrt{-1}\sin\frac{2\pi}{3}\right)\cos\left(\frac{\pi}{6}+x\right) \\ r=2 \ \ldots\ -\cos\left(\frac{\pi}{2}+x\right) \end{cases}$$

$$r'=2\begin{cases} r=0 \ \ldots\ \left(\cos\frac{2\pi}{3}+\sqrt{-1}\sin\frac{2\pi}{3}\right)\cos\left(-\frac{\pi}{2}+x\right) \\ r=1 \ \ldots\ -\cos\left(-\frac{\pi}{6}+x\right) \\ r=2 \ \ldots\ \left(\cos\frac{4\pi}{3}+\sqrt{-1}\sin\frac{4\pi}{3}\right)\cos\left(\frac{\pi}{6}+x\right) \end{cases}$$

which are the nine values of $\dfrac{d^{\frac{1}{3}}\cos x}{dx^{\frac{1}{3}}}$.

They may be written more briefly thus:

$$\cos\left(\frac{\pi}{6}+x\right),\ -\cos\left(\frac{\pi}{2}+x\right),\ -\cos\left(-\frac{\pi}{6}+x\right)$$

$$\left(\cos\frac{\pi}{3}\pm\sqrt{-1}\sin\frac{\pi}{3}\right)\cos\left(\frac{\pi}{2}+x\right),\ \left(\cos\frac{\pi}{3}\pm\sqrt{-1}\sin\frac{\pi}{3}\right)\cos\left(-\frac{\pi}{6}+x\right)$$

and $$\left(\cos\frac{2\pi}{3}\pm\sqrt{-1}\sin\frac{2\pi}{3}\right)\cos\left(\frac{\pi}{6}+x\right)$$

or $\sin x\,,\qquad \cos\left(\frac{\pi}{6}+x\right)\qquad -\cos\left(-\frac{\pi}{6}+x\right)$

$$-\left(\cos\frac{\pi}{3}\pm\sqrt{-1}\sin\frac{\pi}{3}\right)\sin x,\ \left(\cos\frac{2\pi}{3}\pm\sqrt{-1}\sin\frac{2\pi}{3}\right)\cos\left(\frac{\pi}{6}+x\right),$$

$$\left(\cos\frac{\pi}{3}\pm\sqrt{-1}\sin\frac{\pi}{3}\right),\ \cos\left(-\frac{\pi}{6}+x\right)$$

The number of values which r admits of is the same as the denominator of the fraction which stands as the index of differentiation.

Hence, if p be the denominator of the index, both r and r' admit of p different values; and any one value of r may be combined with any one value of r', so that the number of different differential coefficients is p^2.

This result differs from that previously given, M. LIOUVILLE having adopted a very indirect process for obtaining the different values.

24. *To find the differential coefficients of the other circular functions of x.*

We have no other process than that of expansion, which, of course, will not give a *complete* result. Thus, to find the nth differential coefficient of $\tan x$ we may proceed as follows:

$$\frac{d^n\tan x}{dx^n}=\sec x\,\frac{d^n\sin x}{dx^n}+\frac{n\,d}{dx}\sec x\,.\,\frac{d^{n-1}\sin x}{dx^{n-1}}+\ldots$$

the results of the differentiations being supplied by the above formulæ.

The values of the differential coefficients of the inverse functions must be determined by a similar process, but it will not be necessary to write them down.

We shall, however, give the differential coefficient of $\tan^{-1}x$, as it may be done in a very simple manner, when n is positive and greater than 1.

Let $u=\tan^{-1}x.$

$$\therefore\quad \frac{du}{dx}=\frac{1}{1+x^2}=\frac{1}{1+\sqrt{-1}\,.\,x}\cdot\frac{1}{1-\sqrt{-1}\,.\,x}$$

$$=\frac{1}{2}\cdot\left(\frac{1}{1+\sqrt{-1}\,.\,x}+\frac{1}{1-\sqrt{-1}\,.\,x}\right)$$

$$\therefore\quad \frac{d^nu}{dx^n}=\frac{d^{n-1}}{dx^{n-1}}\cdot\frac{du}{dx}$$

$$=\frac{1}{2}\,\frac{d^{n-1}}{dx^{n-1}}\left\{\frac{1}{y}+\frac{1}{y'}\right\}$$

if $\quad y=1+\sqrt{-1}\,.\,x\qquad y'=1-\sqrt{-1}\,.\,x$

Now, $\quad dy^{n-1}=(\sqrt{-1})^{n-1}\,dx^{n-1},\ (dy')^{n-1}=(-\sqrt{-1})^{n-1}\,dx^{n-1}$

$$\therefore\quad \frac{d^nu}{dx^n}=\frac{1}{2}(\sqrt{-1})^{n-1}\,.\,\frac{d^{n-1}}{dy^{n-1}}\cdot\frac{1}{y}+\frac{1}{2}(-\sqrt{-1})^{n-1}.\,\frac{d^{n-1}}{dy'^{\,n-1}}\cdot\frac{1}{y'}$$

$$=\frac{1}{2}(\sqrt{-1})^{n-1}\,.\,(-1)^{n-1}\,.\,\frac{\lfloor n}{y^n}+\frac{1}{2}(-\sqrt{-1})^{n-1}(-1)^{n-1}\frac{\lfloor n}{y'^{\,n}}$$

$$=\frac{1}{2}(-\sqrt{-1})^{n-1}\,\Gamma n\left\{\frac{1}{y^n}+(-1)^{n-1}\cdot\frac{1}{y'^n}\right\}$$

$$=\frac{1}{2}(-\sqrt{-1})^n\,\Gamma n\left\{\frac{1}{(1+\sqrt{-1}\,.\,x)^n}+\frac{(-1)^{n-1}}{(1-\sqrt{-1}\,x)^n}\right\}$$

which may, in certain cases, be reduced by the addition of the quantities

$$(1-\sqrt{-1}\,x)^n \text{ and } (1+\sqrt{-1}\,x)^n(-1)^{n-1}$$

or by the subtraction of $(-1-\sqrt{-1}\,x)^n$ from $(1-\sqrt{-1}\,x)^n$.

If n is negative, or positive and less than 1, the above reasoning does not apply, except in as far as to shew that the result is the expansion of one of the complete results, if there are many.

Section IV.—Expansion of Functions.

25. Our object in this section is the general expansion of a function of $x+h$, by a process analogous to that which constitutes Taylor's theorem.

Let u be any function of x;

u' the same function of $x+h$;

then, if u' be expanded in terms of h, the result will be of the form

$$\Sigma\,A\,h^n$$

where A is a function of x.

Now, $$\frac{d^n u'}{dx^n}=\frac{d^n u'}{dh^n}$$

$$\therefore\quad \Sigma\frac{d^n A}{dx^n}\,.\,h^n=\Sigma\,A\,.\,\frac{d^n h^n}{dh^n}$$
$$=\Sigma\,A\,f(n+1)\,.\,h^o$$

adopting the notation of art. 9.

Now, the only term on the left hand side of this equation which contains h^o, is that which originally contained this power of h;

Call it u_o or $\frac{d^o u}{dx^o}$;

$$\therefore\quad \frac{d^n u_o}{dx^n}=A\,f(n+1)$$

or $$A=\frac{\frac{d^n u_o}{dx^n}}{f(n+1)}$$

$$\therefore\quad u'=\Sigma\frac{d^n u_o}{dx^n}\cdot\frac{h^n}{f(n+1)}.$$

Cor. 1. If the expansion contain no negative powers of h, u_o coincides with u; for if we put $h=0$, we obtain u as one side, and u_o as the other side of the equation.

Cor. 2. Since each coefficient is determined from $\frac{d^o u}{dx^o}$ independently of its connexion with any of the others, it is evident that, when two or more expansions involve the same power of h, we shall obtain all (if any) of them by the above process.

Consequently, should an expansion involve a power of h, which is also involved in the ordinary expansion of TAYLOR's theorem, and in no other, we can at once determine the value of the coefficient, by subtracting from the *general* value of the corresponding differential coefficient, that particular value which is obtained by ordinary differentiation. Should it be demanded to explain how it happens that the general differential coefficient is different from the ordinary one, we should find some difficulty in answering the question. We say there would be considerable difficulty in explaining the reason for this difference; but to shew how it arises is easy enough, as the examples which follow will evince. In fact, the particular cases which form the basis of induction are limited as to the number of their terms, whilst the general form (even when the general symbol has been replaced by one of the particular numbers on which its existence depends) is unlimited; and although a series of the terms may be each zero, it may happen that, under certain circumstances, results may be deducible from them. Zero may, in fact, be a divisor of the form in the final state, and thus its appearance may be chased away.

26. Ex. 1. Let $u=x\sqrt{x-a}$: to find the coefficients of $h^{\frac{1}{2}}$, $h^{\frac{3}{2}}$, &c. in the expansion of u'.

By the formula $$\frac{d^n\,u\,v}{d\,x^n}=u\,\frac{d^n\,v}{d\,x^n}+n\,\frac{d\,u}{d\,x}\,\frac{d^{n-1}\,v}{d\,x^{n-1}}+\text{\&c.}$$

we get $$\frac{d^{\frac{1}{2}}\,x\sqrt{x-a}}{d\,x^{\frac{1}{2}}}=x\,\frac{d^{\frac{1}{2}}\sqrt{x-a}}{d\,x^{\frac{1}{2}}}+\frac{1}{2}\,\frac{d^{-\frac{1}{2}}\sqrt{x-a}}{d\,x^{-\frac{1}{2}}}$$

$$=xf\left(\frac{3}{2}\right)+\frac{1}{2}\,(-1)^{-\frac{1}{2}}\,\frac{\left|\!\underline{\tfrac{3}{2}}\right.}{\left|\!\underline{2}\right.}\,\frac{+1}{\sin\pi}\,(x-a)$$

but $$f\left(\frac{3}{2}\right)=(-1)^{\frac{1}{2}}\,\frac{\left|\!\underline{\tfrac{3}{2}}\right.}{\left|\!\underline{1}\right.}\,\frac{1}{\sin\theta\pi}\quad\left(\because\ \frac{\sin\pi+\theta}{\sin\theta}=-\cos\theta=-1\ \text{when}\ \theta=0\right)$$

$$\therefore\qquad \frac{d^{\frac{1}{2}}\,x\sqrt{x-a}}{d\,x^{\frac{1}{2}}}=xf\left(\frac{3}{2}\right)+\frac{1}{2}f\left(\frac{3}{2}\right)(x-a).$$

Hence the coefficient of $h^{\frac{1}{2}}$ in the expansion of u' is $x+\frac{1}{2}(x-a)$.

But by actual expansion, we obtain

$$(x+h)\sqrt{x-a+h}=(x+h)\sqrt{h}\left\{1+\frac{1}{2}\,\frac{x-a}{h}-\frac{1}{8}\,\frac{(x-a)^2}{h^2}+\dots\right\}$$

$$=h^{\frac{3}{2}}+\left\{x+\frac{1}{2}(x-a)\right\}\sqrt{h}+\left(-\frac{(x-a)^2}{8}+\frac{1}{2}\,x\,(x-a)\right)\frac{1}{\sqrt{h}}+\dots$$

which completely verifies our operations so far as they go.

But further,

$$\frac{d^{\frac{3}{2}}\, x\sqrt{x-a}}{dx^{\frac{3}{2}}} = x\,\frac{d^{\frac{3}{2}}\sqrt{x-a}}{dx^{\frac{3}{2}}} + \frac{3}{2}\,\frac{d^{\frac{1}{2}}\sqrt{x-a}}{dx^{\frac{1}{2}}}$$

$$= x\,.\,\frac{(-1)^{\frac{5}{2}}}{\pi}\;\frac{\Gamma\frac{3}{2}\;\Gamma\frac{1}{2}}{x-a} + \frac{3}{2}\,f\left(\frac{3}{2}\right)$$

Hence, dividing this by $f\left(\frac{5}{2}\right)$ we get as the coefficient of $h^{\frac{3}{2}}$, $\dfrac{\frac{3}{2}f\left(\frac{3}{2}\right)}{f\left(\frac{5}{2}\right)} = 1$.

In both cases our results are obviously correct.

Lastly, to obtain the coefficient of $\dfrac{1}{\sqrt{h}}$.

$$\frac{d^{-\frac{1}{2}}\, x\sqrt{x-a}}{dx^{-\frac{1}{2}}} = \frac{x\,d^{-\frac{1}{2}}\sqrt{x-a}}{dx^{-\frac{1}{2}}} - \frac{1}{2}\cdot\frac{d^{-\frac{3}{2}}\sqrt{x-a}}{dx^{-\frac{3}{2}}}$$

$$= x(-1)^{-\frac{1}{2}}\frac{\Gamma\frac{3}{2}}{\Gamma 2}\;\frac{1}{\sin\pi}\,(x-a) - \frac{1}{2}\,(-1)^{-\frac{3}{2}}\frac{\Gamma\frac{3}{2}}{\Gamma 3}\;\frac{1}{\sin 2\pi}\,(x-a)^2$$

and $$f\left(\frac{1}{2}\right) = (-1)^{-\frac{1}{2}}\,\frac{\Gamma\frac{1}{2}}{\Gamma 1}\;\frac{-1}{\sin 0\,\pi}$$

$$\therefore\qquad \frac{d^{-\frac{1}{2}}u}{dx^{-\frac{1}{2}}}\cdot\frac{1}{f\left(\frac{1}{2}\right)} = \frac{x(x-a)}{2} - \frac{1}{8}\,(x-a)^2.$$

In the same way we might find the values of the other coefficients.

A very natural question to be asked now is this, What is the coefficient of h or of h^2 in this expansion? Should we proceed to the determination of that coefficient, we might expect to find *zero* as the result; but a little consideration will convince us that such a conclusion would be ill founded. In fact, we here determine each coefficient independently of its connection with the others, or of its connection with the actual expansion. Now $(x+h)\sqrt{x-a+h}$ may be expanded in terms of positive integral powers of h; consequently the value of $\dfrac{du}{dx}$ is the coefficient of h in *this* expansion, and not in the expansion above, which does not contain such powers of h.

We confess the subject labours under a slight difficulty in one or two points, to which we shall call attention presently.

Ex. 2. To expand $\dfrac{x^2}{x-a}$ by the theorem.

$$\frac{d^{-1}}{dx^{-1}}\cdot\frac{1}{x-a} = f(0) = (-1)^{-1}\;.\;\Gamma 0$$

$$\frac{d^{-2}}{dx^{-2}}\cdot\frac{1}{x-a} = (-1)^{-2}\,\Gamma(-1)\,(x-a) = -\,\Gamma 0\,.\,(x-a) = +(x-a)\,f(0)$$

$$\frac{d^{-3}}{d\,x^{-3}}\cdot\frac{1}{x-a}=(-1)^{-3}\sqrt{-2}\,(x-a)^2=-\frac{\sqrt{0}}{2}\,(x-a)^2$$

$$\text{\&c.}\qquad=\qquad\text{\&c.}\qquad=+\frac{1}{2}(x-a)^2 f(0)$$

$$\therefore\qquad\frac{d^{-1}}{d\,x^{-1}}\cdot\frac{x^2}{x-a}=x^2 f(0)-2\,x\,(x-a)\,f(0)+(x-a)^2 f(0)$$

hence the coefficient of h^{-1} is $x^2-2\,x\,(x-a)+(x-a)^2$.

The other coefficients may be found in a similar manner. It is remarkable that our formula gives us not only the correct results, but, further, it gives the order in which the different parts occur. If we expand $\frac{(x+h)^2}{x-h+a}$ in negative powers of h, we get not merely a^2 as the coefficient of h^{-1}, but the very terms $x^2-2\,x\,(x-a)+(x-a)^2$.

Let us proceed to find the value of the coefficient of h°.

$$\frac{d^{\circ}\frac{1}{x-a}}{d\,x^{\circ}}=\frac{1}{x-a}$$

$$\therefore\qquad\frac{d^{\circ}\frac{x^2}{x-a}}{d\,x^{\circ}}=\frac{x^2}{x-a}+0\,.\,2\,x\,\frac{d^{-1}\frac{1}{x-a}}{d\,x^{-1}}+\frac{0\,(0-1)}{2}\,\frac{2\,d^{-2}\frac{1}{x-a}}{d\,x^{-2}}$$

$$=\frac{x^2}{x-a}+2\,x\,.\,0 f(0)-0\,(x-a)\,f(0)$$

$$=\frac{x^2}{x-a}+2\,x f(1)-(x-a)\,f(1)$$

and $$f(1)=\frac{d^{\circ}\frac{1}{x^{\circ}}}{d\,x^{\circ}}=1$$

therefore the coefficient of h° is $\frac{x^2}{x-a}+2\,x-\overline{x-a}$.

Now we have two expansions of $\frac{(x+h)^2}{x+h-a}$ involving a term not containing h: we have of course obtained the *sum* of them by our process of expanding the term; consequently that coefficient which we seek is the difference between this quantity and the other coefficient, or u.

Therefore, the coefficient of h° in the expansion which involves negative powers of h is,

$$\left(\frac{d^{\circ}\,u}{d\,x^{\circ}}-u\right)\frac{1}{f(1)}=2\,x-(x-a)\,.$$

To find the coefficient of h^1.

$$\frac{d^1\frac{x^2}{x-a}}{d\,x^1}=-\frac{x^2}{(x-a)^2}+2\,x\,\frac{1}{x-a}+0\,\frac{d^{-1}\frac{1}{x-a}}{d\,x^{-1}}$$

$$= -\frac{x^2}{(x-a)^2} + \frac{2x}{x-a} + 0 f(0)$$

$$= -\frac{x^2}{(x-a)^2} + \frac{2x}{x-a} + 1$$

$$= \frac{du}{dx} + 1$$

hence the coefficient required $= \dfrac{\dfrac{d^1 u}{dx^1} - \dfrac{du}{dx}}{f(2)} = 1.$

In the same way, and with equal facility, we may find the values of the other coefficients. Should we attempt to find the coefficient of such a power of h as $h^{\frac{2}{3}}$, we shall readily find that $\dfrac{d^{\frac{2}{3}} u}{dx^{\frac{2}{3}}}$ is a finite quantity, but that $f\left(1+\dfrac{2}{3}\right)$ is infinite; and, therefore, that the coefficient $= 0$.

27. In Art. 24, we assumed that

$$\frac{d^n \dfrac{1}{(1+\sqrt{-1}\,x)}}{dx^n} = \frac{d^n}{dy^n} \cdot \frac{1}{y} \quad \text{where } y = 1 + \sqrt{-1}\,x.$$

Should any difficulty be experienced respecting this assumption, it will be entirely removed by means of the following proposition.

To find $$d^n \frac{1}{(1+ax)^m}.$$

$$\because \quad \frac{1}{(1+ax)^m} = \frac{1}{(ax)^m \left(1+\dfrac{1}{ax}\right)^m}$$

$$= \frac{1}{a^m x^m}\left\{1 - \frac{m}{ax} + \frac{m(m+1)}{1.2}\,\frac{1}{a^2 x^2} - \&c.\right\}$$

$$\frac{d^n}{dx^n} \cdot \frac{1}{(1+ax)^m} = \frac{1}{a^m}\,\frac{d^n}{dx^n}\left\{\frac{1}{x^m} - \frac{m}{a\,x^{m+1}} + \frac{m(m+1)}{1.2}\,\frac{1}{a^2 x^{m+2}} - \&c.\right\}$$

$$= \frac{1}{a^m}(-1)^n \left\{\frac{\lfloor n+m}{\lfloor m} \cdot \frac{1}{x^{n+m}}\right.$$

$$\left. - \frac{m}{a}\,\frac{\lfloor n+m+1}{\lfloor m+1} \cdot \frac{1}{x^{n+m+1}} + \frac{m(m+1)}{1.2} \cdot \frac{1}{a^2} \cdot \frac{\lfloor n+m+2}{\lfloor m+2 \,.\, x^{n+m+2}} - \&c.\right\}$$

$$= (-1)^n \frac{\lfloor n+m}{\lfloor m}\,\frac{1}{a^m x^{n+m}} \cdot \left\{1 - \frac{m}{a} \cdot \frac{n+m}{m} \cdot \frac{1}{x}\right.$$

$$\left. + \frac{m(m+1)}{1.2} \cdot \frac{1}{a^2} \cdot \frac{(n+m+1)(n+m)}{(m+1)\,m}\,\frac{1}{x^2} - \&c.\right\}$$

$$= (-1)^n \frac{\lfloor n+m}{\lfloor m} \cdot \frac{1}{a^m x^{n+m}} \cdot \left\{1 - \frac{n+m}{ax}\right.$$

$$+\frac{(n+m)(n+m+1)}{1.2}\cdot\frac{1}{a^2 x^2}-\&c.\Big\}$$

$$=(-1)^n.\frac{/\overline{n+m}}{/\overline{m}}\cdot\frac{1}{a^m x^{n+m}}\frac{1}{\left(1+\dfrac{1}{a x}\right)^{n+m}}$$

$$=(-1)^n.\frac{/\overline{n+m}}{/\overline{m}}\cdot\frac{a^n}{(1+a x)^{n+m}}$$

$$=a^n.\frac{d^n\dfrac{1}{y^m}}{d y^n}$$

which is the proposition to be proved.

28. The example given in the last article will furnish us with a ready means of exemplifying a theorem analogous to that of MACLAURIN; for the coefficient of x^{-n} in the expansion of $\dfrac{1}{(1+a x)^m}$ in terms of negative powers of x, is, according to that theorem, supposing it extended to the case before us: (x being $=0$)

$$\frac{d^{-n}\dfrac{1}{(1+a x)^m}}{d x^{-n}}\frac{1}{f(-n+1)}$$ which is equal to

$$(-1)^{-n}\frac{/\overline{-n+m}}{/\overline{m}}\cdot\frac{a^{-n}}{(1+a x)^{-n+m}}\cdot\frac{1}{f(-n+1)}\quad\text{when } x=0$$

$$=(-1)^{-n}\frac{/\overline{-n+m}}{/\overline{m}}\cdot\frac{a^{-n}}{(1+a x)^{-n+m}}\frac{/\overline{n}}{(-1)^{-n}\,/\overline{0}},\quad x=0$$

$$=+\frac{/\overline{-n+m}}{/\overline{m}}\frac{a^{-n}\,/\overline{n}}{/\overline{0}}$$

For instance, if $n \angle m$; since $/\overline{0}=\infty$, the coefficient is zero.

If $n=m$, the coefficient is

$$\frac{/\overline{0}}{/\overline{m}}\cdot\frac{a^{-n}\,/\overline{m}}{/\overline{0}}=a^{-n}$$

If $n=m+r$, it is

$$\frac{/\overline{-r}}{/\overline{m}}\cdot\frac{a^{-(m+r)}\,/\overline{m+r}}{/\overline{0}}=a^{-(m+r)}\,m\,(m+1)\ldots(m+r-1)\times\frac{/\overline{-r}}{/\overline{0}}$$

Now, $/\overline{0}=-1\,/\overline{-1}=(-1)^r\,/\overline{r}\,/\overline{-r}$ if r be a whole number: in this case the coefficient is

$$(-1)^r\frac{m\,(n+1)\;\ldots\;(m+r-1)}{1.2\quad\ldots\quad r}\;\frac{1}{a^{m+r}}.$$

But if r be a fraction, $/\overline{-r}$ is finite and $/\overline{0}$ infinite, therefore the coefficient is zero; results which are all obviously correct. With one more example we shall conclude the present memoir.

Ex. To expand $\frac{1}{\sqrt{1+x}}$ in terms of $\frac{1}{x}$.

Generally
$$d^{-n}\frac{1}{\sqrt{1+x}} = (-1)^{-n}\frac{\Gamma\left(-n+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\,\frac{1}{(1+x)^{-n+\frac{1}{2}}}$$

therefore, the coefficient of x^{-n} is

$$(-1)^{-n}\frac{\Gamma\left(-n+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\,\frac{\Gamma(n)}{(-1)^{-n}\,\Gamma(0)} = \frac{\Gamma\left(-n+\frac{1}{2}\right)\Gamma(n)}{\Gamma\left(\frac{1}{2}\right)\Gamma(0)}.$$

Hence, except n is of the form $\frac{p}{2}$, where p is some odd number, the coefficient is zero; for $\Gamma\left(-n+\frac{1}{2}\right)$ is finite, and $\Gamma(0)$ infinite. But if $n=r+\frac{1}{2}$, the coefficient is

$$\frac{\Gamma(-r)\,\Gamma\left(r+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(0)} = \frac{\Gamma\left(r+\frac{1}{2}\right)}{(-1)^r\,1.2\ldots r\,\Gamma\left(\frac{1}{2}\right)}$$

$$= (-1)^r\,\frac{\left(r-\frac{1}{2}\right)}{1}\,\frac{\left(r-\frac{3}{2}\right)}{2}\cdots\frac{\frac{1}{2}}{r}$$

$$= (-1)^r\,\frac{1.3.5\ldots(2r-1)}{2.4.6\ldots\;2r}$$

$$\therefore\quad \frac{1}{\sqrt{1+x}} = \ldots + (-1)^r\,\frac{1.3.5\ldots(2r-1)}{2.4.6\ldots\;2r}\,\frac{1}{x^{r+\frac{1}{2}}} + \ldots\ldots$$

We now draw the memoir to a conclusion, trusting that it may be deemed worthy of consideration, as well from the generality and completeness of the methods exhibited, as from the simplicity with which they are demonstrated. We could conceive more limited theorems than those which occupy our last section; but as the subject is as yet little studied, and as no person appears to have attempted the application of any such theorems hitherto, I hope partial defects will be excused, and, if possible, remedied by those who enter into the subject.

EDINBURGH, *December* 2. 1839.

XXIX.—*On General Differentiation. Part II. By* The Rev. P. KELLAND, *M.A., F.R.SS. L. & E., F.C.P.S., late Fellow of Queens' College, Cambridge; Professor of Mathematics, &c. in the University of Edinburgh.*

(Read 20th January 1840.)

In a former memoir on this subject, it was my endeavour to exhibit the principles of the science of General Differentiation in a simple, at the same time in a general, point of view. I endeavoured to deduce, from one general formula, results easy of application in all instances; and thus to exhibit the unity of the different parts of the science, and the completeness of its fundamental formulæ, shewing at the same time the facility of their adaptation to particular and varied cases. With the exception of certain expansions by means of a theorem analogous to the series of TAYLOR, I gave no application of the principles to problems of any kind. It is my intention in the present memoir to supply this branch of the subject, without which, indeed, however interesting may be the details, as a portion of pure analysis, they will offer little to interest any but those who attach themselves to the study of analytical combination. We hope, by the exhibition of a few simple mechanical problems, solved by this process, to give to our subject an interest in the eyes of all, derived not from its intrinsic beauty, but from its *use* as a medium of demonstration. It is well known that considerable difficulty hangs over several very simple *inverse* mechanical problems; from the generality of their statement, a direct solution is sometimes impossible by the ordinary methods. We shall shew that by our process such solutions are attainable with the greatest readiness. By this means we hope to give a value to our subject as a branch of knowledge, independent of that value which it must possess from its curious and elegant structure.

I must not conclude my introductory observations, without distinctly disclaiming the merit of having originally conceived the possibility of applying this science to mechanics. M. LIOUVILLE has not only broached the method, but has applied it to a number of cases in his first memoir. The theorem by which my processes are effected is, however, as far as I know, quite new; and one more elegant or simple, considering its comprehensive nature, I can scarcely conceive. But I proceed to its demonstration.

THEOREM.—If $\phi(\theta+\alpha)$ be an integral positive or negative function of $\theta+\alpha$; then will

$$\int_0^z d\theta\,\phi(\theta+\alpha)(z-\theta)^p=(-1)^{p+1}.\,\Gamma(p+1)\,.\cos(p+1)\pi\,.\times\frac{d^{-(p+1)}}{dz^{-(p+1)}}\,.\,\phi(z+\alpha)\,.$$

We shall be able to obtain an equation in other cases by means of the last equality but one in our process.

To prove this theorem:

Let $\int_0^z d\theta\,\phi(\theta+\alpha)(z-\theta)^p$ be denoted by P.

Assume $\theta=z\gamma$ where z is constant;

$$\therefore\quad d\theta=z\,d\gamma$$

and

$$P=\int_0^1 z\,d\gamma\,\phi(\gamma z+\alpha)(z-\theta)^p\,.$$

Let

$$\phi(\gamma z+\alpha)=\Sigma A(\gamma z+\alpha)^m$$

$$\therefore\quad P=\Sigma A\,z\int_0^1 d\gamma(\gamma z+\alpha)^m(z-\theta)^p$$

$$=\Sigma A\,z^{p+1}\int_0^1 d\gamma(\gamma z+\alpha)^m\left(1-\frac{\theta}{z}\right)^p$$

$$=\Sigma A\,z^{p+1}\int_0^1 d\gamma(\gamma z+\alpha)^m(1-\gamma)^p$$

$$=\Sigma A\,z^{p+1}\int_0^1 d\gamma\left\{\gamma^m z^m+m\gamma^{m-1}.\,z^{m-1}\alpha\right.$$

$$\left.+\frac{m(m-1)}{1.2}\gamma^{m-2}z^{m-2}\alpha^2+\ldots\right\}(1-\gamma)^p$$

Now

$$\int_0^1 d\gamma\,\gamma^m(1-\gamma)^p=\frac{\Gamma(p+1)\,\Gamma(m+1)}{\Gamma(m+p+2)}$$

by EULER's and LEGENDRE's theorems.

$$\therefore\quad P=\Sigma A\,z^{p+1}\Gamma(p+1)\left\{\frac{\Gamma(m+1)\,.\,z^m}{\Gamma(m+p+2)}\right.$$

$$\left.+\frac{m\,\Gamma(m)\,z^{m-1}.\,\alpha}{\Gamma(m+p+1)}+\frac{m(m-1)}{1.2}\,\frac{\Gamma(m-1)\,.\,z^{m-2}\alpha^2}{\Gamma(m+p)}+\ldots\right\}$$

and since

$$m\,\Gamma(m)=\Gamma(m+1)$$

$$(m-1)\,m\,\Gamma(m-1)=\Gamma(m+1)$$

$$\ldots=\ldots$$

we get

$$P=\Sigma A\,z^{p+1}\Gamma(p+1)\,\Gamma(m+1)\left\{\frac{z^m}{\Gamma(m+p+2)}+\frac{z^{m-1}\alpha}{\Gamma(m+p+1)}\right.$$

$$\left.+\frac{z^{m-2}}{\Gamma(m+p)}\cdot\frac{\alpha^2}{1.2}+\frac{z^{m-3}}{\Gamma(m+p-1)}\cdot\frac{\alpha^3}{1.2.3}+\ldots\right\}$$

But

$$\frac{d^{-(p+1)}.\,z^m}{dz^{-(p+1)}}=(-1)^{-(p+1)}\frac{\sin m\pi}{\sin(m+p+1)\pi}\,\frac{\Gamma(1+m)\,z^{m+p+1}}{\Gamma(m+p+2)}$$

$$\frac{d^{-(p+1)} \cdot z^{m-1}}{d z^{-(p+1)}} = (-1)^{-(p+1)} \frac{\sin (m-1) \pi}{\sin (m+p) \pi} \frac{\lfloor m \cdot z^{m+p}}{\lfloor m+p+1}$$

$$= (-1)^{-(p+1)} \frac{\sin m \pi}{\sin (m+p+1) \pi} \frac{\lfloor m+1 \cdot z^{m+p}}{m \lfloor m+p+1}$$

&c. = &c.

$$\therefore \quad P = \Sigma A \lfloor p+1 \Big\{ (-1)^{p+1} \frac{\sin \overline{m+p+1}\, \pi}{\sin m \pi} \frac{d^{-(p+1)} \cdot z^m}{d z^{-(p+1)}}$$

$$+ \frac{\alpha}{1} \cdot (-1)^{-(p+1)} \frac{\sin (m+p+1) \pi}{\sin m \pi} m \frac{d^{-(p+1)} \cdot z^{m-1}}{d z^{-(p+1)}}$$

$$+ \frac{\alpha^2}{1 \cdot 2} (-1)^{-(p+1)} \frac{\sin (m+p+1) \pi}{\sin m \pi} m (m-1) \frac{d^{-(p+1)} \cdot z^{m-2}}{d z^{-(p+1)}}$$

$$+ \quad \ldots \quad \Big\}$$

$$= \Sigma A \lfloor p+1 \; (-1)^{p+1} \frac{\sin (m+p+1) \pi}{\sin m \pi} \cdot \frac{d^{-(p+1)}}{d z^{-(p+1)}} \Big\{ z^m$$

$$+ \frac{m}{1} \cdot \alpha z^{m-1} + \frac{m (m-1)}{1 \cdot 2} \alpha^2 z^{m-2} + \&c. \Big\}$$

$$= \Sigma A \lfloor p+1 \; (-1)^{p+1} \frac{\sin (m+p+1) \pi}{\sin m \pi} \frac{d^{-(p+1)}}{d z^{-(p+1)}} \cdot (z+\alpha)^m$$

$$= (-1)^{p+1} \lfloor p+1 \frac{\sin (m+p+1) \pi}{\sin m \pi} \frac{d^{-(p+1)}}{d z^{-(p+1)}} \cdot \phi (z+\alpha)$$

or $$\int_0^z d\theta \phi (\theta+\alpha) (z-\theta)^p = (-1)^{p+1} \lfloor p+1 \cdot \frac{\sin (m+p+1) \pi}{\sin m \pi} \cdot \times \frac{d^{-(p+1)}}{d z^{-(p+1)}} \cdot \phi (z+\alpha).$$

$$= (-1)^{p+1} \lfloor p+1 \cdot \cos (p+1) \pi \frac{d^{-(p+1)}}{d z^{-(p+1)}} \cdot \phi (z+\alpha).$$

if m be a whole number, and p a fraction or whole number.

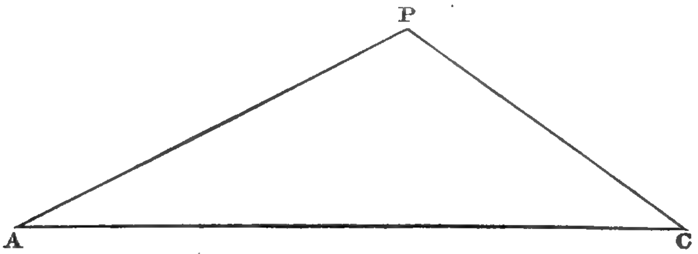
Ex. Let $p=1, \; \phi (\theta+\alpha) = \theta + \alpha$

$$\therefore \quad \int_0^z d\theta (\theta+\alpha)(z-\theta) = \frac{z^3}{6} + \frac{\alpha z^2}{2} \text{ by integration}$$

$$\frac{d^{-2}}{d z^{-2}} (z+\alpha) = \frac{z^3}{2 \cdot 3} + \frac{\alpha z^2}{2} = \text{the same as the other.}$$

It must be observed that the integrations are performed *separately*, as in the demonstration. The problem which led me to this theorem is that of finding the law of force by which the particles of a sphere must act on a point, so that the whole attraction may be the same as though the sphere were collected at its centre of gravity. Unfortunately the question leads to a general differential equation, the solution of which we have not as yet been able to effect. Still we have done all that is requisite in order to exemplify the use of our analysis, by shewing that this question reduces itself to such an equation: we shall, therefore, exhibit our process.

Let C be the centre of a sphere, A the attracted point. $AP=r$, $AC=a$, $CAP=\phi$, $CP=\varrho$, and the radius of the sphere $=R$, $f(r)$ the attraction of an unit at the distance r.



Then the area of an annulus is $2\pi r^2 \sin\phi\, dr\, d\phi$, and its attraction on the point $2\pi r^2\, dr\, d\phi \sin\phi \cos\phi\, f(r)$.

Hence the whole attraction is the following double integral,

$$\pi \int_{a-R}^{a+R} dr\,.\,r^2 f(r) \int_{o}^{\phi_{,}} \sin 2\phi\, d\phi$$

where $$\cos\phi_{,} = \frac{r^2+a^2-R^2}{2ar}$$

and $$\therefore \quad \sin^2\phi_{,} = \frac{4a^2r^2-(r^2+a^2-R^2)^2}{4a^2r^2}.$$

Consequently, the whole attraction is

$$\frac{\pi}{2}\int_{a-R}^{a+R} dr\,.\,r^2 f(r)\,(1-\cos 2\phi_{,})$$

$$=\pi \int_{a-R}^{a+R} dr\,.\,r^2 f(r) \sin^2\phi_{,}$$

$$=\frac{\pi}{4a^2}\int_{a-R}^{a+R} dr\,.\,(4a^2r^2-\overline{r^2+a^2-R^2}^{\,2})\,f(r).$$

This result may be easily reduced to the same form as the first side of the equation which constitutes our theorem, as follows.

Let $$r=a-R+\theta$$

therefore attraction $$=\frac{\pi}{4a^2}\int_{o}^{2R} f(a-R+\theta)\,\Big\{4a^2(a-R+\theta)^2$$

$$-(\overline{a-R+\theta}^{\,2}+a^2-R^2)^2\Big\}\, d\theta$$

$$=\frac{\pi}{4a^2}\int_{o}^{2R} d\theta\,.\,f(a-R+\theta)\,\Big\{4a^2(a+R-\overline{2R-\theta})^2$$

$$-(\overline{a+R-\overline{2R-\theta}}\,\Big|^{2}+a^2-R^2)^2\Big\}$$

$$=\frac{\pi}{4a^2}\int_{o}^{2R} d\theta\, f(a-R+\theta)\,\Big\{4a^2(a+R)^2-8a^2(a+R)\,.\,\overline{2R-\theta}$$

$$+4a^2(2R-\theta)^2-4a^2(a+R)^2-4\overline{a+R}^{\,2}\,.\,\overline{2R-\theta}^{\,2}-\overline{2R-\theta}^{\,4}$$

$$+8a(a+R)^2(2R-\theta)-4a(a+R)(2R-\theta)^2+4\overline{a+R}\,.\,\overline{2R-\theta}^{\,3}\Big\}$$

$$=\frac{\pi}{4a^2}\int_o^{2\mathrm{R}} d\theta f(a-\mathrm{R}+\theta)\ \Big\{ +(8a\,\overline{a+\mathrm{R}}-8a^2)(a+\mathrm{R})(2\mathrm{R}-\theta)$$

$$+\big(4a^2-4(a+\mathrm{R})^2-4a(a+\mathrm{R})\big)(2\mathrm{R}-\theta)^2+4(a+\mathrm{R})\,\overline{2\mathrm{R}-\theta}\,|^3$$

$$-(2\mathrm{R}-\theta)^4\Big\}$$

$$=\frac{\pi}{4a^2}\int_o^{2\mathrm{R}} d\theta f(a-\mathrm{R}+\theta)\ \Big\{8a\mathrm{R}(a+\mathrm{R})(2\mathrm{R}-\theta)$$

$$-4(a^2+3a\mathrm{R}+\mathrm{R}^2)(2\mathrm{R}-\theta)^2+4(a+\mathrm{R})(2\mathrm{R}-\theta)^3-(2\mathrm{R}-\theta)^4\Big\}$$

To exemplify our formula, let us suppose it applied to this proposition; then have we, whole force of attraction

$$=\frac{\pi}{4a^2}\cdot\Big\{8a\mathrm{R}(a+\mathrm{R})\frac{d^{-2}}{dz^{-2}}f(z+\alpha)$$

$$-8(a^2+3a\mathrm{R}+\mathrm{R}^2)\frac{d^{-3}}{dz^{-3}}f(z+\alpha)$$

$$+24(a+\mathrm{R})\frac{d^{-4}}{dz^{-4}}f(z+\alpha)-24\frac{d^{-5}}{dz^{-5}}f(z+\alpha)\Big\}.$$

Now, if $f(z+\alpha)=z+\alpha$, or the force varies as the distance

$$\frac{d^{-2}f(z+\alpha)}{dz^{-2}}=\frac{z^3}{2.3}+\frac{\alpha z^2}{1.2}$$

$$\frac{d^{-3}f(z+\alpha)}{dz^{-3}}=\frac{z^4}{2.3.4}+\frac{\alpha z^3}{2.3}$$

$$\frac{d^{-4}f(z+\alpha)}{dz^{-4}}=\frac{z^5}{2.3.4.5}+\frac{\alpha z^4}{2.3.4}$$

$$\frac{d^{-5}f(z+\alpha)}{dz^{-5}}=\frac{z^6}{2.3.4.5.6}+\frac{\alpha z^5}{2.3.4.5}$$

and $\qquad z=2\mathrm{R},\ \alpha=a-\mathrm{R},$

hence we get whole attraction

$$=\frac{\pi}{4a^2}\Big\{8a\mathrm{R}(a+\mathrm{R})\left(\frac{4\mathrm{R}^3}{3}+(a-\mathrm{R})2\mathrm{R}^2\right)$$

$$-8(a^2+3a\mathrm{R}+\mathrm{R}^2)\left(\frac{2}{3}\mathrm{R}^4+\overline{a-\mathrm{R}}\,.\,\frac{4}{3}\mathrm{R}^3\right)$$

$$+24(a+\mathrm{R})\left(\frac{4}{15}\mathrm{R}^5+\overline{a-\mathrm{R}}\,.\,\frac{2}{3}\mathrm{R}^4\right)$$

$$-\left(\frac{32}{15}\mathrm{R}^6+\overline{a-\mathrm{R}}\,.\,\frac{32}{5}\mathrm{R}^5\right)\Big\}$$

$$=\frac{\pi}{4a^2}\Big\{16a\mathrm{R}(a+\mathrm{R})\left(a\mathrm{R}^2-\frac{\mathrm{R}^3}{3}\right)$$

$$-\frac{16}{3}(a^2+3a\mathrm{R}+\mathrm{R}^2)(2a\mathrm{R}^3-\mathrm{R}^4)$$

$$+16\left(a\mathrm{R}^4-\frac{3}{5}\mathrm{R}^5\right)(a+\mathrm{R})-\frac{32}{5}a\mathrm{R}^5+\frac{64}{15}\mathrm{R}^6\Big\}$$

$$=\frac{\pi}{4a^2}\left\{16a^3R^3-\frac{16}{3}a^2R^4+16a^2R^4-\frac{16aR^5}{3}\right.$$

$$-\frac{32}{3}a^3R^3+\frac{16}{3}a^2R^4-32a^2R^4+16aR^5-\frac{32}{3}aR^5$$

$$+\frac{16}{3}R^6+16a^2R^4-\frac{48}{5}aR^5+16aR^5-\frac{48}{5}R^6$$

$$\left.-\frac{32}{5}aR^5+\frac{64}{15}R^6\right\}$$

$$=\frac{\pi}{4a^2}\left\{\frac{16}{3}a^3R^3+\left(-\frac{16}{3}+16+\frac{16}{3}-32+16\right)a^2R^4\right.$$

$$+\left(-\frac{16}{3}+16-\frac{32}{3}-\frac{48}{5}+16-\frac{32}{5}\right)aR^5$$

$$\left.+\left(-\frac{48}{5}+\frac{16}{3}+\frac{64}{15}\right)R^6\right\}$$

$$=\frac{\pi}{4a^2}\cdot\frac{16}{3}a^3R^3$$

$$=\frac{4}{3}\pi R^3 . a$$

= mass multiplied by distance of centre of gravity of the sphere from the point.

This result is obviously correct.

It will be remarked, that all we have effected by means of our process, is the transformation of a definite integral into an indefinite one. This transformation is, however, of the utmost importance as a general fact, although we make little of it in the present instance.

Next, let the force of attraction be that of the inverse square of the distance, then shall we have to find the integrals of $\frac{1}{(z+\alpha)^2}$.

Now, $$d^{-1}\frac{1}{(z+\alpha)^2}=-\frac{1}{z+\alpha},$$

$$\frac{d^{-2}\dfrac{1}{(z+\alpha)^2}}{dz^{-2}}=-\int\frac{1}{z+\alpha}dz=-\log(z+\alpha),$$

and $$\frac{d^{-3}}{dz^{-3}}\frac{1}{(z+\alpha)^2}=-\int\log(z+\alpha)\,dz$$

$$=-z\log(z+\alpha)+\int\frac{z}{z+\alpha}dz$$

$$=-z\log(z+\alpha)+\int\frac{z+\alpha-\alpha}{z+\alpha}dz$$

$$=-(z+\alpha)\log(z+\alpha)+z,$$

$$\frac{d^{-4}}{dz^{-4}}\frac{1}{(z+\alpha)^2}=-\left(\frac{z^2}{2}+\alpha z\right)\log(z+\alpha)+\frac{z^2}{2}$$

$$+\left(\frac{1}{2}\int dz\,\frac{z^2+2\alpha z}{z+\alpha}=\frac{1}{2}\int dz\,\frac{(z+\alpha)^2-\alpha^2}{z+\alpha}\right)$$

$$=-\left(\frac{z^2}{2}+\alpha z+\frac{\alpha^2}{2}\right)\log(z+\alpha)+\frac{3z^2}{4}+\frac{\alpha z}{2},$$

$$\frac{d^{-5}}{dz^{-5}}\frac{1}{(z+\alpha)^2}=-\left(\frac{z^3}{6}+\frac{\alpha z^2}{2}+\frac{\alpha^2 z}{2}\right)\log(z+\alpha)+\frac{3z^3}{12}+\frac{\alpha z^2}{4}$$

$$+\left(\frac{1}{6}\int\frac{z^3+3\alpha z^2+3\alpha^2 z}{z+\alpha}\,dz=\frac{1}{6}\int\frac{(z+\alpha)^3-\alpha^3}{z+\alpha}\,dz\right)$$

$$=-\frac{1}{6}(z^3+3\alpha z^2+3\alpha^2 z+\alpha^3)\log(z+\alpha)+\frac{z^3}{4}+\frac{\alpha z^2}{4}$$

$$+\frac{1}{6}\left(\frac{z^3}{3}+\alpha z^2+\alpha^2 z\right)$$

$$=-\frac{1}{6}(z^3+3\alpha z^2+3\alpha^2 z+\alpha^3)\log(z+\alpha)+\frac{11z^3}{36}$$

$$+\frac{5\alpha z^2}{12}+\frac{\alpha^2 z}{6}.$$

Which results being substituted in the general formula for the attraction, give

$$\text{attraction}\quad=\frac{\pi}{4a^2}\Big\{-8a\mathrm{R}(a+\mathrm{R})\log(a+\mathrm{R})$$

$$+8(a^2+3a\mathrm{R}+\mathrm{R}^2)(\overline{a+\mathrm{R}}\log\overline{a+\mathrm{R}}-2\mathrm{R})$$

$$-24(a+\mathrm{R})\left(\frac{\overline{a+\mathrm{R}}^2}{2}\log\overline{a+\mathrm{R}}-3\mathrm{R}^2-\overline{a-\mathrm{R}}\,\mathrm{R}\right)$$

$$+24\left(\frac{\overline{a+\mathrm{R}}^3}{6}\log\overline{a+\mathrm{R}}-\frac{22}{9}\mathrm{R}^3-\frac{5}{3}\overline{a-\mathrm{R}}\,\mathrm{R}^2-\frac{\overline{a-\mathrm{R}}^2.\mathrm{R}}{3}\right\}$$

$$=\frac{\pi}{4a^2}\Big\{\Big[(8a^2+16a\mathrm{R}+8\mathrm{R}^2)(a+\mathrm{R})-8(a+\mathrm{R})^3\Big]\times\log(a+\mathrm{R})$$

$$-16\mathrm{R}(a^2+3a\mathrm{R}+\mathrm{R}^2)+24(a+\mathrm{R})(a\mathrm{R}+2\mathrm{R}^2)$$

$$-\frac{80}{3}\mathrm{R}^3-24a\mathrm{R}^2-8a^2\mathrm{R}\Big\}$$

$$=\frac{\pi}{4a^2}.\Big\{-16a^2\mathrm{R}-48a\mathrm{R}^2-16\mathrm{R}^3+24a^2\mathrm{R}$$

$$+48a\mathrm{R}^2+24a\mathrm{R}^2+48\mathrm{R}^3-\frac{80}{3}\mathrm{R}^3-24a\mathrm{R}^2-8a^2\mathrm{R}\Big\}$$

$$=\frac{\pi}{4a^2}\left\{\left(32-\frac{80}{3}\right)\mathrm{R}^3\right\}$$

$$=\frac{4\pi\mathrm{R}^3}{3}\cdot\frac{1}{a^2}.$$

We cannot expect, in our present state of knowledge of the subject, to determine any converse propositions in so general a case as that of the sphere; but as a more simple example will equally illustrate the importance of our formula, we shall give one.

A homogeneous rod of small thickness attracts a point without itself: it is required to find the law of attraction, so that the whole force may vary as the n^{th} power of the reciprocal of the distance of the rod from the point.

Let a be the distance of the point from the nearest point in the rod;

r, $r+dr$, co-ordinates of two points in the rod;

ϱ the distance of the point whose ordinate is r from the attracted point:

Then, if $f(\varrho)$ be the law of attraction, we obtain whole attraction

$$=\int_a^\infty dr f(\varrho)\frac{a}{\varrho}$$

Now $\qquad \varrho^2=a^2+r^2 \qquad \therefore \quad dr=\dfrac{\varrho\, d\varrho}{\sqrt{\varrho^2-a^2}}$

and attraction $\qquad =\displaystyle\int_a^\infty \frac{f(\varrho)\, d\varrho\,.\,a}{\sqrt{\varrho^2-a^2}}$

$$=\frac{1}{2}\int_a^\infty \frac{-a\,.\,\dfrac{d\,.\,\frac{1}{\varrho^2}}{d\varrho}\,d\varrho f(\varrho)\,\varrho^3}{\sqrt{\varrho^2-a^2}}$$

$$=\frac{1}{2}\int_\infty^a \frac{\dfrac{d\,.\,\frac{1}{\varrho^2}}{d\varrho}\,.\,d\varrho f(\varrho)\,\varrho^2}{\sqrt{\dfrac{1}{a^2}-\dfrac{1}{\varrho^2}}}$$

Now let $\quad \dfrac{1}{\varrho^2}=\theta,\ \dfrac{1}{a^2}=z,\ \varrho^2 f(\varrho)=\phi(\theta)$

therefore attraction $\quad =\dfrac{1}{2}\displaystyle\int_0^z \frac{d\theta\,\phi(\theta)}{\sqrt{z-\theta}}\,.$

Hence this form coincides with that in our theorem, and we get

$$\text{attraction} \quad =\frac{(-1)^{\frac{1}{2}}}{2}\sqrt{\tfrac{1}{2}}\ \frac{\sin(m+\frac{1}{2})\pi}{\sin m\pi}\ \frac{d^{-\frac{1}{2}}\phi(z)}{dz^{-\frac{1}{2}}}$$

But, according to hypothesis, the attraction must vary as a^{-n}.

Let it be equal to $\qquad P\,a^{-n}=\dfrac{P}{a^n}=P\,z^{\frac{n}{2}}$

hence $\qquad (-1)^{\frac{1}{2}}\dfrac{\sqrt{\frac{1}{2}}}{2}\ \dfrac{\sin(m+\frac{1}{2})\pi}{\sin m\pi}\ \dfrac{d^{-\frac{1}{2}}\phi z}{dz^{-\frac{1}{2}}}=P\,z^{\frac{n}{2}}$

and
$$(-1)^{\frac{1}{2}}\frac{\sqrt{\pi}}{2}\frac{\sin(m+\frac{1}{2})\pi}{\sin m\pi}\phi(z)=\mathrm{P}\frac{d^{\frac{1}{2}}z^{\frac{n}{2}}}{dz^{\frac{1}{2}}}$$

$$=(-1)^{\frac{1}{2}}\,\mathrm{P}\,\frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}\cdot\frac{\sin\frac{n}{2}\pi}{\sin\frac{n-1}{2}\pi}\,z^{\frac{n-1}{2}}$$

$$\therefore\quad \phi(z)\propto z^{\frac{n-1}{2}}$$

or
$$\phi\left(\frac{1}{a^2}\right)=\frac{1}{a^{n-1}}$$

$$\therefore\quad \phi\left(\frac{1}{\varrho^2}\right)=\frac{1}{\varrho^{n-1}}$$

or
$$\varrho^2 f(\varrho)=\frac{1}{\varrho^{n-1}}$$

$$f(\varrho)=\frac{1}{\varrho^{n+1}}$$

Cor. 1. If $n=1,\ f(\varrho)=\dfrac{1}{\varrho^2}$; the law of nature.

Cor. 2. If $n=0,\ f(\varrho)=\dfrac{1}{\varrho}$, or the law of force, which must hold, in order that an indefinite bar may produce the same effect on points at all distances, is that of the reciprocal of the distance.

Cor. 3. If $\phi(z)=z^{\frac{n-1}{2}}$; we have, by writing $\dfrac{n-1}{2}$ for m,

$$(-1)^{\frac{1}{2}}\frac{\sqrt{\pi}}{2}\frac{\sin\frac{n}{2}\pi}{\sin\frac{n-1}{2}\pi}\cdot z^{\frac{n-1}{2}}=(-1)^{\frac{1}{2}}\,\mathrm{P}\,.\,\frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}\,\frac{\sin\frac{n\pi}{2}}{\sin\frac{n-1}{2}\pi}\,.\,z^{\frac{n-1}{2}}$$

or
$$\mathrm{P}=\frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}\,.$$

If n be odd, this gives

$$\mathrm{P}=\frac{\sqrt{\pi}}{2}\,\frac{\frac{n-1}{2}\cdot\frac{n-3}{2}\cdots\frac{n-\overline{n-2}}{2}\,\Gamma(1)}{\frac{n}{2}\cdot\frac{n-2}{2}\cdots\frac{n-\overline{n-1}}{2}\,\Gamma\left(\frac{1}{2}\right)}$$

$$=\frac{2}{2}\cdot\frac{(n-1)(n-3)\ldots 2}{n\ (n-2)\ldots 1}\cdot\frac{\sqrt{\pi}}{\sqrt{\pi}}$$

$$=\frac{1\,.\,2\,.\,3\,\ldots\,n}{(1\,.\,3\,.\,5\,\ldots\,n)^2}$$

If n be even, we obtain

$$\mathrm{P}=\frac{\sqrt{\pi}}{2}\cdot\frac{\frac{n-1}{2}\cdot\frac{n-3}{2}\cdots\frac{n-\overline{n-1}}{2}\,\Gamma\left(\frac{1}{2}\right)}{\frac{n}{2}\cdot\frac{n-2}{2}\cdots\frac{2}{2}\,\Gamma(1)}$$

$$=\frac{\pi}{2}\ \frac{1.3\ \ldots\ (n-1)}{2.4.6\ldots\ n}$$

$$=\frac{\pi}{2}\cdot\frac{1.2\ \ldots\ n}{(2.4\ \ldots\ n)^2}$$

Cor. 4. If the force of attraction $\propto \log\frac{1}{a}$: let it equal $\log\frac{c}{a^2}$;

$$\therefore \quad \phi(z) \propto \frac{d^{\frac{1}{2}}}{dz^{\frac{1}{2}}} \log C z$$

$$\propto \frac{1}{\sqrt{z}}$$

$$\therefore \quad \phi\left(\frac{1}{a^2}\right) \propto a$$

and $$\phi\left(\frac{1}{\varrho^2}\right) \propto \varrho$$

or $$\varrho^2 f(\varrho) \propto \varrho$$

$$f(\varrho) \propto \frac{1}{\varrho}\ .$$

We have written down this case because at the first sight it appears anomalous. We know indeed that, when the force of attraction is *constant*, $\log . \frac{1}{a}$ enters into the expression for the whole force; but we must remember that this force is infinite, so that it does not in reality vary as $\log . \frac{1}{a}$. But, in addition to this, we found above that, when the force of attraction varies inversely as the distance, the attraction on a point is constant. But the anomaly is easily explained when we reflect that the differential coefficient of a constant to the index $\frac{1}{2}$ is of the same form as that of the logarithm of z; and further, that the actual value of the attraction is expressed in the form of a circular function, viz. $\left(\cos^{-1}\frac{a}{\varrho}\right)$, which is equivalent to $\sqrt{-1}\ \log\ \left(a+\sqrt{\frac{a^2}{\varrho^2}-1}\right)$, a quantity which, when ϱ is ∞, varies as $\log a$.

Let us now pass on to the more ordinary problem of determining the law of attraction, by which the whole attraction of an infinite plane on a point without it, may vary inversely as the n^{th} power of the distance. Retaining the previous notation:

attraction $$= 2\pi\int_0^\infty r\,dr\,f(\varrho)\,\frac{a}{\varrho}$$

$$= 2\pi\int_a^\infty a f(\varrho)\,d\varrho$$

$$= 2\pi a\int_a^\infty f(\varrho)\,d\varrho$$

$$= 2\pi a \int_a^\infty - \varrho^2 \frac{d\frac{1}{\varrho}}{d\varrho} d\varrho f(\varrho)$$

$$= 2\pi a \int_0^{\frac{1}{a}} d\left(\frac{1}{\varrho}\right) . f(\varrho) \varrho^2$$

$$= 2\pi a \int_0^{\frac{1}{a}} d\theta \, \phi(\theta)$$

if $\phi(\theta) = f(\varrho)\varrho^2$, and $\theta = \frac{1}{\varrho}$; therefore also $z = \frac{1}{a}$.

By applying the formula, this gives

$$2\pi a (-1)^{-1} \cos \pi \sqrt{-1} \frac{d^{-1}\phi(z)}{dz^{-1}}.$$

But, according to the hypothesis, the attraction equals $\frac{P}{a^n}$:

$$\therefore \qquad \frac{P}{a^n} = 2\pi a \frac{d^{-1}\phi(z)}{dz^{-1}}$$

$$\text{or} \qquad \frac{d}{dz} \frac{P}{a^{n+1}} = 2\pi \phi(z)$$

$$\text{or} \qquad \frac{d}{dz} P z^{n+1} = 2\pi \phi(z)$$

$$\therefore \qquad \phi(z) = \frac{P}{2\pi}(n+1) z^n$$

$$\text{or} \qquad \phi\left(\frac{1}{a}\right) = \frac{P}{2\pi}(n+1)\frac{1}{a^n}$$

$$\varrho^2 f(\varrho) = \phi\left(\frac{1}{\varrho}\right) = \frac{P(n+1}{2\pi} \frac{1}{\varrho^n}$$

$$f(\varrho) = \frac{(n+1)P}{2\pi} . \frac{1}{\varrho^{n+2}}.$$

Cor. If $n=0$, $f(\varrho) = \frac{1}{\varrho^2}$ the law of nature: hence an infinite plane attracts all points equally, provided they are not in its mass.

From this corollary, it appears that, if a particle of the infinite ether which pervades space (in equilibrium) be moved from its position, the only series of particles by which it will be affected, is that which lies in that plane perpendicular to its line of motion which passes through its position of rest.

Let us next solve a few of the more simple inverse problems of Mechanics. We desire to confine our attention to the more simple, from a wish not to introduce any formula other than that which commences our memoir; and likewise, from a fear of otherwise distracting the attention which we desire to draw to the subject of differentiation itself.

Prob. 1. To find the curve which synchronizes all straight lines drawn through the origin of motion.

Let the line itself be called x; θ the angle which it makes with the vertical; g the force of gravity; then, by the ordinary formula:

$$\text{Time} = \int \frac{dx}{\sqrt{2gx\cos\theta}};$$

the limits of the integral being $x=0$ and $x=z$, where z is the distance from the origin of motion to the synchronizing curve.

Hence, by our formula,

$$\text{Time} = \int_0^z dx\, x^{-\frac{1}{2}} \cdot \frac{1}{\sqrt{2g\cos\theta}}$$

$$= \frac{d^{-1}}{dz^{-1}} \frac{1}{\sqrt{z}\sqrt{2g\cos\theta}}$$

$$= \frac{2\sqrt{z}}{\sqrt{2g\cos\theta}} = \text{constant}$$

$$\therefore \qquad z \propto \cos\theta$$

and the synchronizing curve is the circle.

We can solve this problem by another process, which beautifully illustrates our formula.

Let the origin of measure be the *lowest* point of the line; then the expression for the time is

$$\int_0^z \frac{dx}{\sqrt{2g(z-x)\cos\theta}} \propto \int_0^z \frac{dx}{\sqrt{\cos\theta}} (z-x)^{-\frac{1}{2}}$$

Hence p in the formula is $-\frac{1}{2}$, and $\phi(x+\alpha) = \frac{1}{\sqrt{\cos\theta}}$

$$\therefore \qquad \int_0^z \frac{dx}{\sqrt{\cos\theta}} (z-x)^{-\frac{1}{2}} \propto \frac{d^{-\frac{1}{2}}}{dz^{-\frac{1}{2}}} \cdot \frac{1}{\sqrt{\cos\theta}}$$

$$\propto \frac{z^{\frac{1}{2}}}{\sqrt{\cos\theta}}$$

which being constant by hypothesis $z \propto \cos\theta$ as before.

Prob. 2. To find the tautochronous curve when a body descends by the action of gravity. Retaining the notation of the last problem, measuring from the lowest point,

$$t = \int_0^z \frac{\frac{ds}{dx}\,dx}{\sqrt{2g(z-x)}}$$

Now, by the conditions of the problem, this is to be the independent of z.

Let, therefore, $$\frac{ds}{dx} = \phi(x)$$

and $$t = \int_0^z \frac{\phi(x)\,dx}{\sqrt{2g}\sqrt{z-x}}$$

$$= \mathrm{C} \, . \, \frac{d^{-\frac{1}{2}}}{d z^{-\frac{1}{2}}} \, \phi(z)$$

$$\therefore \quad \frac{d^{-\frac{1}{2}}}{d z^{-\frac{1}{2}}} \, \phi(z) = \text{a const.}$$

$$= \text{A suppose;}$$

$$\text{and} \quad \phi(z) = \frac{d^{\frac{1}{2}}}{d z^{\frac{1}{2}}} \, \mathrm{A}$$

$$= \mathrm{A} \, z^{-\frac{1}{2}}$$

$$\therefore \quad \phi(x) = \frac{\mathrm{A}}{\sqrt{x}}$$

$$\text{or} \quad \frac{d s}{d x} = \frac{\mathrm{A}}{\sqrt{x}}$$

the differential equation to the cycloid.

Lest any difficulty should be felt in this example from the value of A being apparently zero, we think it advisable to write down the full value, which we can easily do by retracing our steps:

$$t = \frac{1}{\sqrt{2g}} (-1)^{\frac{1}{2}} \, \Gamma \tfrac{1}{2} \, \frac{\sin 0 \, \pi}{-\sin \frac{1}{2} \pi} \, . \, \frac{d^{-\frac{1}{2}}}{d z^{-\frac{1}{2}}} \, \phi(z)$$

since $\quad m = -\frac{1}{2} \; p = -\frac{1}{2}$:

hence, if a be the constant time, we get

$$\phi(z) = \frac{\sqrt{2g}}{\sqrt{\pi} \sqrt{-1}} \, \frac{-1}{\sin 0 \, \pi} \, . \, \frac{d^{\frac{1}{2}}}{d z^{\frac{1}{2}}} \, . \, a$$

$$= \frac{\sqrt{2g}}{\sqrt{\pi} \sqrt{-1}} \, . \, \frac{-1}{\sin 0 \, \pi} \, . \, \frac{\sin 0 \, \pi}{\sqrt{\pi}} (-1)^{\frac{3}{2}} \, \frac{a}{z^{\frac{1}{2}}}$$

$$= \frac{\sqrt{2g}}{\pi} \, . \, \frac{a}{\sqrt{z}}$$

Hence $\dfrac{d s}{d x} = \dfrac{\sqrt{2g}}{\pi} \, . \, \dfrac{a}{\sqrt{x}}$ a result which coincides with that obtained by the ordinary process of expansion.

The facility which this process affords in the solution of the more simple converse problems of Geometry is very evident. The following examples will sufficiently illustrate this remark.

Ex. 1. To find a curve such, that the area varies as the n^{th} power of the abscissa. The general expression for the area of any curve is $\int_0^z y \, d x$.

Hence, if $y = \phi(x)$ be the equation to the curve, and $\mathrm{P} z^n$ the function according to which the area is to vary, we shall have

$$\int_0^z y \, d x = \mathrm{P} \, . \, z^n$$

but $$\int_0^z y\,dx = \frac{d^{-1}}{dz^{-1}}\,\phi(z)$$

$$\therefore \quad \frac{d^{-1}}{dz^{-1}}\,\phi(z) = \mathrm{P}\,.\,z^n$$

or $$\phi(z) = n\,\mathrm{P}\,z^{n-1}$$

hence $y=\phi(x)=n\,\mathrm{P}\,.\,x^{n-1}$ is the equation to the curve.

Ex. 2. To find a curve such that the area shall vary as the logarithm of the abscissa.

In this case, we must suppose the origin to be at a distance from the place at which the area commences, in order to prevent the appearance of ∞ in the operations.

Let, therefore, $y=\phi(x+\alpha)$ be the equation to the curve, the limits of integration being $x=0$ $x=z$,

then $$\int_0^z \phi(x+\alpha)\,dx = \mathrm{P}\,.\log z \qquad \text{by the question}$$

or $$\frac{d^{-1}}{dz^{-1}}\,\phi(z+\alpha) = \mathrm{P}\,.\log z$$

$$\phi(z+\alpha) = \frac{d}{dz}\,\mathrm{P}\,.\log z$$

$$= \frac{\mathrm{P}}{z}$$

or $$y = \frac{\mathrm{P}}{x}$$

$$x\,y = \mathrm{P}$$

which is the equation to the hyperbola.

Ex. 3. To find a curve such that the volume of the solid generated by its revolution round the axis of x shall be a certain function of x.

Let $y^2=\phi(x)$ be its equation;

$$\therefore \quad \text{volume} \quad = \pi\int_0^z \phi(x)\,dx$$

$$= \pi\,\frac{d^{-1}}{dz^{-1}}\,\phi(z) = f(z)$$

$f(x)$ being the *given* function of x.

$$\therefore \quad \phi(z) = \frac{1}{\pi}\,\frac{d}{dz}\,f(z)$$

$$y = \frac{1}{\sqrt{\pi}}\sqrt{\frac{d}{dz}\,f(z)}$$

is the equation required.

Cor. If $f(z)=\mathrm{P}\,z^n$

$$y = \frac{1}{\sqrt{\pi}}\,\sqrt{n\,\mathrm{P}\,.\,z^{n-1}}$$

$$= \sqrt{\frac{n\,\mathrm{P}}{\pi}}\,.\,z^{\frac{n-1}{2}}$$

Ex. 4. A curve is described having a line of given length as its axis. From the further extremity of the line is described a reversed parabola, having a common axis with that of the curve. A third curve is then described, whose ordinate is a mean proportional between the ordinates of the former curves, and such that the volume of the solid described by it between the limits of the line in question is a certain given function of the length of the line: Required the equations to the two curves?

Let $y=\phi(x)$ be the equation to the first mentioned curve;

z the length of the line, which is made the axis of x;

$f(z)$ the function of z according to which the volume of the solid swept out by the last curve varies.

Then $y^2=\sqrt{m(z-x)}\,.\,\phi x$ is the equation to this curve; m being the *latus rectum* of the parabola.

Therefore $\pi\sqrt{m}\int_0^z dx\sqrt{z-x}\,\phi(x)$ is the value of the volume of the solid swept out; so that $\pi\sqrt{m}\int_0^z dx\sqrt{z-x}\,\phi(x)=f(z)$ by the question;

or $$\pi\sqrt{m}(-1)^{\frac{3}{2}}\,\sqrt{\tfrac{3}{2}}\,\frac{\sin\left(m+\frac{3}{2}\right)\pi}{\sin m\pi}\,\frac{d^{-\frac{3}{2}}}{dz^{-\frac{3}{2}}}\,\phi(z)=f(z)$$

$$\therefore \quad \phi(z)=\mathrm{A}\,\frac{d^{\frac{3}{2}}}{dz^{\frac{3}{2}}}f(z)$$

A being some constant.

And consequently $\phi(x)=\mathrm{A}\,\dfrac{d^{\frac{3}{2}}}{dx^{\frac{3}{2}}}f(x)$ is the equation to the first curve.

The second is immediately deducible from it.

Cor. 1. Let $$f(z)=z^n$$

$$\therefore \quad \frac{d^{\frac{3}{2}}}{dx^{\frac{3}{2}}}f(x)=\mathrm{C}\,x^{n-\frac{3}{2}}$$

and $$\phi(x)\propto x^{n-\frac{3}{2}}\,.$$

Cor. 2. If $n=2$ $\phi(x)\propto x^{\frac{1}{2}}$.

In this case both the curves are parabolas, and the volume of the solid varies as the area of a circle, whose diameter is the given line.

I shall now conclude the series of examples. It was originally my intention to have exemplified the theorem of expansion given in my preceding memoir; but, on consideration, I deem it advisable to confine the present series to the illustration of the theorem which forms the commencement of the paper. I hope at some future period, should no one render it unnecessary, to return to this subject; and look in the mean time for the fruit which shall be produced by a more extended culture of the science.

Edinburgh, January 20. 1840.

XXX.—*On Sulphuret of Cadmium, or Greenockite, a new Mineral.* By ARTHUR CONNELL, *Esq., F.R.S.E.*

(Read 16th March 1840.)

This mineral is found embedded in small crystals in prehnite, at Bishoptown, in Renfrewshire. It had been long supposed by mineralogists to be a variety of zinc-blende; but it was first distinguished from that mineral by Lord GREENOCK, who communicated his opinion to Professor JAMESON; and two small crystals, together less than a grain in weight, were sent to me for chemical examination by the latter, who concurred in the supposition that it was a new mineral.

The crystals sent appeared to be six-sided pyramids, having the faces transversely streaked. Their colour was wine-yellow. Fracture, conchoidal. Lustre, shining or splendent, and vitreous. Hardness about that of calcareous spar; streak orange-red; semitransparent.

A small fragment heated in a glass-tube acquired a beautiful deep carmine-red colour, and on cooling recovered its yellow tint. At a red heat it did not fuse nor volatilize. These reactions at once distinguished the mineral from the native sulphurets of arsenic, to which it bore some external resemblance. In an open glass-tube, the appearances were exactly the same, even when urged by the blowpipe; it became as before deep red, and on cooling recovered its yellow colour, retaining its lustre and transparency. When a somewhat larger fragment was heated in a glass-tube, it decrepitated violently before assuming the red tint, but no evolution of vapour was observed; and when the particles into which it separated were collected together into one place, and heated till almost black over a spirit-lamp, and then shifted into a different part of the tube, every depth of tint of red was observed according to the temperature.

In powder it was readily soluble in muriatic acid, by the aid of heat, exhaling a strong smell of sulphuretted hydrogen. Carbonate of soda caused a white precipitate, dissolved by ammonia. The muriatic solution by evaporation afforded a white prismatic crystallization, not deliquescing in an ordinary atmosphere. This character distinguished the mineral from zinc-blende, with which the previous reactions had closely corresponded, and suggested the idea that it might be sulphuret of cadmium, a supposition farther strengthened by finding the above-mentioned change of colour by heat described by BERZELIUS as a character of the artificial sulphuret of cadmium. It was next found that the precipitates caused by potash and by carbonate of ammonia were not dissolved by excess of the precipitants. These reactions all tended to confirm the above idea of the nature of the contained metal; but what left the matter no longer doubtful, when taken in conjunction with the above-mentioned characters, was observing that a muriatic solution with excess of acid, gave a fine yellow precipitate, with a current of sulphuretted hydrogen, exactly similar to that obtained by the same means from a similar solution of metallic cadmium. When a precipitate was no longer caused by the current of sulphuretted hydrogen, and the solution of the mineral was then neutralized by ammonia, a few dark flocks fell of sulphuret of iron. Through the neutralized liquid a fresh current of sulphuretted hydrogen was passed, but no farther precipitation ensued, shewing the absence of zinc, a conclusion farther confirmed by finding that the excess of potash and of carbonate of ammonia, used as precipitants, took up nothing.

The muriatic solution of the mineral gave a yellow precipitate with hydrosulphuret of ammonia, and white precipitates with prussiate of potash, oxalate of ammonia, and phosphate of soda; and no precipitate with sulphuric acid. A piece of zinc threw down reduced metal as a grey ramification.

These various reactions left no doubt that the mineral under examination was sulphuret of cadmium; that it contained no sensible admixture of zinc; and that the only impurity which could be detected was a slight trace of iron. The different observations were farther confirmed by comparative trials made on a solution of metallic cadmium. It was therefore quite evident that the mineral was not only a new one, but one of much interest, since, so far as I know, no *separate* ore of cadmium had ever before been discovered; that metal having hitherto been found merely as a constituent, or more probably as an admixture, in certain ores of zinc, to the extent of a few per cents.

The materials sent to me by Professor JAMESON gave no farther means of prosecuting the examination of the mineral, either chemically or in relation to specific gravity; but by the kindness of Lord GREENOCK, I was furnished for these purposes with the largest, although not the most perfectly formed, crystal which I have yet had an opportunity of seeing. His Lordship has also lately obtained one very finely crystallized specimen, although not of a large size, which is evidently a six-sided pyramid, without any transverse streaking of the faces, and terminating in a short six-sided prism; but as the crystalline form of the mineral is under investigation by Professor JAMESON, and there are some modifications which will require a careful examination, I wish to say nothing farther on the subject of its crystallization, except as respects the particular crystal analyzed.

This large crystal was a somewhat imperfectly formed six-sided pyramid, one of the faces of the pyramid being apparently obliterated by the extension of the two contiguous; the faces being transversely streaked, and with traces also of a six-sided prism. It possessed a slightly reddish yellow colour, and considerable transparency, except in one or two small points, which were dark coloured and opaque. Its streak was orange-red as that of the others. When detached it weighed 3.68 grains. Suspended in distilled water by a fine hair, it lost .76 of a grain, giving its specific gravity as 4.842 at 60° F.; which thus considerably exceeds the specific gravity of zinc-blende. I then detached from it the darker and opaque particles, and substituted for them a small quantity of yellow and transparent portions from another crystallized specimen, also given me by Lord GREENOCK.

3.71 grains thus selected were reduced to somewhat coarse powder, and fuming nitric acid was poured on them, drop by drop, in a deep flask. The action was violent, and attended by a copious evolution of red fumes, but not the least smell of sulphuretted hydrogen was observed. An excess of nitric acid was then added, and the whole digested till all the sulphur which had separated was dissolved. Water was then added, and the sulphuric acid thrown down by muriate of barytes. The sulphate of barytes, after being well washed with hot water, was dried and ignited, and weighed 6.07 grains, equivalent to .837 of sulphur.

The excess of barytes was then removed from the liquid, after concentration by heat, by sulphuric acid. After again concentrating, carbonate of ammonia was added in excess. The carbonate of cadmium was separated by filtration, and well washed, dried, and ignited. The oxide of cadmium thus obtained had an ochre-yellow colour, and weighed 3.28 grains, equivalent to 2.868 of cadmium. A little of it dissolved in muriatic acid, was entirely taken up by excess of ammonia.

The filtered liquid was then evaporated by heat, but no precipitation had taken place when all smell of carbonate of ammonia had disappeared; thereby confirming the previous observations as to the absence of zinc. The evaporation was carried to dryness, and the ammoniacal salt driven off by heat. A residue of .04 remained, of a reddish-white colour, which, in so far as its small quantity permitted examination, was found to be, in part at least, a subsulphate of iron, insoluble in water, and scarce soluble even in acids till previously boiled with potash; but as the proportion of its constituents could not be determined on so little material, the iron could not be computed in any other way than by stating it as a trace in the mineral, its amount, on any view, being very small; and if more than such, this was not the stage of the analysis in which it ought to have been obtained.

We have thus, in the 3.71 grains of the mineral under analysis,

Sulphur,	.837	22.56
Cadmium,	2.868	77.30
Iron, traces,		
	3.705	99.86

which agrees completely with the theoretical composition of

1 atom Sulphur,	201.16	22.40
1 atom Cadmium,	696.76	77.59
	897.92	99.99

The mineral is thus a protosulphuret of cadmium, and its formula Cd S.

It is thus evidently, both physically and chemically, a perfectly well characterized and distinct species. The mineral which it ranks nearest in the system is zinc-blende, but from this it differs essentially, not only in its chemical nature, but in its external characters, such as specific gravity, and form of crystallization. I believe a ready mode of distinguishing it from the transparent yellow blende, which it resembles a good deal, and with which it was long confounded, is afforded by the streak, that of the latter being white, whilst that of sulphuret of cadmium is orange-red. The property of becoming red by heat, and returning to yellow on cooling, is possessed in a slight degree by yellow zinc-blende; but the colour which this latter mineral acquires is not carmine, but a sombre rose tint, and never becomes very deep; and by a repetition of the heating process, when carried to redness, it gradually loses the property altogether, along with its transparency; whereas the sulphuret of cadmium may be ignited as often as thought proper, without losing the property or its transparency. This quality, to the extent in which zinc-blende possesses it, does not appear to depend in that mineral on the presence of cadmium, for I was unable to detect that metal in a crystallized and transparent yellow zinc-blende from the Hartz, belonging to Mr Rose of this city, which acquired a rose tint by heat; and it is scarce necessary to say, that the quality is not possessed by the artificial white sulphuret of zinc thrown down by sulphuretted hydrogen. It therefore appears to depend, in zinc-blende, on the arrangement of particles in crystallization; whilst, in sulphuret of cadmium, it is an inherent quality of the substance, being possessed in perfection by the artificial sulphuret of cadmium, got by sulphuretted hydrogen.

The reactions before the blowpipe may also serve to distinguish the two minerals. It is difficult to act on sulphuret of cadmium, *per se*, from its decrepitating property; but when this can be accomplished on charcoal, the usual yellowish-red ring, arising from the oxidation of sublimated cadmium, is formed around the fragment. When mixed with soda, and acted on, on charcoal, this ring continues to be formed to the last, whilst, with zinc-blende and soda, a white ring is formed; and in those zinc-blendes which contain a little cadmium, the red ring, according to BERZELIUS, is formed at first only, and is succeeded by the white sublimate of zinc. With borax, the sulphuret of cadmium gives a transparent yellow glass.

I was extremely happy to find, that the idea had occurred to Professor JAMESON, which was early suggested to myself, that the mineralogical name of this substance should be derived from that of the distinguished nobleman who first observed it; and it is satisfactory to think, that the beauty of this mineral, and its interesting nature and properties, render it not altogether unworthy of being associated with Lord GREENOCK's name. I cordially concur in the name of Greenockite, which has been already proposed for it by Professor JAMESON.

It is already known that the artificial sulphuret of cadmium may be used as a pigment. This will also apply to the native, if it could be got in sufficient quantity. I have had some experiments made on a minute scale with this view, with the finely ground powder of the mineral, used as a water colour. Its tint is an orange-yellow, differing from that of any of the ordinary yellow pigments. Mixed with blue it gives a green.

Since the above paper was read, Mr NICOL has informed me that he has satisfied himself that Greenockite possesses the property of depolarizing light. Professor FORBES has also obtained distinct proof of the same fact; and has farther observed a curious effect of dichroism by polarized light, which does not take place with common. These observations lead to the same conclusion which its apparent crystalline form suggested, that the mineral does not belong to the tessular system. But farther investigation is probably necessary to determine whether the form belongs to the rhombohedral system, as the fine crystal already referred to would seem to indicate, or to the prismatic, as the observations of Mr BROOKE, published by Professor JAMESON (Ed. Phil. Journ.) since this paper was read, would appear to suggest.

XXXI.—*Solution of a Functional Equation, with its Application to the Parallelogram of Forces, and to Curves of Equilibration.* By William Wallace, *LL.D., F.R.S.E., F.R.A.S., M. Camb. Phil. S., Hon. M. Inst. Civ. Engin., Emeritus Professor of Mathematics in the University of Edinburgh.*

(Read 2d December 1839.)

Article 1. The introduction of the notion of a *function* of a variable quantity into the mathematics, without any regard to its particular form, has given vast extension to the science, and been the germ of some of its most important theories. The doctrine of curve lines, no doubt, produced that of functions, for the former may be made the visible expression of the latter: thus, either of the co-ordinates of a curve being taken as the representation of the *variable*, the other co-ordinate is a *function* of the variable; so also are the arc of the curve, and its area. Indeed, in contemplating functions, and discussing their properties, it is convenient to substitute in our reasonings the geometrical representation for the abstract notion of the function.

2. In addition to the aid which geometry gives us in forming distinct notions of the relations of functions, the notation of modern analysis affords farther assistance in discussing their properties. In our Trigonometrical Tables, the cosine, sine, tangent, &c., are all regarded as functions of the angle; and the calculus of sines is, in fact a creation of the mind, called into existence by the power of a few abbreviations of the words *sine, cosine, &c.*, which, as symbols, serve in our processes of reasoning to represent the things they signify.

3. From the notion of a function, which we acquire from geometrical extension, combined with the use of the arbitrary symbols of analysis, we learn that it presents to the mind two distinct objects; namely, its form, and its properties: for example, the function *log x*, that is the logarithm of a number x, may be represented geometrically by the ordinates of a curve; also, by spaces between a hyperbola and its asymptote; these ordinates and spaces have certain relations to each other, which are the *properties of the function.* Among its analytical properties there is this one,

$$\log x + \log y = \log (xy) ,$$

which is deducible from this definition of the function, that *it is the exponent of the power of some given number, which number being raised to that power, produces x.*

4. There being a necessary connection between the form of a function and its properties, by which, from the former, we may deduce the latter, it follows that, reversely, when the properties of a function are known, we may from these deduce its form.

Hence it appears that, relatively to the form and properties of a function, there may be a *direct* and an *inverse* theory; by the one, the properties are deduced from the form: and by the other, the form from the properties. These will be analogous to other reverse theories; as involution and evolution, or the direct and inverse methods of fluxions, &c. But here again it happens, as in the theories just mentioned, that the difficulties to be encountered in the inverse are greater than those presented in the direct theory.

5. Let us take as an example the function $y=\log x$; from this, by the definition of a logarithm, $x=a^y$, a being a given number; we have similarly $y'=\log x'$, and $x'=a^{y'}$; hence, $xx'=a^y \, . \, a^{y'}=a^{y+y'}$, and $y+y'=\log(xx')$; that is,

$$\log x+\log x'=\log(x x') \, . \qquad (1)$$

Here we have easily deduced a property of the function from its form.

The reverse problem requires that we find a function of x whose form is unknown, and which, being expressed by the symbol $f(x)$, has this property.

$$f(x)+f(x')=f(xx') \, . \qquad (2)$$

But the algebraic analysis that so readily applied to the former problem, does not so easily apply to the latter. This last equation (2), in which the form of the function $f(x)$ is unknown, is called a *functional equation*; and it is *resolved*, when the equation (1) has by a legitimate process been deduced from it.

6. In physical inquiries, functional equations may occur, by the solution of which the physical laws and their consequences may be discovered. I propose in this memoir to give two examples of such an application of this theory, to the doctrine of *statics*. In the first I shall deduce the known law of the equilibrium of three pressures applied at a point from a functional equation; and, in the second, from the same equation, investigate some elegant properties of curves of equilibration, which are applicable to the construction of bridges.

7. Let x denote a variable quantity, and $f(x)$ a function of the variable; also let $x_{\circ}$ and $x_{,}$ denote two values of x, which are entirely independent of each other, and c a constant quantity. Let us suppose the function $f(x)$ to be such as satisfies this equation,

$$f(x_{\circ})\,.\,f(x_{,})=c\,\{f(x_{\circ}+x_{,})+f(x_{\circ}-x_{,})\} \quad . \quad . \quad . \qquad \text{A}$$

It is proposed to determine all the possible forms of the function.

8. There is a very simple property of a function, from which I propose to deduce the solution. It is this:

The partial differential coefficient of a function which is the sum of two inde-

pendent variables, is the very same function, whichever of the two be reckoned variable, the other continuing constant. That is, supposing x_0 and $x_,$ to be independent variables, and y any function of $x_0+x_,$ represented by $f(x_0+x_,)$, then

$$\frac{dy}{dx_0}=f'(x_0+x_,)\ ; \qquad \frac{dy}{dx_,}=f'(x_0+x_,)=\frac{dy}{dx_0}.$$

This property, which is sufficiently known, may be exemplified by a particular case. Suppose $y=(x_0+x_,)^n$, then, making x_0 variable, and $x_,$ constant,

$$\frac{dy}{dx_0}=n(x_0+x_,)^{n-1}\ ;$$

and making $x_,$ variable, and x_0 constant,

$$\frac{dy}{dx_,}=n(x_0+x_,)^{n-1}.$$

9. Applying now this property to the functional equation (A); making x_0 variable, and $x_,$ constant, we have

$$\frac{df(x_0)}{dx_0}f(x_,)=cf'(x_0+x_,)+cf'(x_0-x_,),$$

$$\frac{d^2f(x_0)}{dx_0^2}f(x_,)=cf''(x_0+x_,)+cf''(x_0-x_,).$$

Again, differentiating the same function, and making x_0 constant, we have

$$\frac{df(x_,)}{dx_,}f(x_0)=cf'(x_0+x_,)-cf'(x_0-x_,),$$

$$\frac{d^2f(x_,)}{dx_,^2}f(x_0)=cf''(x_0+x_,)+cf''(x_0-x_,).$$

Now the right hand side of the second differential equation being the same on either hypothesis, we have

$$\frac{d^2f(x_0)}{dx_0^2}f(x_,)=\frac{d^2f(x_,)}{dx_,^2}f(x_0)\ ;$$

and, putting y_0 for $f(x_0)$, and $y_,$ for $f(x_,)$,

$$\frac{d^2y_0}{dx_0^2}\,\frac{1}{y_0}=\frac{d^2y_,}{dx_,^2}\,\frac{1}{y_,}.$$

The two sides of this equation are functions of the same form, the one of x_0 and the other of $x_,$, and, by hypothesis, these quantities are independent of each other; therefore, each must necessarily be equal to some constant quantity, which is the same for both: so that we have

$$\frac{d^2y_0}{dx_0^2}\cdot\frac{1}{y_0}=a\ constant\ ;$$

and, in general, denoting $f(x)$ by y,

$$\frac{d^2y}{dx^2}\cdot\frac{1}{y}=a\ constant.$$

10. We may consider this differential equation, and the functional equation

$$f(x_\circ)f(x_,)=c\,\{f(x_\circ+x_,)+f(x_\circ-x_,)\}$$

as the representatives of each other, so that if x and y be co-ordinates of a curve, the functional equation will express a property of that curve. Now, considering y as a function of x, the function and its variable may either increase together, or else y may decrease while x increases: therefore, it may be, that the function which satisfies the equation (A) will have different forms; for if y decrease while x increases, the differential coefficient $\frac{dy}{dx}$ will be negative; if, however, y and x increase together, then it will be positive.

11. Let us first consider the case in which y decreases while x increases. The differential equation to be resolved may then be expressed thus, (putting c^2 for a constant),

$$\frac{d^2y}{dx^2}\cdot\frac{1}{y}=-\frac{1}{c^2},$$

or

$$d\left(\frac{dy}{dx}\right)=-\frac{y\,dx}{c^2};$$

and, multiplying both sides by $\frac{dy}{dx}$,

$$\frac{dy}{dx}\cdot d\left(\frac{dy}{dx}\right)=-\frac{y\,dy}{c^2};$$

and taking the integrals,

$$\left(\frac{dy}{dx}\right)^2=b-\frac{y^2}{c^2}.$$

Now, $\frac{dy}{dx}$ expresses the tangent of the angle which a line touching the curve makes with the axes to which y is a perpendicular ordinate: and since $\frac{y^2}{c^2}$ manifestly cannot exceed b, therefore y must have a *maximum* value, which must satisfy the equation $\frac{dy}{dx}=0$, and putting a for that *maximum* value of y, we have $b-\frac{a^2}{c^2}=0$: and $b=\frac{a^2}{c^2}$, therefore, $\left(\frac{dy}{dx}\right)^2=\frac{a^2-y^2}{c^2}$;

and

$$\frac{dx}{c}=\frac{dy}{\sqrt{(a^2-y^2)}}=\frac{\frac{dy}{a}}{\sqrt{\left(1-\frac{y^2}{a^2}\right)}};$$

from this, by integration, we get

$$\cos\left(\alpha-\frac{x}{c}\right)=\frac{y}{a},\ \text{or}\ \sin\left(\alpha-\frac{x}{c}\right)=\sqrt{\left(1-\frac{y^2}{a^2}\right)}:$$

We may assume that, when y is a *maximum*, and $=a$, then $x=0$; therefore

$$\cos\alpha=1,\ \text{and}\ \sin\alpha=0.$$

Now, $$\frac{x}{c} = \alpha - \left(\alpha - \frac{x}{c}\right);$$

and $$\cos\frac{x}{c} = \cos\alpha\cos\left(\alpha - \frac{x}{c}\right) + \sin\alpha\sin\left(\alpha - \frac{x}{c}\right):$$

that is, $$\cos\frac{x}{c} = \frac{y}{a}.$$

Hence it appears, that one form of the function y is

$$y = f(x) = a\cos\frac{x}{c}:$$

here a is the value of $f(x)$ when $x=0$, and c is an arbitrary constant.

12. The second case of the differential equation, in which x and y increase together, is this,

$$\frac{d^2y}{dx^2}\cdot\frac{1}{y} = \frac{1}{c^2} \quad \text{or} \quad d\left(\frac{dy}{dx}\right) = \frac{y\,dx}{c^2}:$$

hence, as before, multiplying both sides by $\frac{dy}{dx}$, and integrating, we get

$$\left(\frac{dy}{dx}\right)^2 = \frac{y^2}{c^2} - b:$$

Let a be the value of y when it is the least possible; this must satisfy the equation $\frac{dy}{dx} = 0$; and we have

$$\frac{a^2}{c^2} - b = 0, \quad \text{or} \quad b = \frac{a^2}{c^2}:$$

we may assume, as before, that a is the value of y when $x=0$: we have now

$$\left(\frac{dy}{dx}\right)^2 = \frac{y^2 - a^2}{c^2},$$

and $$\frac{dx}{c} = \frac{dy}{\surd(y^2 - a^2)}.$$

To integrate this equation, let us assume that

$$y = \frac{a}{2}\left(u + \frac{1}{u}\right);$$

then $$dy = \frac{a}{2}\left(1 - \frac{1}{u^2}\right)du = \frac{a}{2}\cdot\frac{u^2-1}{u^2}du;$$

$$y^2 - a^2 = \frac{a^2}{4}\left(u^2 - 2 + \frac{1}{u^2}\right) = \frac{a^2}{4}\left(u - \frac{1}{u}\right)^2;$$

$$\surd(y^2 - a^2) = \frac{a}{2}\left(u - \frac{1}{u}\right) = \frac{a}{2}\cdot\frac{u^2-1}{u};$$

and by substitution in the differential equation,

$$\frac{dy}{\surd(y - a^2)} = \frac{1}{2}a\cdot\frac{u^2-1}{u^2}\cdot\frac{2}{a}\cdot\frac{u}{u^2-1}du = \frac{du}{u};$$

therefore $$\frac{dx}{c} = \frac{du}{u}, \quad \text{and} \quad \frac{x}{c} = \log u + \log b.$$

Now, when $x=0$, then $y=a$, and since $y=\frac{a}{2}\left(u+\frac{1}{u}\right)$, therefore $u+\frac{1}{u}=2$, and $u^2+2+\frac{1}{u^2}=4$, and $u^2-2+\frac{1}{u^2}=0$, and $u-\frac{1}{u}=0$; hence $u^2=1$, and $2\log u=0$, and $\log b=0$. On the whole,

$$\frac{x}{c}=\log u;$$

and
$$u=e^{\frac{x}{c}}, \text{ and } \frac{1}{u}=e^{-\frac{x}{c}},$$

and
$$y=\frac{a}{2}\left(u+\frac{1}{u}\right)=\frac{a}{2}\left\{e^{\frac{x}{c}}-e^{-\frac{x}{c}}\right\},$$

here a is the value of y when $x=0$, and e is the base of NEPER's system of logarithms.

13. We have now found that the functional equation

$$f(x_0)f(x_{,})=\mathrm{C}\{f(x_0+x_{,})+f(x_0-x_{,})\}$$

may be satisfied in two ways, viz. by making

$$f(x)=a\cos\frac{x}{c}: \qquad (1)$$

or
$$f(x)=\frac{a}{2}\left\{e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right\}. \qquad (2)$$

If we make $e^{\frac{1}{c}}=r$, that is, $\frac{1}{c}=\text{Nep.}\log r$, the second function may also be expressed thus,

$$f(x)=\frac{a}{2}\left\{r^x+r^{-x}\right\}: \qquad (2)$$

thus our problem (Art. 7) is completely resolved.

APPLICATIONS OF THE FUNCTIONAL EQUATION.

I. TO THE FUNDAMENTAL THEOREM IN STATICS.

14. The foundation of Statics is the theorem implied in the expression, *The Parallelogram of Forces.* This proposition, which in substance is due to STEVINUS, has been proved in three different ways.

(1.) By the principle of virtual velocities, which is the foundation of Dynamics.

(2.) More legitimately from the Theory of the Lever, first established by ARCHIMEDES.

(3.) By means of a few axioms of Statics, as simple and self-evident as those of Geometry.

This last way of treating the subject was first given by DANIEL BERNOUILLI, in the first volume of the Petersburgh Commentaries: his demonstration was afterwards improved by D'ALEMBERT in the first and sixth volumes of his *Opuscules*. It has also been adopted by late writers, and, in particular, by POISSON, in his *Traité de Mecanique*, and by WHEWELL, in his *Analytical Statics*. It is the third method which I mean to follow; and, excepting the particular mode of establishing the analytical principle, my demonstration will differ but little from POISSON'S: he has resolved the functional equation in two different ways in the two editions of his book, but my solution is different from both.

15. The axioms of statics, on which the following investigation is to rest, are these:

(1.) The direction of the resultant of any two forces is in the plane of the forces; and when they are equal, it bisects the angle made by the straight lines, which indicate their direction.

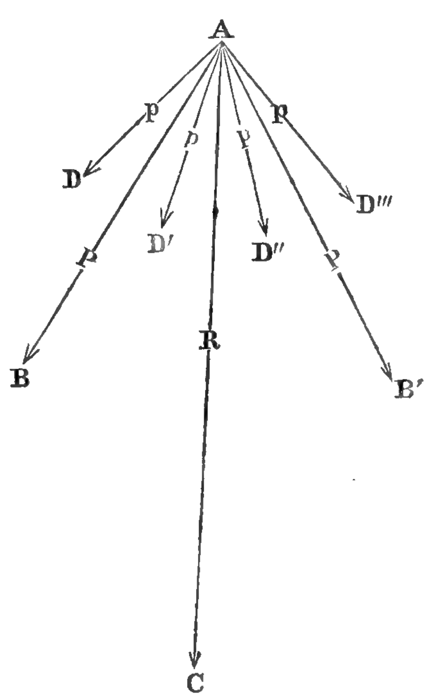
(2.) When the directions of the constituent and resultant forces coincide, this last is equal to both the others. And if the angles which the constituents make with the resultant be supposed to increase, the resultant will decrease continually, until it become $=0$. The directions of the constituents will then be perpendicular to that of the resultant.

(3.) If each of the constituent forces be increased or diminished in any ratio, the same for both, the resultant will be increased or diminished in the same ratio: that is, if the forces P, Q, and their resultant R, change their values, and become P′, Q′, R′, and if $\frac{P}{P'}=\frac{Q}{Q'}$; then shall $\frac{R}{R'}=\frac{P}{P'}=\frac{Q}{Q'}$; also $\frac{R}{P}=\frac{R'}{P'}$, and $\frac{R}{Q}=\frac{R'}{Q'}$.

The general problem now to be resolved is this.

PROBLEM.—*To find* R, *the resultant of any two given forces* P *and* Q, *which act at a point; also, its direction.*

Fig. 1.



We shall begin with the case in which the given forces are equal.

16. CASE I. Let P and P be two equal forces, which act in the directions AB, AB′, and R their resultant, which acts in the direction AC, a line bisecting the angle BAB′: It is required to find the magnitude of the force R.

It is evident from our third axiom, that, while the angles BAC, B′AC, continue the same, $\frac{R}{P}$ must be a constant quantity, but this quantity will change if the angle change. Therefore, $\frac{R}{P}$ must be some function of the angle BAC; and, denoting the angle by $x_\circ$, we may express the relation between $\frac{R}{P}$ and $x_\circ$ thus.

$$\frac{R}{P}=f(x_\circ). \qquad (1)$$

Now, like as R is the resultant of two equal forces P, P, we may assume that P and P are each the resultants of two equal forces p, p, which make equal angles with them; one pair acting in the directions AD, AD′, and another pair in the directions AD″, AD‴. Let each of the four equal angles DAB, BAD′, D″AB′, B′AD‴, be denoted by $x_{,}$, and we have, because P is the resultant of p, p,

$$\frac{P}{p}=f(x_{,}). \qquad (2)$$

therefore, taking the product of equations (1), (2),

$$\frac{R}{p}=f(x_\circ).f(x_{,}) \qquad (3)$$

Now, the force R, which is equivalent to the equal forces P, P, must also be equivalent to the four forces which compose P, P; two of these are forces p, p, which make with R angles each equal to $x_\circ+x_{,}$, and the other two, p, p, make also with R angles each equal to $x_\circ-x_{,}$. Let the resultant of the first pair be R′, and the resultant of the other pair R″, we have then

$$\frac{R'}{p}=f(x_\circ+x_{,}), \qquad \frac{R''}{p}=f(x_\circ-x_{,}),$$

and

$$\frac{R'+R''}{p}=f(x_\circ+x_{,})+f(x_\circ-x_{,}).$$

Now, the forces R′, R″, which constitute the force R, lie in the same direction with it; therefore, R=R′+R″, and so we have

$$\frac{R}{p}=f(x_\circ+x_{,})+f(x_\circ-x_{,}). \qquad (4)$$

We have now, from equations (3) and (4),

$$f(x_\circ)f(x_{,})=f(x_\circ+x_{,})+f(x_\circ-x_{,});$$

and multiplying both sides by C^2 a constant

$$Cf(x_\circ).Cf(x_{,})=C\{Cf(x_\circ+x_{,})+Cf(x_\circ-x_{,})\};$$

and putting simply $f(x_\circ)$, and $f(x_{,})$, and $f(x_\circ+x_{,})$, and $f(x_\circ-x_{,})$, instead of the same symbols multiplied by the constant C (this, because of the indefinitude of the symbol f, is evidently allowable), we have this functional equation

$$f(x_\circ).f(x_{,})=C\{f(x_\circ+x_{,})+f(x_\circ-x_{,})\},$$

of which we have found two solutions (Art. 13); the first of these, however, only will apply to the present case, because $f(x)=\frac{R}{P}$ decreases while the angle x increases; thus we have

$$\frac{R}{P}=f(x)=a\cos\left(\frac{x}{c}\right);$$

here a is the value of $f(x)$ when $x=0$; now, when $x=0$, then $c\,x=0$, and $\cos(c\,x)=1$, and $\mathrm{R}=2\,\mathrm{P}$, and $\frac{\mathrm{R}}{\mathrm{P}}=f(x)=2$, therefore $a=2$, and $\mathrm{R}=2\,\mathrm{P}\cos\frac{x}{c}$.

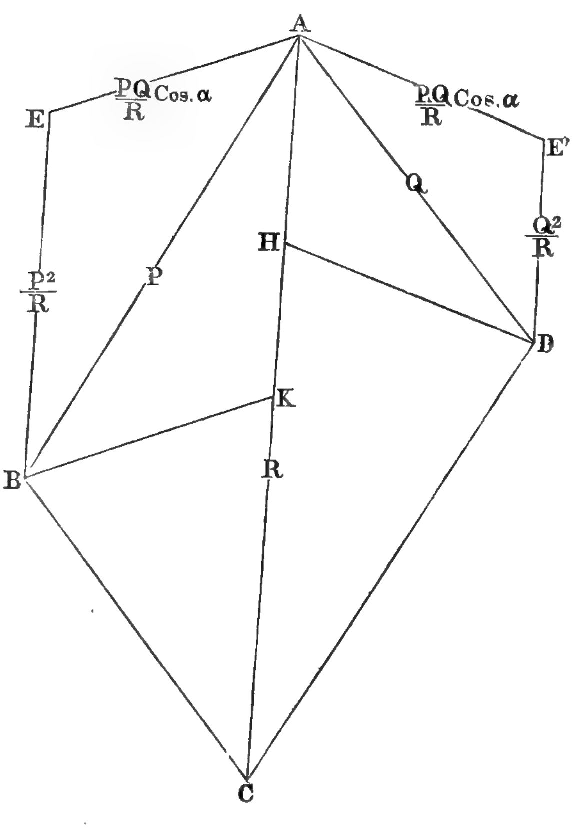
There is yet an indeterminate quantity c; to find the value of which, we must consider that, P being supposed given, while x increases from 0 to $\frac{1}{2}\pi$, the resultant R must decrease continually from 2 P to 0; now, this can only happen when $c=1$, for if c were less than 1, the resultant would vanish before x became a right angle; and if c were greater than 1, it would not vanish when x was a right angle; therefore, $c=1$; and in the case when the forces are equal,

$$\mathrm{R}=2\,\mathrm{P}\cos x: \quad . \quad . \quad . \quad . \quad \text{(A)}$$

thus the first case of the problem is resolved.

17. Case 11. Let us next suppose that P and Q are any two forces whose directions are AB and AD, and R their resultant, whose direction is AC, the angle BAD being any whatever; we have to determine the force R, and the direction of the line AC relatively to AB and AD.

Fig. 2.



Put the angles $\mathrm{BAC}=\phi$, $\mathrm{CAD}=\theta$, then $\mathrm{BAD}=\alpha=\phi+\theta$. At the point A in the line AC, make the angles CAE, CAE′, each equal to BAD; then the angle BAE will be equal to $\mathrm{CAD}=\theta$, and $\mathrm{DAE'}=\mathrm{CAB}=\phi$.

Make the lines AB, AD, AC, proportional to the forces P, Q, R, or such, that $\mathrm{P}:\mathrm{R}=\mathrm{AB}:\mathrm{AC}$, $\mathrm{R}:\mathrm{Q}=\mathrm{AC}:\mathrm{AD}$, and, therefore, $\mathrm{P}:\mathrm{Q}=\mathrm{AB}:\mathrm{AD}$. In the line AC, take $\mathrm{AH}=\frac{\mathrm{AD}^2}{\mathrm{AC}}$, and $\mathrm{AK}=\frac{\mathrm{AB}^2}{\mathrm{AC}}$, and make $\mathrm{AE}=\mathrm{AE'}=\frac{\mathrm{AB}\,.\,\mathrm{AD}}{\mathrm{AC}}$, and draw the lines CB, CD, DE′, DH, BK, BE.

The triangles BAC, BAK have a common angle, and by construction $\mathrm{CA}:\mathrm{AB}=\mathrm{AB}:\mathrm{AK}$, so that the sides about that angle are proportionals; therefore the triangles are similar, and have their remaining angles equal, viz. $\mathrm{ACB}=\mathrm{ABK}$, and $\mathrm{ABC}=\mathrm{AKB}$.

The triangles BAE, CAD have their angles BAE, CAD equal, and by construction the sides about these angles proportionals, for $\mathrm{AC}:\mathrm{AD}=\mathrm{AB}:\mathrm{AE}$; therefore the triangles are similar, and hence the angles $\mathrm{ABE}=\mathrm{ACD}$, and $\mathrm{AEB}=\mathrm{ADC}$.

And since the angle $\mathrm{ABK}=\mathrm{ACB}$, and $\mathrm{ABE}=\mathrm{ACD}$, therefore $\mathrm{EBK}=\mathrm{BCD}$;

now AEB has been proved equal to ADC, and by construction KAE = BAD; therefore the figures BADC, KAEB are equiangular; they are also made up of similar triangles, therefore they are similar.

In the same way it may be proved that the figure BADC is similar to E′AHD, the equal angles being E′AH = BAD and AHD = ADC, HDE′ = DCB and AE′D = ABC.

Thus it appears that the three figures BADC, KAEB, E′AHD are similar, and that the lines AB, AC, AD have to each other the same ratios as the lines AK, AB, AE have to each other; also the same ratios as the lines AE′, AD, AH, have to each other. And since, by hypothesis, a force represented by AC, and acting on the point A in the direction of that line, is equivalent to two forces represented by the lines AB and AD, acting at A in their directions; so, by reason of the similarity of figures, a force represented by AB, and acting in its direction, will be equivalent to forces represented by AK, AE acting in the direction of these lines. Also a force represented by AD, and acting in the direction AD, will be equivalent to forces represented by AH, AE′ acting in the directions of AH, AE′.

It now appears that the force expressed by AC, which is the resultant of the two forces AB, AD, may also be considered as the resultant of the forces AK, AH, together with the resultant of the equal forces AE, AE′. But the force AK is by construction $=\frac{AB^2}{AC}=\frac{P^2}{R}$; and the force $AH=\frac{AD^2}{AC}=\frac{Q^2}{R}$; and the two equal forces AE, AE′, which are each equal to $\frac{AB.AD}{AC}$, and make with AC angles each equal $BAD=\alpha$, have been proved by our first case to compose a force equal to $2AE.\cos EAC=\frac{2PQ}{R}\cos\alpha$; therefore, on the whole,

$$R=\frac{P^2}{R}+\frac{Q^2}{R}+\frac{2PQ}{R}\cos\alpha\ ,$$

and

$$R^2=P^2+Q^2+2PQ\cos\alpha. \quad \ldots\ldots \quad \text{(B)}$$

Now, by the elements of geometry, this last expression is the diagonal of a parallelogram whose sides about one of its angles are P and Q, and the contained angle α: hence we have this proposition,

THEOREM.—*Two forces which act on a point in the directions of the sides of a parallelogram, and which are represented in magnitude by these sides, are equivalent to a single force acting in the direction of the diagonal, and represented in magnitude by that diagonal.*

In this way, by the theories of analysis and Geometry, the proposition which is the foundation of statics is derived from a few axioms, which are analogous to those of Geometry, and which seem to be necessary consequences of our primary notions of a force.

I am next, from the same source, but with the aid of the proposition that has just now been demonstrated, to establish the theory of curves of equilibration, a practical application of mathematics which is of essential importance in the construction of bridges.

II. ARCH OF EQUILIBRATION.

18. The construction of a bridge of considerable length, such as those across the Thames at London, or that over the Menai Strait at Bangor, is one of the noblest achievements of human power, whether we consider its conception or execution. It has long exercised the ingenuity of mechanicians in devising arches which shall unite the properties of stability with elegance of form.

19. There are two chief theories regarding the proper form of an arch, both resting on the principles of Geometry and Statics. One of these, the more ancient, is the *wedge* theory. In this, the arch is formed of a series of stone-wedges, which ought to be so adapted to each other, in regard to weight and position, that they shall have no tendency to move in any direction; the pressures throughout the arch being either counteracted by equal opposite pressures, or else exerted against fixed points of support.

The French mathematician La Hire explained this theory in a treatise on mechanics, printed in 1695. It was followed by other French engineers and mathematicians; as by Parent, in the Memoirs of the French Academy for 1704; by Couplet, in the same work for 1729; by Bouguer and Bossut, in 1774 and 1776; and again by this last writer, in the third volume of his *Cours de Mathematiques;* and in this country by the late Mr Atwood, who published a *Dissertation on the Construction and Properties of Arches* in 1801; to this a *Supplement* was given in 1804.

20. The second theory of an arch is that deduced from the properties of the curve, formed by a cord or chain hanging freely in a vertical plane from two fixed points, which, because of the way in which it is formed, is called the *Catenaria* or *Catenary.* This curve was first noticed by Galileo, who, however, did not precisely comprehend its nature, for, in his second dialogue on motion, he says that it is a parabola; but again, in his fourth dialogue, he says that, to a certain extent in the lower part of the curve, it differs very little from a parabola. I notice this because it has been said that he believed it to be exactly a parabola.* The discovery of the true nature of the curve was hardly within the power of the mathematical science of Galileo's time. The method of fluxions of Newton, and the differential calculus of Leibnitz, however, enabled mathematicians to surmount this, and many other difficulties in statics.

* Leslie's Geometrical Analysis. *The Catenary.*

21. JAMES BERNOUILLI, in the year 1690, proposed in the *Leipsic Acts* "to find the nature of the curve formed by a rope which hangs freely suspended between two fixed points." This problem was resolved by HUYGENS, LEIBNITZ, and JOHN BERNOUILLI. In England, DAVID GREGORY gave a complete solution of BERNOUILLI's problem.* In his memoir, he says, "The *catena*, placed in an inverted position, maintains its figure, and does not fall downwards, so makes a thin arc or *fornix*;" and he afterwards adds, "The *catenaria* are the only true arches or *fornices*, and an arch of any other figure is sustained for this reason only, because a catenaria is included in its thickness." In this assertion GREGORY went too far; for, as JOHN BERNOUILLI truly said, an arch may have the form of a circle or an ellipse, or indeed any curve whatever, and be perfectly secure, by adapting the mass which it supports to its form.

22. Writers on the Method of Fluxions have exemplified the use of that calculus by applying it to the catenary, and arches of equilibration. Thus, EMERSON gave their theory in various parts of his works; and Dr CHARLES HUTTON has also explained it in his *Treatise on Bridges*. He there also adverts to the wedge theory, which had been delivered before by ATWOOD.

Dr J. ROBISON, formerly Professor of Natural Philosophy in our University, adopted the theory of equilibrated curves, in a valuable article on *Arches* which he contributed to the Supplement to the fourth edition of the Encyclopædia Britannica.† Since that essay was published, a period of nearly forty years has elapsed, and in this time Bridges of Suspension have come much into use. To these the simple catenary, which is inapplicable to stone bridges, finds an important application. On this subject the late DAVIES GILBERT, Esq., President of the Royal Society of London, gave a memoir, which is published in its *Transactions* for 1826. This contains *tables of the co-ordinates and arch of a catenary;* the numbers extend to eight decimal places, supposing the parameter to be an unit. Such tables, if correct, must be highly useful to engineers in the construction of bridges; it so happens, however, that, instead of the numbers being true to eight decimal places, they are only exact in general to about five. If the last three figures of each be rejected, the remaining figures will be nearly correct.

23. The roadway along a bridge should be nearly a horizontal straight line. An exact catenarian arch, with such a roadway, would require to be of great

* Philosophical Transactions, No. 231, (vol. i. p. 39 of LOWTHORP's Abridgment), GREGORY's Memoir, which was in Latin, was translated and published in *Miscellanea Curiosa*, edited (I believe) by Dr DERHAM.

† His articles in that edition of the Encyclopædia and its Supplement, were, in 1822, collected and published in 4 vols. 8vo. The article on *Arches* is republished in the seventh edition of the Encyclopædia, to which I added a short supplement on *Equilibrated Curves*.

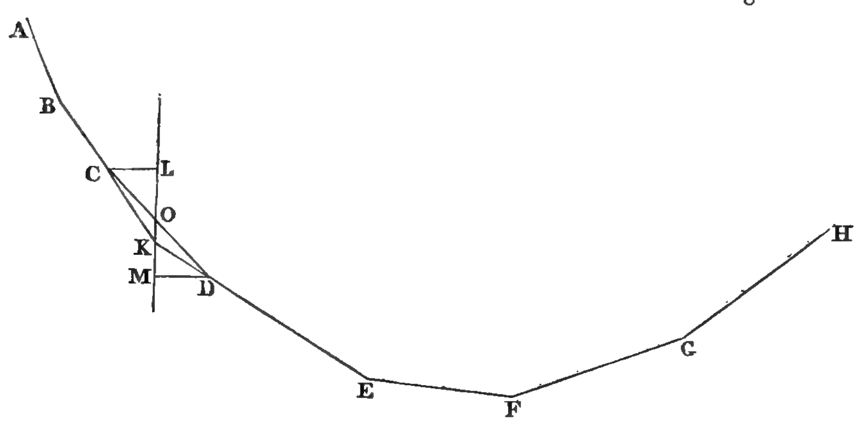
thickness at the crown in respect to the rise of the arch. From the catenary, however, we can construct an equilibrated arch that shall have any height and span and thickness at the crown that may be required. Hence a table of co-ordinates of the catenary will serve for the construction of bridges which are rigid in all their parts, as well as for bridges of suspension. In fact, curves of suspension and the catenary belong to the same family of curves, and are nearly related. The nature of the former is expressed by a formula involving two parameters; but, when these are supposed equal, the general analytical expression for a curve of equilibration becomes the equation of the catenary.

24. To avoid reference to any but the most elementary theories, I shall begin with investigating a property of an equilibrated polygon.

Equilibrated Polygon.

Problem.

Fig. 3.



Let ABCDEFGH be a chain formed by straight rods of any length, which turn with perfect freedom about their extremities as joints. Suppose that the chain hangs vertically from two fixed points, A, H; and, abstracting from its own weight, that it is loaded at the joints with given weights, or masses of matter. It is proposed to determine the geometrical condition that must be satisfied by the position of the rods when the whole constitute an equilibrium.

Let BC, CD, DE be any three adjoining rods; these would manifestly form an equilibrium, independently of the others, if the extreme points B and E were fixed, the links BC, ED turning freely about them as centres: Produce BC, and ED, the extreme links of these three, until they meet in K: Let m and m' denote the weights on the chain at C and D, and let O be their centre of gravity.

The rod CD is urged by three forces, viz. two in the directions KB, KE, and the vertical pressures of the masses m and m', which are equivalent to a single mass pressing the rod vertically downward at O their centre of gravity. The direction of this last force must, in the case of equilibrium, pass through K, the intersection of the directions of the other two. Draw the vertical line OK, and from C and D draw perpendiculars CL, DM. Put ϕ, ϕ', ϕ'' for the angles LCK, LCD (or its equal CDM), and KDM, these being the angles which the links BC, CD, DE make with the plane of the horizon.

By the nature of the centre of gravity, and similar triangles,

$$m : m' = \mathrm{DO} : \mathrm{CO} = \mathrm{DM} : \mathrm{CL} = \frac{1}{\mathrm{CL}} : \frac{1}{\mathrm{DM}} = \frac{\mathrm{KO}}{\mathrm{CL}} : \frac{\mathrm{KO}}{\mathrm{DM}}.$$

Now $$\frac{\mathrm{KO}}{\mathrm{CL}} = \frac{\mathrm{KL}}{\mathrm{CL}} - \frac{\mathrm{OL}}{\mathrm{CL}} = \tan \mathrm{KCL} - \tan \mathrm{OCL} = \tan \phi - \tan \phi' ;$$

and $$\frac{\mathrm{KO}}{\mathrm{DM}} = \frac{\mathrm{MO}}{\mathrm{DM}} - \frac{\mathrm{MK}}{\mathrm{DM}} = \tan \mathrm{ODM} - \tan \mathrm{KDM} = \tan \phi' - \tan \phi'' :$$

Therefore, $$m : m' = \tan \phi - \tan \phi' : \tan \phi' - \tan \phi''.$$

Hence we have this proposition :

THEOREM.—*In an equilibrated polygon, the loads on any two joints are proportional to the difference of the tangents of the angles* ϕ, ϕ' *which the sides about that joint make with the plane of the horizon : And if* c *be put to denote some constant force or pressure,*

$$\tan \phi - \tan \phi' = \frac{m}{c} . \qquad \text{(1)}$$

This is the condition required.

COROLLARY.—Hence, if the angle which any one of the rods makes with the horizon be given, the like angles which all the others make will also be known.

25. By a theorem in the calculus of sines,

$$\tan (\phi - \phi') = \frac{\tan \phi - \tan \phi'}{1 + \tan \phi \tan \phi'} ;$$

therefore, $$\tan \phi - \tan \phi' = \tan (\phi - \phi') (1 + \tan \phi \tan \phi').$$

Hence, in the equilibrated polygon, m and c representing the things already specified,

$$\frac{m}{c} = \tan (\phi - \phi') (1 + \tan \phi \tan \phi') \qquad \text{(2)}$$

If the number of links of the chain be very great, so that the angle made by any two adjoining links is very obtuse, the tangent of the difference of the angles will be almost proportional to the difference of the angles themselves; and the product of the tangents will be almost the square of either. In this case,

$$1 + \tan \phi \tan \phi' = 1 + \tan^2 \phi = \sec^2 \phi,$$

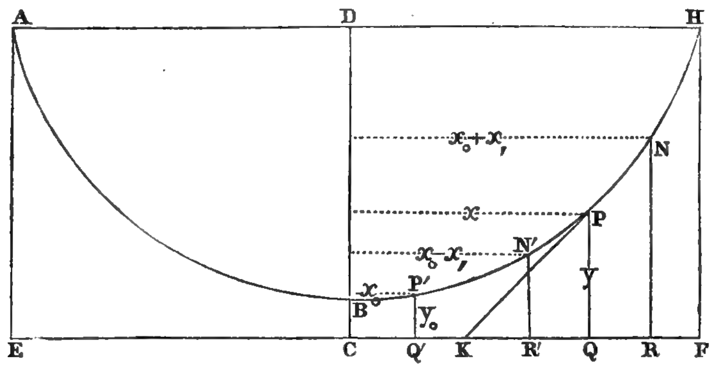
and $$\frac{m}{c} = (\phi - \phi') \sec^2 \phi \quad \text{nearly.}$$

When the number of links is infinitely great, so that the figure which they form may be considered as a curve, $\phi - \phi'$ is $d\phi$, the differential of ϕ and

$$\frac{m}{c} = \sec^2 \phi \, d\phi = d (\tan \phi) \qquad \text{(3)}$$

Let ABH be that curve, which is now the figure of the chain; draw a horizontal line ECF, and through B, the lowest point of the curve, draw a vertical

Fig. 4.



line CBD, meeting EF in C. From any point P in the curve draw PQ perpendicular to CF, and PK touching the curve at P and meeting EF in K; then PKQ will be equal to the angle which an element of the curve makes with a horizontal line at P: Put CQ$=x$ and PQ$=y$ and the angle PKQ$=\phi$.

In all curves $\tan\phi=\dfrac{dy}{dx}$, and (making dx constant) $d\tan\phi=\dfrac{d^2y}{dx}$: we have therefore

$$\frac{d^2y}{dx}=\frac{m}{c}\,,\ \text{and}\ \frac{d^2y}{dx^2}=\frac{m}{c\,dx}\,.$$

Now, if the vertical pressure on the curve at P be a column of matter whose base is dx and altitude y, we have $m=y\,dx$; and, on this hypothesis,

$$\frac{d^2y}{dx^2}=\frac{y\,dx}{c\,dx}=\frac{y}{c}\,;$$

and

$$\frac{d^2y}{dx^2}\cdot\frac{1}{y}=\frac{1}{c}\,,\ =a\ constant.$$

This differential equation is identical with that deduced from the functional equation

$$f(x_0)\,.\,f(x_{,})=\mathrm{C}\,\{\,f(x_0+x_{,})+f(x_0-x_{,})\,\}:$$

the latter must therefore express a property of the former. In this case, x and y increase together, and the value of y when $x=0$ is the perpendicular BC from the lowest point of the chain, this is the quantity equivalent to C in the functional equation. We have now (independently of the integral equation deduced from the differential equation in article 12) this elegant proposition in statics.

26. Theorem.—Let ABH be a perfectly flexible chain of uniform thickness, and composed of infinitely small links suspended, in a vertical plane, from two fixed points A, H: Suppose that an infinite number of infinitely thin columns or rods, PQ, &c., are attached to the chain, and hang freely, and quite contiguous to each other, with their lower ends in a horizontal straight line ECF, thereby forming a continuous plane surface between that line and the plane curve ABH. Assuming now the straight line ECF

as an *axis*, the hanging rods PQ, &c. will be *ordinates* to that axis: Let B be the lowest point of the curve, and BC the shortest ordinate: Let CQ, the distance of any ordinate PQ from C be denoted by x, and PQ, the ordinate, by $f(x)$: Similarly, $x_{\circ}$ and $x_{,}$ being any values of x, whose sum is $x_{\circ}+x_{,}$ and difference $x_{\circ}-x_{,}$, let the ordinates corresponding to these values of x be denoted by

$$f(x_{\circ}), \quad f(x_{,}), \quad f(x_{\circ}+x_{,}), \quad f(x_{\circ}-x_{,});$$

and let the shortest ordinate BC, or $f(x=0)$, be denoted by a; then shall

$$\frac{2f(x_{\circ})f(x_{,})}{a}=f(x_{\circ}+x_{,})+f(x_{\circ}-x_{,}).$$

This property is altogether similar to a property of cosines of angles; also to the lines in an ellipse and an hyperbola, which are analogous to cosines. In the case of angles, it is known that the radius or cosine of *zero*, being denoted by a, and any two angles by $x_{\circ}$ and $x_{,}$,

$$\frac{2\cos x_{\circ}\cos x_{,}}{a}=\cos(x_{\circ}+x_{,})+\cos(x_{\circ}-x_{,}).$$

In the ellipse and hyperbola, if there be four sectors,

$$x_{\circ}, \quad x_{,}, \quad x_{\circ}+x_{,}, \quad x_{\circ}-x_{,},$$

the third and fourth of which are the sum and the difference of the first and second, and if these be contained between the semi-transverse axis and other semidiameters, from the vertices of which ordinates are drawn to the conjugate axis, these ordinates being expressed by a like notation, viz.

$$\text{ord}(x_{\circ}), \quad \text{ord}(x_{,}), \quad \text{ord}(x_{\circ}+x_{,}), \quad \text{ord}(x_{\circ}-x_{,});$$

then, in both curves, putting a for the semitransverse axis,

$$\frac{2\,\text{ord}(x_{\circ})\,\text{ord}(x_{,})}{a}=\text{ord}(x_{\circ}+x_{,})+\text{ord}(x_{\circ}-x_{,}).^{*}$$

27. From the perfect identity of the relations between the semiordinates of the circle and ellipse, also the hyperbola, and the ordinates of the curve we are now considering, it must follow that all the consequences deducible from the formulæ which express the relations of the semiordinates of the conic sections may, without farther investigation, be enunciated as properties of the equilibrated curve.

Thus, putting $n\,x$ instead of $x_{\circ}$, and x instead of $x_{,}$, also y instead of $f(x)$ or $f(x_{,})$; we have

$$\frac{2y\,.f(n\,x)}{a}=f\{(n+1)x\}+f\{(n-1)x\}:$$

and hence again,

$$f\{(n+1)x\}=\frac{2y}{a}f(n\,x)-f\{(n-1)x\}.$$

* See my paper in this volume, page 436.

From this last we form the following table of formulæ:

$$f(o\,x)=a\,,$$
$$f(1\,x)=y\,,$$
$$f(2\,x)=\frac{1}{a}\,\{2\,y^2-a^2\}\,,$$
$$f(3\,x)=\frac{1}{a^2}\,\{4\,y^3-3\,a^2\,y\}\,,$$
$$f(4\,x)=\frac{1}{a^3}\,\{8\,y^4-8\,a^2\,y^2+a^4\}\,,$$
$$f(5\,x)=\frac{1}{a^4}\,\{16\,y^5-20\,a^2\,y^3+5\,a^4\,y\}\,,$$
$$f(6\,x)=\frac{1}{a^5}\,\{32\,y^6-48\,a^2\,y^4+18\,a^4\,y^2-a^6\}\,,$$
$$f(7\,x)=\frac{1}{a^6}\,\{64\,y^7-112\,a^2\,y^5+56\,a^4\,y^3-7\,a^6\,y\}\,;$$

and in general,*

$$2\,a^{n-1}f(n\,x)=(2\,y)^n-n\,a^2\,(2\,y)^{n-2}+\frac{n\,(n-3)}{1\,.\,2}\,a^4\,(2\,y)^{n-4}\,.\,-\frac{n\,(n-4)\,(n-5)}{1\,.\,2\,.\,3}\,a^6\,(2\,y)^{n-6}+\&c.$$

By this formula, supposing any number of ordinates to stand at equal distances along the axis x; and the parameter a, also y the first ordinate, to be given; then all the remaining ordinates, to the last, may be found.

28. It has been found (Article 12), that $x=$CQ, and $y=$PQ, being co-ordinates at any point P of the curve, and $a=$BC, the least ordinate, then

$$\frac{d\,y}{d\,x}=\frac{\surd\,(y^2-a^2)}{c}\,.$$

Now PK being a straight line that touches the curve at P, and meets the axis CE in K; and ϕ denoting the angle PKQ; in all curves

$$\frac{d\,y}{d\,x}=\tan\phi\,;$$

therefore, putting t to denote $\tan\phi$,

$$t=\frac{\surd\,(y^2-a^2)}{c}\,.$$

Hence again $y^2-c^2\,t^2=a^2$, and $y\,d\,y=c^2\,t\,d\,t$.

Now, $t\,d\,x=d\,y$, and $y\,t\,d\,x=y\,d\,y=c^2\,t\,d\,t$; therefore, $c^2\,d\,t=y\,d\,x$.

We have now $\quad c\,d\,t=y\,\dfrac{d\,x}{c}\,,\quad d\,y=t\,d\,x=c\,t\,\dfrac{d\,x}{c}\,.$

And again, from these equations,

$$d\,y+c\,d\,t=(y+c\,t)\,\frac{d\,x}{c}\,;$$

$$d\,y-c\,d\,t=-\frac{d\,x}{c}\,(y-c\,t)\,;$$

* For the mode of deduction, see the paper just quoted.

$$\frac{d\,y+c\,d\,t}{y+c\,t}=\frac{d\,x}{c}, \qquad \frac{d\,y-c\,d\,t}{y-c\,t}=-\frac{d\,x}{c}:$$

and taking the integrals, so that when $x=0$, then $y=a$, and $t=0$, we have

$$\log\frac{y+c\,t}{a}=\frac{x}{c},$$

$$\log\frac{y-c\,t}{a}=-\frac{x}{c}:$$

Hence, e being the number which is the base of NEPER's logarithms,

$$\frac{y+c\,t}{a}=e^{\frac{x}{c}};$$

$$\frac{y-c\,t}{a}=e^{-\frac{x}{c}}.$$

By adding and subtracting, there is obtained

$$y=\frac{a}{2}\left\{e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right\}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$t=\frac{a}{2c}\left\{e^{\frac{x}{c}}-e^{-\frac{x}{c}}\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

These equations, which involve in them this other equation

$$y^2-c^2\,t^2=a^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

express the nature of the curve of equilibration.

29. It was found (Art. 28) that $c^2\,d\,(\tan\phi)=y\,d\,x=d$ (area BPQC). Hence, by integration, putting s to denote the area BPQC,

$$s\begin{cases}=c^2\tan\phi\\[4pt] =\dfrac{a\,c}{2}\left\{e^{\frac{x}{c}}-e^{-\frac{x}{c}}\right\}\end{cases}$$

30. By trigonometry, the subtangent QK is equal to PQ . cot PKC; therefore,

$$\text{subtan. QK}=c\,.\,\frac{e^{\frac{x}{c}}+e^{-\frac{x}{c}}}{e^{\frac{x}{c}}-e^{-\frac{x}{c}}}=c\,.\,\frac{e^{\frac{2x}{c}}+1}{e^{\frac{2x}{c}}-1}.$$

In this formula, the number $e=2.7182818284$. The numerator and the denominator of the fraction $\frac{\text{QK}}{c}$ will, therefore, both increase continually with x. The ratio of QK to c will, however, evidently approach to that of equality. Thus it appears, that c is the value of the subtangent when x is infinite.

31. It appears that the equation of the curve of equilibration contains two *constants*, a and c, like those of the ellipse and hyperbola, the constants of which are the semiaxes; these enter similarly into the equation of their curves, but here the constants do not enter similarly, for one, viz. a, enters as a coefficient, and the other, $\frac{1}{c}$, as an exponent. We have already named a the *parameter* of

the curve; we may, to distinguish the constants from each other, now name c its *modulus.* We have seen that a curve of equilibration has some properties absolutely identical with those of an hyperbola, and quite analogous to those of the circle and ellipse; such is that given in Art. 27, and I shall now investigate others.

32. It has been found that the relation between s, the curvilineal space BPQC, and t, the tangent of the angle PKQ, is expressed by the formula

$$\frac{s}{c} = tc.$$

Let either of these equal quantities be designated by the symbol F (x), which indicates a function of the variable amplitude x; let $x_\circ$ and $x_,$ be any values of x, then

$$f(x_\circ),\quad f(x_,),\quad f(x_\circ+x_,),\quad f(x_\circ-x_,),$$
$$F(x_\circ),\quad F(x_,),\quad F(x_\circ+x_,),\quad F(x_\circ-x_,)$$

will denote the same functions of

$$x_\circ,\quad x_,,\quad x_\circ+x_,,\quad x_\circ-x_,,$$

that $f(x)$ and F (x) are of x.

To abridge, let us put r to denote $e^{\frac{1}{c}}$. We have found that

$$2f(x_\circ) = a\left\{r^{x_\circ}+r^{-x_\circ}\right\},$$
$$2f(x_,) = a\left\{r^{x_,}+r^{-x_,}\right\},$$
$$2F(x_\circ) = a\left\{r^{x_\circ}-r^{-x_\circ}\right\},$$
$$2F(x_,) = a\left\{r^{x_,}-r^{-x_,}\right\}.$$

From these formulæ, by multiplying corresponding sides of the equations, we get

$$4f(x_\circ).f(x_,) = a^2\left\{r^{x_\circ}+r^{-x_\circ}\right\}\left\{r^{x_,}+r^{-x_,}\right\} \qquad (1)$$
$$4F(x_\circ).F(x_,) = a^2\left\{r^{x_\circ}-r^{-x_\circ}\right\}\left\{r^{x_,}-r^{-x_,}\right\} \qquad (2)$$
$$4f(x_\circ).F(x_,) = a^2\left\{r^{x_\circ}+r^{-x_\circ}\right\}\left\{r^{x_,}-r^{-x_,}\right\} \qquad (3)$$
$$4F(x_\circ).f(x_,) = a^2\left\{r^{x_\circ}-r^{-x_\circ}\right\}\left\{r^{x_,}+r^{-x_,}\right\} \qquad (4)$$

Now, $a^2\left\{r^{x_\circ}+r^{-x_\circ}\right\}\left\{r^{x_,}+r^{-x_,}\right\} = a^2\left\{r^{x_\circ+x_,}+r^{-(x_\circ+x_,)}\right\} + a^2\left\{r^{x_\circ-x_,}+r^{-(x_\circ-x_,)}\right\}:$

Again, $a^2\left\{r^{x_\circ+x_,}+r^{-(x_\circ+x_,)}\right\} = 2af(x_\circ+x_,),$

And $a^2\left\{r^{x_\circ-x_,}+r^{-(x_\circ-x_,)}\right\} = 2af(x_\circ-x_,);$

Therefore, $4f(x_\circ).f(x_,) = 2a\left\{f(x_\circ+x_,)+f(x_\circ-x_,)\right\}.$

The remaining equations (2), (3), (4), may be treated exactly in the same way as equation (1), and like results will be obtained; and from the four we obtain these formulæ

$$f(x_\circ+x_,)+f(x_\circ-x_,)=\frac{2}{a}f(x_\circ)\,.\,f(x_,)\,; \tag{1}$$

$$f(x_\circ+x_,)-f(x_\circ-x_,)=\frac{2}{a}\,\mathrm{F}(x_\circ)\,.\,\mathrm{F}(x_,)\,; \tag{2}$$

$$\mathrm{F}(x_\circ+x_,)-\mathrm{F}(x_\circ-x_,)=\frac{2}{a}f(x_\circ)\,.\,\mathrm{F}(x_,)\,; \tag{3}$$

$$\mathrm{F}(x_\circ+x_,)+\mathrm{F}(x_\circ-x_,)=\frac{2}{a}\,\mathrm{F}(x_\circ)\,.f(x_,)\,. \tag{4}$$

From these again, by addition and subtraction, we find

$$a\,.f(x_\circ+x_,)=f(x_\circ)\,.f(x_,)+\mathrm{F}(x_\circ)\,.\,\mathrm{F}(x_,)\,, \tag{5}$$

$$a\,.f(x_\circ-x_,)=f(x_\circ)\,.f(x_,)-\mathrm{F}(x_\circ)\,.\,\mathrm{F}(x_,)\,, \tag{6}$$

$$a\,.\,\mathrm{F}(x_\circ+x_,)=\mathrm{F}(x_\circ)\,.f(x_,)+f(x_\circ)\,.\,\mathrm{F}(x_,)\,, \tag{7}$$

$$a\,.\,\mathrm{F}(x_\circ-x_,)=\mathrm{F}(x_\circ)\,.f(x_,)-f(x_\circ)\,.\,\mathrm{F}(x_,)\,, \tag{8}$$

These formulæ are absolutely identical with those given for the ellipse and hyperbola in my Memoir, already quoted, on the Analogy between the co-ordinates of these Curves, the variable line x here coming in the place of the elliptic or hyperbolic sectors.

33. Considering the subtangent KQ as a function of the amplitude x, let it be denoted by the symbol $f'(x)$, then (Art. 30),

$$\frac{\mathrm{KQ}}{c}=\frac{f'(x)}{c}=\frac{f(x)}{\mathrm{F}(x)}\,;$$

and
$$\frac{f'(x_\circ+x_,)}{c}=\frac{f(x_\circ+x_,)}{\mathrm{F}(x_\circ+x_,)}=\frac{f(x_\circ)\,f(x_,)+\mathrm{F}(x_\circ)\,\mathrm{F}(x_,)}{\mathrm{F}(x_\circ)\,f(x_,)+f(x_\circ)\,\mathrm{F}(x_,)}\,.$$

Now,
$$f(x_\circ)=\frac{\mathrm{F}(x_\circ)\,f'(x_\circ)}{c}\,,\ \text{and}\ f(x_,)=\frac{\mathrm{F}(x_,)\,f'(x_,)}{c}\,:$$

Therefore, substituting and dividing the numerator and denominator by $\mathrm{F}(x_\circ)\,\mathrm{F}(x_,)$, we find

$$f'(x_\circ+x_,)=\frac{f'(x_\circ)\,f'(x_,)+c^2}{f'(x_\circ)+f'(x_,)}\,. \tag{9}$$

And similarly,
$$f'(x_\circ-x_,)=\frac{f'(x_\circ)\,f'(x_,)-c^2}{f'(x_\circ)-f'(x_,)}\,. \tag{10}$$

All these formulæ are perfectly analogous to properties of the ellipse and hyperbola, particularly the latter of these two curves.

34. Resuming the formula of Art. 28, viz.

$$e^{\frac{x}{c}}=\frac{y+c\,t}{a}\,,\qquad e^{-\frac{x}{c}}=\frac{y-c\,t}{a}\,;$$

let n be any number whatever, positive or negative, whole or fractional, then

$$e^{\frac{nx}{c}}=\left\{\frac{y+c\,t}{a}\right\}^n,\qquad e^{-\frac{nx}{c}}=\left\{\frac{y-c\,t}{a}\right\}^n;$$

and by adding and subtracting,

$$\frac{a}{2}\left\{e^{\frac{nx}{c}}+e^{-\frac{nx}{c}}\right\}=\frac{1}{2a^{n-1}}\left\{(y+ct)^n+(y-ct)^n\right\},$$

$$\frac{a}{2}\left\{e^{\frac{nx}{c}}-e^{-\frac{nx}{c}}\right\}=\frac{1}{2a^{n-1}}\left\{(y+ct)^n-(y-ct)^n\right\}.$$

These formulæ, by our functional notation, may be expressed thus:

$$f(nx)=\frac{1}{2a^{n-1}}\Big[\left\{f(x)+F(x)\right\}^n+\left\{f(x)-F(x)\right\}^n\Big],$$

$$F(nx)=\frac{1}{2a^{n-1}}\Big[\left\{f(x)+F(x)\right\}^n-\left\{f(x)-F(x)\right\}^n\Big].$$

They denote a property of the curve of equilibration quite analogous to that of the conic sections which is expressed by DEMOIVRES theorem. From these formulæ, by putting nx instead of x, and $\frac{1}{n}$ instead of n, we obtain two others,

viz.
$$f(x)=\frac{a^{\frac{n-1}{n}}}{2}\Big[\left\{f(nx)+F(nx)\right\}^{\frac{1}{n}}+\left\{f(nx)-F(nx)\right\}^{\frac{1}{n}}\Big];$$

$$F(x)=\frac{a^{\frac{n-1}{n}}}{2}\Big[\left\{f(nx)+F(nx)\right\}^{\frac{1}{n}}-\left\{f(nx)-F(nx)\right\}^{\frac{1}{n}}\Big].$$

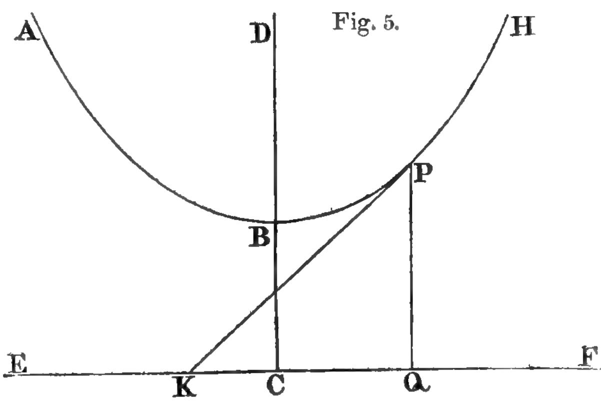
By these formulæ, combined with this,

$$\left\{f(nx)\right\}^2-\left\{F(nx)\right\}^2=a^2,$$

we may find $f(nx)$ and $F(nx)$ from $f(x)$ and $F(x)$, and the contrary.

35. As in an ellipse or hyperbola, which, like a curve of equilibration, have two parameters, if these be supposed equal, the curve becomes a circle or equilateral hyperbola, which have each only one parameter: so, in like manner, we may assume that a and c, the parameter and modulus of a curve of equilibration, are equal. Then the equations of the curve are a little more simple, they being, putting $BC=a$, $CQ=x$,

Fig. 5.



(1) $$PQ=f(x)=y=\frac{a}{2}\left\{e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right\},$$

(2) $$F(x)=ta=\frac{a}{2}\left\{e^{\frac{x}{a}}-e^{-\frac{x}{a}}\right\},$$

(3) $$s=\frac{a^2}{2}\left\{e^{\frac{x}{a}}-e^{-\frac{x}{a}}\right\},$$

$$y^2=(1+t^2)a^2 \quad .$$

In this case, putting z for BP, the length of the curve between the least ordinate and y, since

$$\frac{dy}{dx}=t=\frac{1}{2}\left\{e^{\frac{x}{a}}-e^{-\frac{x}{a}}\right\},$$

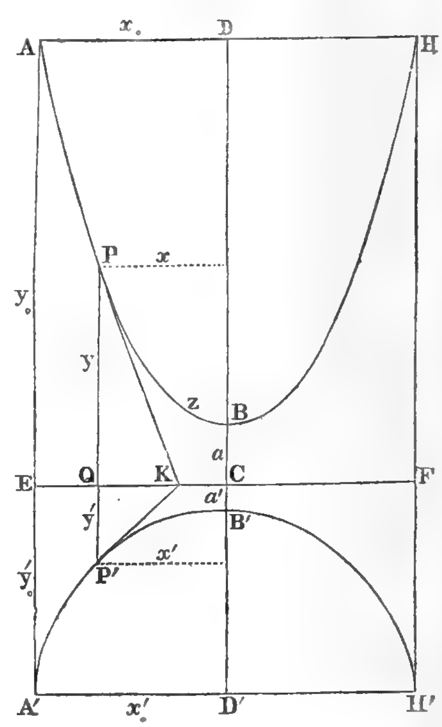
therefore $$\frac{dz^2}{dx^2}=\frac{dx^2+dy^2}{dx^2}=\frac{1}{4}\left\{e^{\frac{2x}{a}}+e^{-\frac{2x}{a}}+2\right\},$$

and $$\frac{dz}{dx}=\frac{1}{2}\left\{e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right\}=\frac{y}{a},$$

and $$a\,dz=y\,dx.$$

36. By hypothesis (art. 26), from every point of a chain of uniform thickness, a rod is suspended, whose weight may be expressed by $y\,dx$, and here we have found that the rod is equivalent in weight to $a\,dz$, which may represent an *element* of the chain; hence, it follows that, whether the chain be loaded, according to the hypothesis, with rods, or be composed of some perfectly flexible material, like gossamer, of uniform thickness, and not loaded, the curve it forms will be the very same, that is, it will be a *catenary*. So that the properties which have been proved to belong to the equilibrated curve, in its general form, may be all affirmed to be true of the simple catenary, that is, a curve formed by a chain or cord of uniform thickness, hanging in a vertical plane from two fixed points.

Fig. 6.



37. Let APBH be a common catenary, and A′P′B′H′ a curve of equilibration, such as it has been defined in art. 25, which have a common horizontal axis EF, and their vertical axes is the same straight line; let PQ, P′Q be ordinates which have the same amplitude CQ; let ϕ denote the angle which a straight line PK, touching the catenary ABH at the top of the ordinate, makes with the axis EF, and ϕ' the angle which a straight line drawn at P′ the top of the other ordinate, touching the curve A′B′H′, makes with the same axis: Put x for CQ, the common amplitude, y for the ordinate PQ, and y' for the ordinate P′Q, and a and a' for CB, BC′ the parameters of the curves. Because $\tan\phi=\frac{dy}{dx}$, and $\tan\phi'=\frac{dy'}{dx}$, therefore $\tan\phi:\tan\phi'=dy:dy'$: Now, y having to y' a constant ratio, viz. that of a to a', we have $y:y'=dy:dy'=a:a'$;

therefore $$\tan\phi:\tan\phi'=a:a';$$

and $$\cot\phi:\cot\phi'=a':a;$$

and $$y\cot\phi:y'\cot\phi'=y\,a':y'\,a.$$

Now, $y\,a'=y'\,a$; therefore $y'\cot\phi=y\cot\phi'$, but $y'\cot\phi$ and $y\cot\phi'$ express the segments of the axis between the ordinates y, y', and the lines touching the curves. On the whole, then, we have these two propositions.

If a catenary ABH, *and curve of equilibration* A′B′H′, *have a common horizontal axis* EF, *and their vertical axes* CD, CD′, *in the same straight line; and if an ordinate* PQ, P′Q, *to each curve, pass through the same point in the horizontal axis, then,*

1. (1) *Straight lines drawn touching the curves at the tops of the ordinates, shall intersect each other in the horizontal axis.*

2. (2) *The tangents of the angles which the touching lines make with the horizontal axis, shall have to each other the ratio of the parameters of the curves.*

38. These are entirely analogous to known properties of a circle and ellipse, also of an equilateral hyperbola, and any other hyperbola; and as an ellipse may be constructed from a circle, and any hyperbola from an equilateral hyperbola, when the ratio of the axes is given, so, in like manner, a curve of equilibration, whose parameter and modulus are known, may be constructed from a catenary. For, from what has been shewn, it is manifest that, supposing an equilibrated curve and a catenary to have the same horizontal axis and their vertical axes on a straight line, any ordinates of the two curves, which have the same amplitude, will have to each other the constant ratio of the parameter of the curves.

39. Deferring, then, for the present, the farther consideration of curves of equilibration, having two parameters; let ABH be a catenary (Fig. 5 or Fig. 6), of which EF is the horizontal axis, CBD the vertical axis, CB the parameter, PQ any ordinate corresponding to the amplitude CQ; let PK touch the curve, and meet the horizontal axis in K.

Let the parameter BC $= a$,
the amplitude CQ $= x$,
the ordinate PQ $= y = f(x)$,
the arc PB $= z = \mathrm{F}(x)$,
the space BCQP $= s$,
the angle PKQ $= \phi$.

In addition to the properties of the curve stated in art. 35, it has these; $x_\circ$ and $x_{,}$ being any two amplitudes.

$$af(x_\circ + x_{,}) = f(x_\circ) f(x_{,}) + \mathrm{F}(x_\circ)\, \mathrm{F}(x_{,}); \qquad (1)$$

$$af(x_\circ - x_{,}) = f(x_\circ) f(x_{,}) - \mathrm{F}(x_\circ)\, \mathrm{F}(x_{,}); \qquad (2)$$

$$a\,\mathrm{F}(x_\circ + x_{,}) = \mathrm{F}(x_\circ) f(x_{,}) + f(x_\circ)\, \mathrm{F}(x_{,}); \qquad (3)$$

$$a\,\mathrm{F}(x_\circ - x_{,}) = \mathrm{F}(x_\circ) f(x_{,}) - f(x_\circ)\, \mathrm{F}(x_{,}); \qquad (4)$$

$$\text{area of space } s = a\,\mathrm{F}(x); \qquad (5)$$

$$y = f(x) = a \sec\phi; \quad z = \mathrm{F}(x) = a \tan\phi. \qquad (6)$$

We may enunciate the formulæ of art. 34, which give the values of $f(nx)$ and $\mathrm{F}(nx)$, as properties of the catenary, simply by assuming that c, the modulus of the equilibrated curve, is equal to a, its parameter; and, in addition, we shall generalise the properties given in this article.

40. Putting $r=e^{\frac{1}{a}}$, (e the base of NEPER'S logarithms), the equations of the catenary are,

$$f(x)=\frac{a}{2}\left\{r^{x}+r^{-x}\right\},\ \mathrm{F}(x)=\frac{a}{2}\left\{r^{x}-r^{-x}\right\};$$

$f(x)$ being the ordinate, and $\mathrm{F}(x)$ the arc, corresponding to the amplitude x.

Let $x_{,}$, x_2, x_3, ... x_n, be any values of x, we have

$$\begin{array}{ll} a\,r^{x_{,}}=f(x_{,})+\mathrm{F}(x_{,}), & a\,r^{-x_{,}}=f(x_{,})-\mathrm{F}(x_{,}), \\ a\,r^{x_2}=f(x_2)+\mathrm{F}(x_2), & a\,r^{-x_2}=f(x_2)-\mathrm{F}(x_2), \\ a\,r^{x_3}=f(x_3)+\mathrm{F}(x_3), & a\,r^{-x_3}=f(x_3)-\mathrm{F}(x_3), \\ \cdots & \\ a\,r^{x_n}=f(x_n)+\mathrm{F}(x_n), & a\,r^{-x_n}=f(x_n)-\mathrm{F}(x_n), \end{array}$$

The sum and difference of the products of the sides of these two sets of equations being taken, and it being observed that

$$a\,r^{\pm(x_{,}+x_2+x_3\dots+x_n)}=f(x_{,}+x_2+x_3\ \&\text{c.})\pm\mathrm{F}(x_{,}+x_2+x_3\ \&\text{c.})$$

we have, by substituting,

$$2\,a^{n-1}.f(x_{,}+x_2+x_3+\&\text{c.})=\left\{\begin{array}{l}\{f(x_{,})+\mathrm{F}(x_{,})\}\{f(x_2)+\mathrm{F}(x_2)\}\{f(x_3)+\mathrm{F}(x_3)\}\ \&\text{c.}\\ +\{f(x_{,})-\mathrm{F}(x_{,})\}\{f(x_2)-\mathrm{F}(x_2)\}\{f(x_3)-\mathrm{F}(x_3)\}\ \&\text{c.}\end{array}\right\}\quad(7)$$

$$2\,a^{n-1}.\mathrm{F}(x_{,}+x_2+x_3+\&\text{c.})=\left\{\begin{array}{l}\{f(x_{,})+\mathrm{F}(x_{,})\}\{f(x_2)+\mathrm{F}(x_2)\}\{f(x_3)+\mathrm{F}(x_3)\}\ \&\text{c.}\\ -\{f(x_{,})-\mathrm{F}(x_{,})\}\{f(x_2)-\mathrm{F}(x_2)\}\{f(x_3)-\mathrm{F}(x_3)\}\ \&\text{c.}\end{array}\right\}\quad(8)$$

These two formulæ comprehend in them, as particular cases, the expansions of $f(x_{\circ}\pm x_{,})$ and $\mathrm{F}(x_{\circ}\pm x_{,})$ given in art. 39.

41. Let $\phi_{,}$, ϕ_2, ϕ_3, ... ϕ_n, denote the angles which lines touching the curve at the tops of the ordinates $f(x_{,})$, $f(x_2)$, $f(x_3)$, ... $f(x_n)$, make with the horizontal axis; and let ϕ denote the angle which the tangent at the top of the ordinate $f(x_{,}+x_2+x_3\dots+x_n)$ makes with that axis;

because $\quad a\,.\,e^{\frac{x_{,}}{a}}=f(x_{,})+\mathrm{F}(x_{,})=a\,(\sec\phi_{,}+\tan\phi_{,})=a\tan(45^{\circ}+\tfrac{1}{2}\phi_{,})$

therefore

$$\begin{aligned} x_{,}&=a\log\tan(45^{\circ}+\tfrac{1}{2}\phi_{,}),\\ x_2&=a\log\tan(45^{\circ}+\tfrac{1}{2}\phi_2),\\ x_3&=a\log\tan(45^{\circ}+\tfrac{1}{2}\phi_3),\\ &\cdots\\ x_n&=a\log\tan(45^{\circ}+\tfrac{1}{2}\phi_n). \end{aligned}$$

By adding into one sum the sides of these equations, and observing that

$$x_{,}+x_2+x_3\dots x_n=a\log\tan(45^{\circ}+\tfrac{1}{2}\phi);$$

and passing from the logarithms to the numbers, we obtain

$$\tan(45° + \tfrac{1}{2}\phi) = \tan(45° + \tfrac{1}{2}\phi_{,}) \tan(45° + \tfrac{1}{2}\phi_2) \tan(45° + \tfrac{1}{2}\phi_3) \ldots \tan(45° + \tfrac{1}{2}\phi_n). \quad (9)$$

And because θ being any angle, $\tan(45 + \theta)\tan(45° - \theta) = 1$; therefore

$$\tan(45° - \tfrac{1}{2}\phi) = \tan(45° - \tfrac{1}{2}\phi_{,}) \tan(45° - \tfrac{1}{2}\phi_2) \tan(45° - \tfrac{1}{2}\phi_3) \ldots \tan(45° - \tfrac{1}{2}\phi_n). \quad (10)$$

These formulæ express elegant properties of the catenary, which are not less general and remarkable than properties of a circle, which are contemplated with high satisfaction by geometers.

42. Because $e^{\frac{x}{a}} + e^{-\frac{x}{a}} = \frac{2y}{a}$; by subtracting 4 from the squares of these equals, and taking the square roots of the results, we find

$$e^{\frac{x}{a}} - e^{-\frac{x}{a}} = \frac{2\sqrt{(y^2 - a^2)}}{a};$$

Therefore
$$e^{\frac{x}{a}} = \frac{y + \sqrt{(y^2 - a^2)}}{a};$$

and
$$\frac{x}{a} = \text{Nep. log}\,\frac{y + \sqrt{(y^2 - a^2)}}{a} = m \text{ com. log}\,\frac{y + \sqrt{(y^2 - a^2)}}{a}, \quad (11)$$

and because
$$z^2 = y^2 - a^2, \quad \text{and} \quad y = \sqrt{(a^2 + z^2)};$$

therefore
$$\frac{x}{a} = \text{Nep. log}\,\frac{z + \sqrt{(a^2 + z^2)}}{a} = m \text{ com. log}\,\frac{z + \sqrt{(a^2 + z^2)}}{a}, \quad (12)$$

By these formulæ, x may be found from either y or z.

We may also express x by ϕ; for since

$$y = a \sec\phi, \quad \sqrt{(y^2 - a^2)} = a \tan\phi;$$

and
$$y + \sqrt{(y^2 - a^2)} = a(\sec\phi + \tan\phi) = a \tan(45° + \tfrac{1}{2}\phi);$$

therefore
$$\frac{x}{a} = \text{Nep. log}\left\{\frac{\tan(45° + \frac{1}{2}\phi)}{\text{rad.}}\right\} = m \text{ com. log}\left\{\frac{\tan(45° + \frac{1}{2}\phi)}{\text{rad.}}\right\}. \quad (13)$$

In these formulæ, $m = .43429448$, and $\log m = 9.6377843$.

43. The properties of the catenary which have been hitherto found are all expressed in finite terms; some of them, however, may be expressed by series, which have remarkable properties; these we are now to investigate.

Resuming the equation of article 12, and putting $\tan\phi$ for $\frac{dy}{dx}$, and making the parameter $=1$, we have

$$y\,dx = d\tan\phi = \sec^2\phi\,d\phi;$$

Now
$$y = \sec\phi,$$

therefore
$$dx = \sec\phi\,.\,d\phi = \frac{d\phi}{\cos\phi};$$

and integrating, so that x and ϕ may begin together,

$$2x = \text{Nep. log}\,\frac{1 + \sin\phi}{1 - \sin\phi};$$

and $$e^{2x}=\frac{1+\sin\phi}{1-\sin\phi}, \text{ and } \sin\phi=\frac{e^{2x}-1}{e^{2x}+1},$$

and $$\cos\phi=\frac{2e^x}{e^{2x}+1}=\frac{2}{e^x+e^{-x}}:$$

Now $$\cos\phi=\frac{d\phi}{dx}, \text{ and } dx=\frac{d\phi}{\cos\phi},$$

therefore $$\tfrac{1}{2}(e^x+e^{-x})=\frac{dx}{d\phi}, \text{ and } d\phi=\frac{dx}{\tfrac{1}{2}(e^x+e^{-x})}.$$

Now $$\cos\phi=1-\frac{\phi^2}{1.2}+\frac{\phi^4}{1.2.3.4}-\frac{\phi^6}{1.2.3.4.5.6}+\&c.$$

and $$\frac{1}{2}(e^x+e^{-x})=1+\frac{x^2}{1.2}+\frac{x^4}{1.2.3.4}+\frac{x^6}{1.2.3.4.5.6}+\&c.$$

Put C_2 for $\frac{1}{1.2}$, C_4 for $\frac{1}{1.2.3.4}$, C_6 for $\frac{1}{1.2.3.4.5.6}$, &c., and we have $$dx=\frac{d\phi}{1-C_2\phi^2+C_4\phi^4-C_6\phi^6+\&c.}:\quad d\phi=\frac{dx}{1+C_2x^2+C_4x^4+C_6x^6+\&c.}.$$

It is a remarkable property of these expressions, that the coefficients of the terms in the denominators, excepting the signs, are identical; and it is easy to see that the reciprocals of these series will be recurring series which will have the very same property. The reciprocal of the denominator of the first of these expressions (viz. cos ϕ) is the secant of ϕ; and the law of the terms is known to be this: *

Let $$\alpha=1,$$
$$\beta=\frac{2.1}{1.2}\alpha=1,$$
$$\gamma=\frac{4.3}{1.2}\beta-\frac{4.3.2.1}{1.2.3.4}\alpha=5,$$
$$\delta=\frac{6.5}{1.2}\gamma-\frac{6.5.4.3}{1.2.3.4}\beta+\frac{6\ldots.1}{1\ldots.6}\alpha=61,$$
$$\epsilon=\frac{8.7}{1.2}\delta-\frac{8.7.6.5}{1.2.3.4}\gamma+\frac{8\ldots.3}{1\ldots.6}\beta-\frac{8\ldots.1}{1\ldots.8}\alpha=1385,$$
$$\zeta=50521,\ \eta=2702765,\ \theta=199360981,\ \iota=19391512145,\ \&c.$$

Then $$\sec\phi=1+\frac{\alpha}{1.2}\phi^2+\frac{\beta}{1.2.3.4}\phi^4+\frac{\gamma}{1.2.3.4.5.6}\phi^6+\&c.$$

We have now $$dx=d\phi\left\{1+\frac{\alpha}{1.2}\phi^2+\frac{\beta}{1.2.3.4}\phi^4+\frac{\gamma}{1.2.3.4.5.6}\phi^6+\&c.\right\};$$
$$d\phi=dx\left\{1-\frac{\alpha}{1.2}x^2+\frac{\beta}{1.2.3.4}x^4-\frac{\gamma}{1.2.3.4.5.6}x^6+\&c.\right\};$$

* EULER, *Calculus Differentialis*, Pars ii. cap. viii.; also LEGENDRE, *Exercices de Calcul Integral*, tome ii. p. 144.

and hence, by integrating, and substituting for $\alpha.\beta$ &c. their numeral values

$$\left.\begin{aligned} x&=\phi+\frac{\phi^3}{1.2.3}+\frac{5\,\phi^5}{1.2.3.4.5}+\frac{61\,\phi^7}{1.2.3.4.5.6.7}+\frac{1385\,\phi^9}{1.2.3.4.5.6.7.8.9}+\&c.\\ \phi&=x-\frac{x^3}{1.2.3}+\frac{5\,x^5}{1.2.3.4.5}-\frac{61\,x^7}{1.2.3.4.5.6.7}+\frac{1385\,x^9}{1.2.3.4.5.6.7.8.9}-\&c.\end{aligned}\right\}\quad(14)$$

44. In the application of these formulæ, it must be remembered that ϕ is expressed in parts of the tabular radius of the trigonometrical tables: therefore, if the angle be expressed in minutes, it must be multiplied by the number 3437.74677 (the radius reduced to minutes). If the angle ϕ be considerable, the series will converge too slow to be useful.

A convenient expression, as an approximation to the value of x, may be found from the series by the following process: We found that

$$x=\phi+\frac{\phi^3}{6}+\frac{5\,\phi^5}{120}+\frac{61\,\phi^7}{5040}-\&c.$$

Now $$\sin\phi=\phi-\frac{\phi^3}{6}+\frac{\phi^5}{120}-\frac{\phi^7}{5040}+\&c.$$

and $$\tan\phi=\phi+\frac{2\,\phi^3}{6}+\frac{16\,\phi^5}{120}+\frac{272\,\phi^7}{5040}+\&c.$$

Therefore $$\tan\phi-\sin\phi=\frac{3\,\phi^3}{6}+\frac{15\,\phi^5}{120}+\frac{273\,\phi^7}{5040}+\&c.$$

and $$\tfrac{1}{3}(\tan\phi-\sin\phi)=\frac{\phi^3}{6}+\frac{5\,\phi^5}{120}+\frac{91\,\phi^7}{5040}+\&c.$$

By subtracting the sides of this last equation from those of the first, and transposing, we have

$$x=\phi+\tfrac{1}{3}(\tan\phi-\sin\phi)-\frac{\phi^7}{168}\ \&c.$$

If the angle ϕ be not very great, we have, as an approximation, putting a for the parameter,

$$x=a\{\phi+\tfrac{1}{3}(\tan\phi-\sin\phi)\},\qquad(15)$$

This in many cases may be sufficiently near to the value of x.

Suppose, as an example, that $a=100$ feet and $\phi=42°$; the calculation will be as follows:

	ϕ (in parts of radius) $=.7330388$
From	$\tan\phi=.9004040$
Subtract	$\sin\phi=.6691306$
Divide by	3)2312734(.0770911
	.81013 :.

$$x=100\{\phi+\tfrac{1}{3}(\tan\phi-\sin\phi)\}=81.013 \text{ feet.}$$

The more correct value of x is 80.916 feet; the corresponding value of $y=a\sec\phi$ is 134.563 feet; and the catenary arc $=a\tan\phi=90.040$ feet.

It is easy to see how, from the formula, an approximate geometrical determination of points in the catenary may be obtained.

45. It has been found that, x denoting the amplitude of any point in a catenary, y the ordinate at that point, and ϕ the angle which a line touching the curve at the top of the ordinate makes with the horizontal axis, then (Art. 43),

$$dx = a \sec\phi \,.\, d\phi,$$

and
$$x = a \int \sec\phi \,.\, d\phi \,.$$

Suppose *sec* ϕ to be expressed, not in decimal parts of the radius, as in the common trigonometrical table, but in units, each of which is the arc that measures an angle of one minute of a degree; of these, the radius contains 3437.74677. Let n denote this number, and suppose $d\phi$ to be one of these units. The integral $\int \sec\phi \, d\phi$ will be approximatively expressed by the series

$$\frac{1}{n}\{\sec 1' + \sec 2' + \sec 3' + \sec 4' + \&c.\}$$

and
$$x = \frac{a}{n}\{\sec 1' + \sec 2' + \sec 3' + \sec 4' + \&c.\}$$

Now the sum of the series continued to as many terms as there are minutes in the angle ϕ, is known to express the length of the enlarged meridian in WRIGHT's, or as it is called (improperly) MERCATOR's projection of the sphere; and these sums are given in nautical tables under the name of meridional parts, therefore, putting M (ϕ) to denote the meridional parts of a latitude ϕ, and this angle ϕ being found from either of these formulæ,

$$\sec\phi = \frac{f(x)}{a}, \ \tan\phi = \frac{F(x)}{a}; \text{ we have } x = a \,.\, \frac{M\phi}{n};$$

Or we may first find ϕ, and then $f(x)$ and F (x), from these formulæ,

$$\left.\begin{array}{ll} (1) & M(\phi) = \dfrac{nx}{a}; \\ (2) & f(x) = a \sec\phi = \dfrac{a}{\cos\phi}; \\ (3) & F(x) = a \tan\phi \,. \end{array}\right\} \quad (16)$$

EXAMPLE. Let the parameter of a catenary be 100 feet; it is proposed to find the ordinate $f(x)$ and the arc F (x) to the amplitude $x = 125$ feet.

	Log.		Log.		Log.
$n = 3437.7$	3.53627	a	2.00000	$\tan\phi$	10.20477
$x = 125$	2.09691	$\cos\phi$	9.72381	a	2.00000
$a = 100$ Ar. comp.	8.00000	$f(x) = 188.83$	2.27619	$F(x) = 160.24$	2.20477
$M(\phi) = 4297.1$	3.63318				
$\phi = 58° 2'$.					

Here we first find M (ϕ) to be 4297.1, which, by inspection in a table of meridional parts,* gives $\phi = 58° 2'$. The angle ϕ being known, $f(x)$ and F (x) are

* MENDOZA RIOS' *Collection of Tables for Navigation*; or any treatise on navigation.

found by (16). Greater accuracy may be obtained: this, however, is sufficient to shew the process of calculation.

As a table of catenarian co-ordinates and arcs may be made from a table of meridional parts; so, on the other hand, a table of meridional parts might be made *experimentally* from a catenary. This would indeed be a singular way of finding the course a ship should steer from a given place, to reach a port whose latitude and longitude were known. The solution in this way is evidently possible.

46. From the analogy which has been shewn to subsist between catenarian co-ordinates and the meridional parts of latitudes, and the properties of the former, we have (by the way) this property of the enlarged meridians in nautical charts.

THEOREM.—Let $\phi_1, \phi_2, \phi_3, \ldots \phi_n$ be latitudes of parallels on the sphere; and $M(\phi_1), M(\phi_2), M(\phi_3), \ldots M(\phi_n)$ their meridional parts;

Let ϕ be a parallel whose meridional parts $=M(\phi_1)+M(\phi_2)+M(\phi_3)\ldots+M(\phi_n)$;

Then, $\tan(45°+\frac{1}{2}\phi)=\tan(45°+\frac{1}{2}\phi_1)\tan(45°+\frac{1}{2}\phi_2)\tan(45°+\frac{1}{2}\phi_3)\ldots\tan(45°+\frac{1}{2}\phi_n)$.

EXAMPLE.

Merid. Parts.		Log tangent.
$M(\phi_1=12°)=\ \ 725.32$	$45°+\ 6°=51°$	10.091631
$M(\phi_2=14°)=\ \ 848.49$	$45\ +\ 7\ =52$	10.107190
$M(\phi_3=20°)=1225.14$	$45\ +10\ =55$	10.154773
$M(\phi_n=30°)=1888.38$	$45\ +15\ =60$	10.238561
$M(\phi=61°18\frac{1}{2}')=4687.33$	$45\ +\frac{1}{2}\phi=75°39'\,11''$	10.592155

Here the theorem is verified; for the sum of the meridional parts of 12°, 14°, 20°, 30° is the meridional parts of $61°\ 18\frac{1}{2}'=\phi$; and the continual product of the tangents of the halves of these angles, each increased by 45°, is equal to the tangent of $75°\ 39'\ 11''=45°+\frac{1}{2}\phi$ nearly.

One obvious use of this last formula would be, to construct a table of enlarged meridians, having a common difference of one minute; the latitudes being placed against their meridional parts.

RELATED PROPERTIES OF A CATENARY AND A PARABOLA.

47. The ancient geometers, in treating of curve lines, endeavoured to shew how they might be exhibited by an organic construction. It may be supposed that, with this view, they defined lines of the second order by sections of a cone, and conchoids by the motion of a point restrained to a certain course by an instrument. DIOCLES defined his *Cissoid* by shewing how points might be found in it; but NEWTON, probably supposing this imperfect, took the trouble to invent an instrument for describing it by continued motion, like the *conchoid*. The geometers who first treated of the catenary (viz. GREGORY and BERNOUILLI),

shewed how it might be constructed by a parabola and hyperbola. GREGORY'S construction is, however, complex,* and probably was never employed in delineating a catenary. I propose here to shew how the curve may be actually generated from a parabola alone, and my analysis will not require the integral calculus, it being derived from the property investigated in art. 25, that the increment of the curve at any point is always as the increment of the tangent of the angle which a line touching the curve at that point makes with the horizontal axis.

Fig. 7.

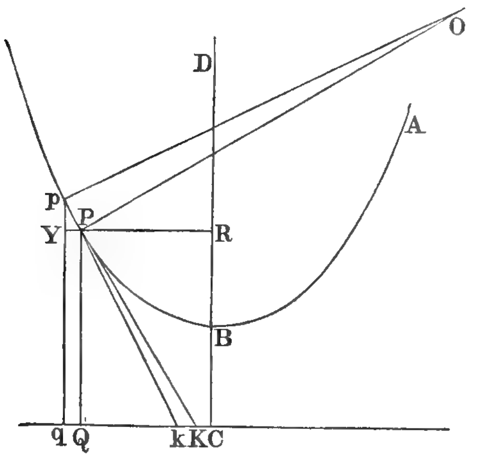
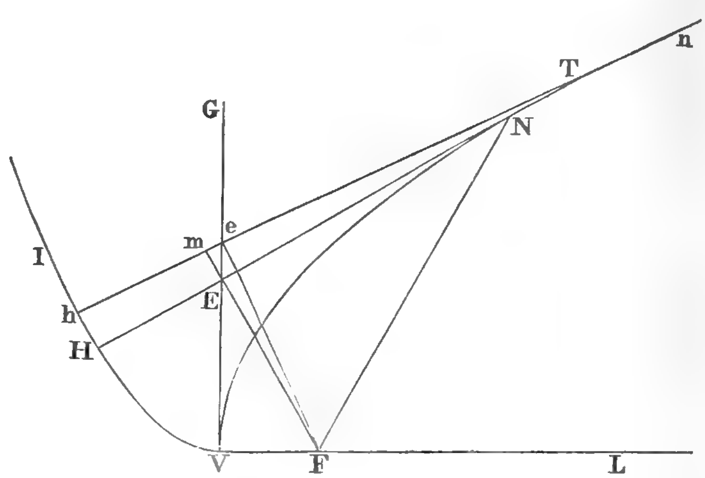


Fig. 8.



48. (Fig. 7.) Let ABP be a catenary, CQ its horizontal, CD its vertical axis, and BC its parameter. From P, p, two points, comprehending between them an infinitely small arc of the curve, draw ordinates PQ, pq, and straight lines PK, $p\,k$, touching the curve, and meeting CQ in K and k. Take a straight line VL, terminated at V (Fig. 8); to this line draw VG, a perpendicular, and in VL take VF, equal to BC, the parameter of the catenary. At the point F, make the angles VFE, VFe, equal to the angles PKQ, $p\,k\,q$.

By the nature of the catenary (art. 25), (see Figs. 7 and 8),

$$\text{arc P}p = \text{BC}\,(\tan \text{K} - \tan k) = \text{FV}\,(\tan e\text{FV} - \tan \text{EFV}):$$

But $$\text{FV}\,(\tan e\text{FV} - \tan \text{EFV}) = e\text{V} - \text{EV} = \text{E}e\,;$$

therefore, Ee, the increment of the line VE, is equal to Pp, the increment of the arc BP. Now, by construction, the straight line VE, and the catenary arc BP, must begin to be generated together; therefore, they are always equal.

In Fig. 8, draw EN perpendicular to FE, and $e\,n$ to Fe, and produce FE to meet $n\,e$ in m; and, in Fig. 7, draw p Y parallel to KQ. The infinitely small right-angled triangles PpY (Fig. 7), and eEm (Fig. 8), are similar, because the angle pPY is equal to the angle PKQ, that is, by construction, to the angle EFV, which

* *Philosophical Transactions*, as quoted at art. 21.

again is equal to Eem. Now, it was shewn that the lines eE, Pp, are equal; therefore, em is equal PY or Qq, the increment of CQ; and Em to pY, the increment of PQ.

It is a known property of a parabola, that the common intersection of a tangent to the curve, and a perpendicular to the tangent from its focus, is in a straight line touching the parabola at its vertex.* Hence it follows, that if a parabola be described about F as a focus, with its vertex at V, so that VE touches the curve at its vertex, the lines EN, En, will touch that parabola at points N, n.

Suppose, now, that the parabolic curve nNV is the edge of a mould of some solid material, such as in practice is used for tracing the curve, and that a thread is applied along that curved edge, beginning at its vertex V, and extending indefinitely to some point in the curve, where it is fixed to the mould; if the thread be gradually unlapped from the curve, the extremity of the thread that leaves the vertex V will generate a curve VHh, which will be the *involute* of the parabola. The lines EH, eh, will be *normals* to this curve, and em, the increment of the normal; but em has been proved to be equal to Qq, the increment of CQ, the abscissa of the catenary; therefore, that abscissa, and the normal EH, which begin to be generated together, will always be equal. It has been shewn also that Em, which is the increment of the line FE, is equal to pV, the increment of the ordinate PQ of the catenary; therefore, on the whole, we have this proposition.

THEOREM (Figs. 7 and 8).—Let VN be a parabola (Fig. 8), of which F is the focus, FL the axis, V the vertex, and VG a perpendicular to the axis at V. Suppose a thread to be applied along the curve, with one end at V, and the other fastened to the curve at some point indefinitely remote. Let this thread be unwound from the curve, and kept tight, so that its extremity V may describe a curve line VHI: this will be the *involute* of the parabola.

Take any point E in the line VG; draw EF to the focus, and EH perpendicular to EF; meeting the involute in H. Assume C a given point, as an origin in a straight line CD given in position (Fig. 7); in that line take CR equal to FE, draw RP perpendicular to CR, and equal to EH: The point P will be in a catenary, whose parameter CB is equal to FV in the parabola; and the arc BP of the curve, between the axis CB and P, is equal to the straight line VE.

49. This construction gives a perfectly distinct notion of the catenary: besides, for a practical purpose, it is easy, requiring merely the correct construction of a mould for making a parabolic curve.

* See my *Treatise on Conic Sections*, Part i. proposition 14.

Theoretically, a single point in the catenary is all that is required to determine any number of pairs of co-ordinates: For, let x and $f(x)$ be co-ordinates at a given point of the curve, then

$$\tfrac{1}{2}x;\ \text{and}\ f(\tfrac{1}{2}x) = \frac{\sqrt{[a^2 + \{f(x)\}^2]}}{a},$$

will be another pair, which may be found from the former by a geometrical construction; and any number $f(2x)$, $f(3x)$ &c. from $y=f(x)$ by the formulæ of Art. 27. Also, having given three of these four ordinates $f(x_\circ)$, $f(x_{,})$, $f(x_\circ + x_{,})$, $f(x_\circ - x_{,})$, the fourth is obtained by the relation

$$2f(x_\circ)f(x_{,}) = a\{f(x_\circ + x_{,}) + f(x_\circ - x_{,})\}.$$

50. Returning to the parabola and catenary (Figs. 7 and 8); since the triangle FE$e = \frac{1}{2}$FE . em is the increment of the triangle FVE; and the space PQqY = PQ . Qq is the increment of the curvilineal space BPQC; and, since FE = PQ, and em = Qq, therefore the triangle FEV is half the space BPQC, and that space is equal to CB × arc PB.

And because the triangles EVF, PQK are similar, EV : VF = PQ : QK. Now EV = arc PB, and VF = BC; therefore, in the catenary, the subtangent QK is a fourth proportional to the arc PB, the parameter BC, and the ordinate PQ.

51. At the points P, p, which are infinitely near, draw PO, po perpendiculars to the tangents PK, pk; these will meet at O, the centre of the *circle of curvature* at P: and the angle contained by the normals OP, Op will be equal to that contained by the tangents KP, kp at their intersection; but that angle is equal to the angle EFe in the parabola, which again is equal to the angle made by the lines EN, en, tangents to the parabola at T, their intersection; therefore, the isosceles triangles POp, ETm are similar, and

Em : Pp = ET : PO, that is, since Pp = Ee, Em : Ee = ET : PO;

Join FN, and because the triangle Eem is similar to EFV, which again is similar to FEN (*Conic sections*), so that

Em : Ee = EV : EF = EF : NF;

Therefore (since ultimately ET = EN), EN : PO = EN : NF:

Hence PO, the radius of curvature of the catenary at P, is equal to the line NF in the parabola: Now $\mathrm{FN} = \frac{\mathrm{FE}^2}{\mathrm{FV}} = \frac{\mathrm{PQ}^2}{\mathrm{CB}}$; hence it appears that the radius of curvature at any point in a catenary is a third proportional to the parameter, and an ordinate to the horizontal axis at that point.

51. From the four preceding articles, we derive the following proposition:

THEOREM (Figs. 7 and 8).—Let VN be a parabola, of which V is the vertex, and

F the focus: Let a straight line NE touch the curve at any point N, and let FE be a perpendicular from the focus on this line: Let z denote the parabolic arc VN, t the segment NE of the touching line between the point of contact and perpendicular, and p the perpendicular: Let ABC be a catenary, of which CQ is the horizontal axis, and BC $= a$ the parameter, which is equal to FV, one-fourth of the parameter of the parabola: Let CQ $= x$, and PQ $= f(x)$ be co-ordinates at any point P of the catenary: The parabola and catenary are so related, that if $x = z - t$, then $f(x) = p$.

52. Suspended bridges are now very common; and there is a species of bridge coming into use, the arch of which is convex upward, and formed by uniting several bended planks with oak trenails; this kind of bridge is, in some places, carried across ravines in the line of railways. I know not whether engineers erect these upon the principle of equilibrium, but I believe it quite possible that such arches may advantageously have the form of curves of equilibration, with straight roadways.

53. The construction of a catenary, also a curve of equilibration, must be greatly facilitated by a table of co-ordinates of a catenary; and I have already stated, that such a table has been actually given by the late DAVIES GILBERT, Esq.* The formulæ of this memoir give great facilities for the construction of such tables, and I have computed those here given by the following formulæ.

Continuing the notation of art. 39, and assuming the parameter a to be $= 1$, we have found

$$y = f(x) = \tfrac{1}{2}(e^{x} + e^{-x});\quad z = \mathrm{F}(x) = \tfrac{1}{2}(e^{x} - e^{-x}).$$

These expressions, by development, give

$$f(x) = 1 + \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} + \frac{x^6}{1.2.3.4.5.6} + \&c$$

$$\mathrm{F}(x) = x + \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} + \frac{x^7}{1.2.3.4.5.6.7} + \&c.$$

Some of the numbers in the tables were found by these series, as

$$f(1) = 1 + \frac{1}{1.2} + \frac{1}{1.2.3.4} + \&c. \qquad \mathrm{F}(1) = 1 + \frac{1}{1.2.3} + \frac{1}{1.2.3.4.5} + \&c.$$

$$f(.1) = 1 + \frac{1}{10.20} + \frac{1}{10.20.30.40} + \&c. \qquad \mathrm{F}(.1) = \frac{1}{10} + \frac{1}{10.20.30} + \&c.$$

$$f(.01) = 1 + \frac{1}{100.200} + \&c. \qquad \mathrm{F}(.01) = \frac{1}{100} + \frac{1}{100.200.300} + \&c.$$

$$f(.0001) = 1 + \frac{1}{10000.20000} + \&c. \qquad \mathrm{F}(.0001) = \frac{1}{10000} + \&c.$$

* *Philosophical Transactions* for 1826, Part iii. I have been told that the very ingenious author of this memoir did not himself compute the numbers, which are almost all incorrect.

54. The values of $f(1.1)$, $F(1.1)$; $f(.11)$, $F(.11)$, &c. were found from the formulæ

$$f(x_\circ+x_,)=f(x_\circ)f(x_,)+F(x_\circ)F(x_,);\ F(x_\circ+x_,)=F(x_\circ)f(x_,)+f(x_\circ)F(x_,).$$

Thus the first and second terms of a series of values of $f(x)$ and $F(x)$ were obtained; from these the following terms were deduced, by a formula investigated as follows.

In the formula $f(x_\circ+h)+f(x_\circ-h)=2f(x_\circ)f(h)$, put $x+h$ instead of $x_\circ$, and we have

$$f(x+2h)+f(x)=2f(x+h)f(h):$$

Now, $$f(h)=1+\frac{h^2}{1.2}+\frac{h^4}{1.2.3.4}+\frac{h^6}{1.2.3.4.5.6}+\&c.$$

Let $x_,=x+h$, $x_2=x_,+h$, $x_3=x_2+h$, &c. be successive values of x, which go on increasing by differences, each equal to h, and put

$$P=h^2+\frac{h^4}{3.4}+\frac{h^6}{3.4.5.6}+\frac{h^8}{3.4.5.6.7.8}+\&c.$$

Then, from what has been just shewn, we have

$$f(x_2)+f(x)=2f(x_,)+Pf(x_,),$$

$$\left.\begin{array}{ll}\text{and} & f(x_2)=f(x_,)+\{f(x_,)-f(x)\}+Pf(x_,);\\ \text{similarly,} & f(x_3)=f(x_2)+\{f(x_2)-f(x_,)\}+Pf(x_2),\\ \text{and} & f(x_4)=f(x_3)+\{f(x_3)-f(x_2)\}+Pf(x_3),\end{array}\right\}\ (\alpha)$$

Thus, all the numbers in the series $f(x), f(x_,), f(x_2)$, &c., which follow the first two, are derived from them simply by subtraction and addition, after the terms $Pf(x_,)$, $Pf(x_2)$, $Pf(x_3)$, have been found. In the computation of the tables, h was assumed to be 1, or $\frac{1}{10}$, or $\frac{1}{100}$, or $\frac{1}{10000}$.

Let t denote any term in the series of values $f(x), f(x_,), f(x_2)$, &c.

When $h=1$, then $Pt=t+\dfrac{t}{3.4}+\dfrac{t}{3.4.5.6}+\&c.$

When $h=\dfrac{1}{10}$, $\quad Pt=\dfrac{t}{100}+\dfrac{t}{300.400}+\dfrac{t}{300.400.500.600}+\&c.$

&c.

These series converge very fast, and their terms are readily found each from that before it: thus, $\dfrac{t}{300.400}$ is found from $\dfrac{t}{100}$ by dividing the latter by 1200, and so on.

55. For the corresponding series of arcs of the catenary, we have this formula, $F(x_\circ+h)+F(x_\circ-h)=2f(h)F(x_\circ)$, which, putting $x+h$ for $x_\circ$, gives

$$F(x+2h)+F(x)=2F(x+h)f(h):$$

Hence, putting $x_,=x+h$, $x_2=x_,+h$, &c., and P for the same series as before, we have

$$F(x_2)=F(x_,)+\{F(x_,)-Fx\}+PF(x_,) \qquad (\beta).$$

This formula, compared with formulæ (α) in last article, shews that the arcs $F(x_2)$, $F(x_3)$, &c., are to be found from $F(x)$ and $F(x_{,})$, exactly as $f(x_2)$, $f(x_3)$, &c. are from $f(x)$ and $f(x_{,})$.

56. As an example, let it be required to find the numeral values of the series of ordinates $f(.2)$, $f(.3)$, $f(.4)$, &c., and arcs $F(.2)$, $F(.3)$, $F(.4)$, &c. having given

$$f(0)=1\,,\ f(.1)=1.005004168\,;\quad F(0)=0\,,\ F(.1)=100166750$$

The calculation may stand thus:

$f(0)$	1.000000000	$F(0)$	0.000000000
$A=f(.1)$	1.005004168	$a=F(.1)$	0.100166750
$f(.1)-f(0)$	5004168	$F(.1)-F(0)$	.100166750
$B=\frac{A}{100}$	10050042	$b=\frac{a}{100}$	1001668
$C=\frac{B}{1200}$	8375	$c=\frac{b}{1200}$	835
$\frac{C}{3000}$	3	$\frac{c}{3000}$	0
$A=f(.2)$	1.020066756	$a=F(.2)$	0.201336003
$f(.2)-f(.1)$	15062588	$F(.2)-F(.1)$	.101169253
$B=\frac{A}{100}$	10200668	$b=\frac{a}{100}$	2013360
$C=\frac{B}{1200}$	8501	$c=\frac{b}{1200}$	1678
$\frac{C}{3000}$	3	$\frac{c}{3000}$	0
$A=f(.3)$	1.045338516	$a=F(.3)$	0.304520294
$f(.3)-f(.2)$	25271759	$F(.3)-F(.2)$	.103184291
$B=\frac{A}{100}$	10453385	$b=\frac{a}{100}$	3045203
$C=\frac{B}{1200}$	8711	$c=\frac{b}{1200}$	2538
$\frac{C}{3000}$	3	$\frac{c}{3000}$	1
$f(.4)$	1.081072374	$F(.4)$	0.410752327

These values of $f(.2)$, $f(.3)$, $f(.4)$, and $F(.2)$, $F(.3)$, $F(.4)$, are true to seven decimal places. In this way tables I. and II. were constructed; but the values were found to more decimal places. Precautions were also used as checks to bring out ten figures correct throughout the whole; but the principle of calculation was the same as has been here explained.

57. The Tables which are to follow require hardly any explanation. In them all, the parameter, that is $f(0)$, is unity. The first gives the values of $f(x)$, $F(x)$,

and the angle ϕ, to a series of values of x, from $x=0$ to $x=.01$, the common difference of the values of x being .0001. The second gives the values of $f(x)$ and F (x) and ϕ, from $x=.01$ to $x=1$, the common difference being .01 ; and farther, from $x=1$ to $x=5$, the common difference being .05. In the third Table, instead of a series of values of x increasing by a common difference, there are given the values of x, $f(x)$, and F (x) to a series of angles ϕ, increasing by a common difference of half a degree. In this table the values of x are NEPER's logarithms of the tangents of $(45°+\frac{1}{2}\phi)$. The ordinate $f(x)$ is the natural secant, and the arc F (x) the natural tangent of that angle. These tables, I presume, are sufficient for all applications of the catenary to the construction of bridges of suspension and of equilibration.

58. The second table alone gives the values of the ordinate and arc of the curve to values of x, which differ by $\frac{1}{100}$th of the parameter from $x=0$ to $x=1$; but, by the first and second tables used together, we may find the same to values of x which differ by $\frac{1}{10000}$th of the parameter, by the formula for $f(x+h)$, and F $(x+h)$: here x expresses the tenths and hundredths of the given value of x, and h the thousandths and ten thousandths.

As an example, let the values of $f(x)$ and F (x) be required to $x+h=.8327$: In this case,

$$x=.83, \qquad fx=1.36468{,}40133, \qquad F(x)=0.92863{,}47270\,;$$
$$h=.0027, \qquad f(h)=1.00000{,}36450 \qquad F(h) \quad 0.00270{,}00033.$$

And the formulæ for calculation are

$$f(x+h)=f(x)\,.\,f(h)+F(x)\,F(h): \qquad F(x+h)=F(x)f(h)+f(x)\,F(h).$$

We may be satisfied with seven correct figures of the result, then we may neglect two figures of each tabular number, and, using contracted multiplication, have

$$\begin{array}{rlrl} f(x)f(h)= & 1.36468898 & F(x)f(h)= & 0.92863810 \\ F(x)\,F(h)= & \underline{.00250731} & f(x)\,F(h)= & \underline{0.00270001} \\ f(.8327)= & 1.3671963\,. & F(.8327) & 0.9313381\,. \end{array}$$

If, instead of seven, no more decimal places are required than are given of the value of x (viz. four), we may then take only five figures of the given tabular numbers, and now we have

$$\begin{array}{rlrl} f(x)f(h)= & 1.36468 & F(x)f(h)= & 0.92863 \\ F(x)\,F(h)= & \underline{.00251} & f(x)\,F(h)= & \underline{0.00270} \\ f(.8327)= & 1.3662\,. & f(.8327)= & 0.9313\,. \end{array}$$

From the first and second tables a more extensive one may be formed by interpolation and prolongation; indeed it was partly with this view that the numbers have been carried on to so many places of decimals.

Construction of Curves of Equilibration by the Tables.

59. Various problems may be proposed respecting the construction of a catenary and equilibrated arches; but of these, I believe the two which follow are the most useful.

Problem I.—*A chain of a given length hangs freely between two points, which are at a given distance in a horizontal line; to find the position of its lowest point, and the parameter of the catenary.*

Suppose the chain to be 100 feet in length, and the distance between the points of suspension to be 60 feet.

Applying our notation: in a catenary of which a is the parameter, x the amplitude of $f(x)$ an ordinate, and $\mathrm{F}(x)$ the corresponding arc of the curve, there are given $x = 30$ feet, and $\mathrm{F}(x) = 50$ feet; to find a and $f(x)-a$.

Assuming a to be $= 1$, the problem requires that a tabular value of x be found, which shall satisfy the condition $\frac{\mathrm{F}(x)}{x}=\frac{50}{30}=1.66667$. Now, the quantity $\frac{\mathrm{F}(x)}{x}$, at first $\doteq 1$, increases continually: and it appears from our second table, that to $x_0=1.8$, $\frac{\mathrm{F}(x_0)}{x_0}=1.63454$, and to $x_,=1.85$, $\frac{\mathrm{F}(x_,)}{x_,}=1.67637$,

therefore $$\frac{\mathrm{F}(x)}{x}-\frac{\mathrm{F}(x_0)}{x_0}=.03213\text{; and }\frac{\mathrm{F}(x_,)}{x_,}-\frac{\mathrm{F}(x_0)}{x_0}=.04183.$$

Now, as an approximation, the first of these differences will be to the second nearly as $x-x_0$ to $x_,-x_0$:

Therefore, $$4183 : 3213 = x_,-x_0 : x-x_0 = .05 : x-x_0\,;$$

and $$x-x_0=\frac{3213\times .05}{4183}=.03841,\text{ and } x=1.8+03841=1.83841\,;$$

and $$\mathrm{F}(x)=\frac{5x}{3}=3.06402.$$

This is the tabular value of $\mathrm{F}(x)$ when the parameter $=1$, but to the parameter a, we have $x=1.83841\,a$, and $\mathrm{F}(x)=3.06402\,a$. In the catenary formed by the chain, $\mathrm{F}(x) = 50$ feet; therefore, $a=\frac{50}{3.06402}=16.318$ feet. Now, when $a=1$, $f(x)$ is the secant of an arc ϕ, of which $\mathrm{F}(x)$ is the tangent: therefore, $\tan\phi=3.06402$, and $\phi=71^\circ\,55'\,30''$; and $f(x) = \sec\phi=3.22308$, and $f(x)-1=2.22308$. Hence the distance between the lowest point of the chain and the line joining the points of suspension, is $\frac{2.22308\times 50}{3.06402} = 36.277$ feet. Now, the parameter of the curve has been found, therefore the curve may be constructed by co-ordinates either from the table, or by the geometrical construction given in article 48.

60. The problem may be otherwise solved as follows: Putting x and $f(x)$ to denote the co-ordinates of a tabular catenarian arc, similar to the half of that formed by the chain, and $\mathrm{F}(x)$ for the tabular arc, the parameter being unity, let the angle made by a line touching the curve and the horizontal axis be ϕ: Put 2 C for the length of the chain in feet, and 2 D for the distance between its points of support; these are, by hypothesis, given numbers.

By the nature of the catenary (art. 39),

$$x = \text{Nep. log}\left\{\frac{\tan(45^\circ + \frac{1}{2}\phi)}{\text{rad.}}\right\}, \text{ and } \mathrm{F}x = \tan\phi:$$

Therefore,
$$\cot\phi \, . \, \text{Nep. log}\left\{\frac{\tan(45^\circ + \frac{1}{2}\phi)}{\text{rad.}}\right\} = \frac{x}{\mathrm{F}(x)} = \frac{\mathrm{D}}{\mathrm{C}}.$$

Now N denoting any number,

$$\text{Nep. log N} : \text{Com. log N} = \text{Nep. log } 10 : \text{Com. log } 10.$$

Again,
$$\text{Nep. log } 10 = 2.3025851 = \frac{1}{.43429448},$$

therefore, ϕ must satisfy this condition;

$$\text{Nat. cot } \phi \, . \, \text{Com. log}\left\{\frac{\tan(45^\circ + \frac{1}{2}\phi)}{\text{rad.}}\right\} = .43429448 \, \frac{\mathrm{D}}{\mathrm{C}}.$$

The value of ϕ is to be found by successive trials in the trigonometrical tables.

In the example of this problem $\frac{\mathrm{D}}{\mathrm{C}} = \frac{3}{5}$; therefore, the angle ϕ must satisfy this condition

$$\frac{\text{Com. log tan}(45^\circ + \frac{1}{2}\phi)}{\text{Nat. tan } \phi} = .2605767,$$

which is nearly true when $\phi = 71^\circ\ 55'\ 30''$; for

$$\log\tan(45^\circ + \tfrac{1}{2}\phi) = \tan 80^\circ\ 57'\ 45'' = .7984515;$$

and
$$\text{Nat. tan } \phi = 3.064031;$$

and
$$\frac{.7984515}{3.064031} = .26059.$$

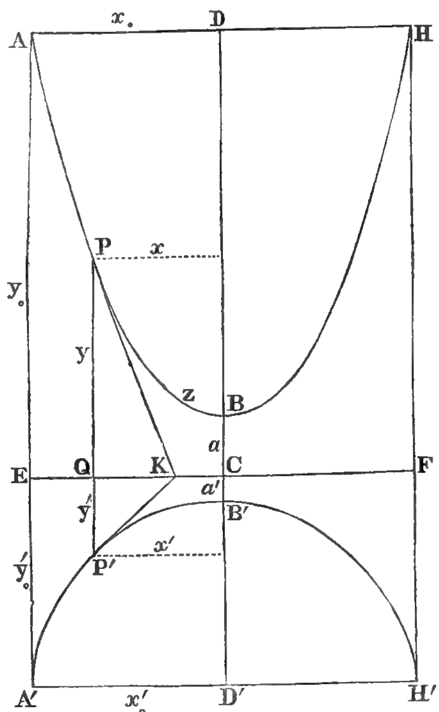
When ϕ is known, the things required may be found as by the other method.

61. Problem II.—*The span and height of an equilibrated arch are given: the roadway over it is to be a straight line: the parameter of the curve, which is a line equal to its thickness at the crown, is also given: to find the numeral values of ordinates to the curve.*

Let the figure bounded by the straight lines A′E, EF, FH′, and the curve

A′P′B′H′, be a section of the arch and the materials of which it is composed. Assuming the roadway EF for the horizontal axis of the curve, and CB′D′, a perpendicular to EF through B, the crown of the arch, as the vertical axis; let CQ $= x$, and P′Q $= y'$ be co-ordinates at any point P′ of the curve: let a' denote CB′ the thickness at the crown, which is the parameter of the curve; and let a, a constant line, be its modulus (see art. 31).

Fig. 9.



The equation of the curve is

$$y' = \frac{a'}{2}\left\{e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right\} \quad \text{(Art. 28)}.$$

Let ABH be a catenary whose parameter BC$=a$ is equal to the modulus of the equilibrated arch A′B′H′; let the two curves have the same horizontal axis EF, and their vertical axes CD′, CD, in the same straight line; and let $x =$ CQ, and $y =$ PQ, be co-ordinates of the catenary at any point P. Its equation was found to be

$$y = \frac{a}{2}\left\{e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right\}.$$

From these equations, it appears (as has been shewn, art. 37), that

$$y' : y = a' : a.$$

By this property, the ordinates of the curve of equilibration may be found from those of the catenary; and for these, there are given in this memoir tables sufficiently extensive, and more than sufficiently accurate, for all practical purposes.

Before we can employ the tables, however, the numeral value of a, the parameter of the catenary must be known. Produce A′E, H′F, the ordinates of the equilibrated arch, until they meet the catenary in A and H.

Put CE (half the span of the arch) $= x_\circ$; A′E (the height of the roadway above the base of the arch) $= y'_\circ$; AE (the distance of the extremity of the catenary from the roadway) $= y_\circ$. Because $a' : y'_\circ = a : y_\circ$, and that a' and $y'_\circ$ are given, the ratio of a to $y_\circ$ is given: hence, if a be found, $y_\circ$, the ordinate of the catenary, will be known. Again, because $\frac{y'_\circ}{a'} = \frac{y_\circ}{a} = f\left(\frac{x_\circ}{a}\right)$, the tabular value of the ordinate in a catenary whose modulus is unity, the amplitude being $\frac{x_\circ}{a}$; therefore, the value of $\frac{y'_\circ}{a'}$ may be found nearly by the table, just as x was found in the last problem; and thence, $\frac{x_\circ}{a}$ the amplitude of the function $f\left(\frac{x_\circ}{a}\right)$.

Now $x_\circ$ is known, therefore a becomes known; and $y_\circ = \frac{a\,y'_\circ}{a'}$, the ordinate of the catenary is known.

Besides this way of finding $y_\circ$ by the table, there are two direct methods given in art. 42. From the first of these, considering that $\frac{y_\circ}{a} = \frac{y'_\circ}{a'}$, we have

$$\frac{x_\circ}{a} = (\text{Nep. log } 10) \times \text{Com. log } \frac{y'_\circ + \sqrt{(y'^2_\circ - a'^2)}}{a'}.$$

Now, a' and $y'_\circ$, and Nep. log 10 = 2.3025851, are given; therefore $\frac{x_\circ}{a}$ is given, and $x_\circ$ is also given; therefore a is given. And since $y_\circ = \frac{a}{a'}\,y'_\circ$; therefore $y_\circ$, either ordinate of the catenary, at the end of the roadway, is given.

By the second method, putting ϕ to denote the angle which a straight line touching the subsidiary catenary at the top of either of its extreme ordinates $y_\circ$, makes with the horizontal axis, we find that angle, and thence a and $y_\circ$ by these formulæ (to which logarithms are particularly applicable),

$$\cos\phi = \frac{a}{y_\circ} = \frac{a'}{y'_\circ}; \qquad \frac{x_\circ}{a} = 2.3025851\,.\,\text{Com. log}\left\{\frac{\tan(45^\circ + \frac{1}{2}\phi)}{\text{rad.}}\right\};$$

$$a = x_\circ \div \frac{x_\circ}{a}; \qquad y_\circ = \frac{a}{a'}\,y'_\circ.$$

In this way, by either method, we determine the catenary whose parameter is the modulus of the equilibrated arch; and then, the ordinates of the latter by those of the former.

Example.—Find co-ordinates of an equilibrated arch A′B′H′ (Fig. 9), having given its span A′H′ = 100 feet; its height B′D′ = 40 feet; the thickness at the crown B′C′ = 6 feet; and therefore A′E the height of the roadway above the base of the arch = 46 feet.*

In this case,

$$CE = x_\circ = 50, \quad A'E = y'_\circ = 46, \quad B'C = a' = 6.$$

Calculation by the first formula:

$$\sqrt{(y'^2_\circ - a'^2)} = \sqrt{2080} = 45.6070170,$$

$$\frac{\sqrt{(y'^2_\circ - a'^2)} + y'_\circ}{a'} = 15.267836.$$

The common logarithm of this number is 1.1837775;

* These are nearly the dimensions of the middle arch of Blackfriars' Bridge, London.

$$\frac{x_\circ}{a} = 2.3025851 \times 1.1837775 = 2.725748\,;$$

$$a = x_\circ \div \frac{x_\circ}{a} = \frac{50}{2.725748} = 18.3436 \text{ feet,}$$

$$y_\circ = \frac{a}{a'} y'_\circ = \frac{18.3436 \times 46}{6} = 140.6343.$$

We have now found CB $= a$, the modulus of the equilibrated curve (which is also the parameter of the catenary), to be 18.3436 feet, and AE $=$ HF $= y_\circ = 140.6343$ feet.

Logarithmic calculation by the second formula :

		Logarithms.
	$a' = 6$	0.7781513
	$y'_\circ = 46$	1.6627578
$\frac{a}{y_\circ} = \frac{a'}{y'_\circ} = \cos$	$(\phi_\circ = 82°\ 30'\ 19'')$	9.1153935
	$45° + \frac{1}{2}\phi_\circ = 86°\ 15'\ 9\frac{1}{2}''$	
com. log. $\frac{\tan(45° + \frac{1}{2}\phi_\circ)}{\text{rad}}$	$= 1.1837772$	0.0732701
	Nep. log $10 = 2.3025851$	0.3622157
	$\frac{x_\circ}{a} = 2.725748$	0.4354858
	$x_\circ = 50$ feet	1.6989700
$a = x_\circ \div \frac{x_\circ}{a}$	$= 18.343585$	1.2634842
	$\frac{a'}{a} = .3270898$	9.5146671
$y_\circ = y'_\circ \frac{a}{a'}$	$= 140.6344$	2.1480907

In the catenary, we have now its parameter a = 18.343585 feet; and, to construct it, we may set off from C both ways, in the line EF, distances each equal to a, and divide each of these into 100 equal parts. If now x denote the number of these divisions between C and any point in the scale CE, the ordinate of the catenary at that point will be $y = a \,.\, f(x)$; here $f(x)$ denotes the tabular value of the ordinate whose amplitude is x. The corresponding ordinate of the equilibrated curve will be $a' \,.\, f(x)$, for then $y : y' = a : a'$. There is, however, no necessity for actually constructing the catenary; it is merely subsidiary, and it has been introduced here only as a geometrical representation of the relation between the tabular co-ordinates x and $f(x)$. We have found its extreme ordinates

$y_\circ = a f(x_\circ)$ to be each 140.6343 feet; these have the same amplitude as $y_\circ' = 46$ feet, the ordinate of the equilibrated curve.

The following Table shews the length of forty-five ordinates at as many points of the arch on either side of the crown. The first ten stand at equal distances of 1.834 feet along the roadway; the remainder are distant from each by half that extent, viz. .917.

Co-ordinates of an Equilibrated Arch.

Tabular Co-ordinates of Catenary.		Co-ordinates of Arch in Feet.		Tabular Co-ordinates of Catenary.		Co-ordinates of Arch in Feet.	
x	y	x	y'	x	y	x	y'
0	1.00000	0.000	6.000	1.65	2.69951	30.267	16.197
.1	1.00500	1.834	6.030	1.70	2.82832	31.184	16.970
.2	1.02007	3.669	6.120	1.75	2.96419	32.101	17.785
.3	1.04534	5.503	6.272	1.80	3.10747	33.018	18.645
.4	1.08107	7.337	6.486	1.85	3.25853	33.936	19.551
.5	1.12763	9.172	6.766	1.90	3.41773	34.853	20.506
.6	1.18547	11.006	7.113	1.95	3.58548	35.770	21.513
.7	1.25517	12.840	7.531	2.00	3.76220	36.687	22.573
.8	1.33743	14.675	8.025	2.05	3.94832	37.604	23.690
.9	1.43309	16.509	8.599	2.10	4.14431	38.522	24.866
1.	1.54308	18.343	9.258	2.15	4.35067	39.439	26.104
1.05	1.60379	19.261	9.623	2.20	4.56791	40.356	27.407
1.10	1.66852	20.178	10.011	2.25	4.79657	41.273	28.779
1.15	1.73741	21.095	10.424	2.30	5.03722	42.190	30.223
1.20	1.81066	22.012	10.864	2.35	5.29047	43.107	31.743
1.25	1.88842	22.929	11.330	2.40	5.55695	44.025	33.342
1.30	1.97091	23.847	11.825	2.45	5.83732	44.942	35.024
1.35	2.05833	24.764	12.350	2.50	6.13229	45.859	36.794
1.40	2.15090	25.681	12.905	2.55	6.44259	46.776	38.656
1.45	2.24884	26.598	13.493	2.60	6.76901	47.693	40.614
1.50	2.35241	27.515	14.114	2.65	7.11234	48.610	42.674
1.55	2.46186	28.433	14.771	2.70	7.47347	49.528	44.841
1.60	2.57746	29.350	15.465			50.000	46.000

The first two columns of the table express the length of the co-ordinates of a catenary whose parameter is unity; these are just the numbers of our second table. The third column contains the values of the numbers in the first column reduced to feet, by multiplying each by the number $a = 18.343585$, and putting down the results true to thousandth parts of a foot.

The second column, or values of y reduced to feet by multiplying each number by a, would express the ordinates of the catenary; and any ordinate $(a\,y)$ of the catenary, having to the corresponding ordinate of the equilibrated arch the ratio of a to a', that is, $a\,y : y' = a : a'$, it follows that $a\,a'\,y = a\,y'$, and $y' = a'\,y$. Now $a' = 6$, therefore the numbers in the fourth column are found from those in the second by multiplying each by 6.

The numbers in this table have the general properties which belong to the function $f(x)$ in our catenary tables, so that, if $f(x-h)$, $f(x)$, $f(x+h)$ be three ordinates whose amplitudes have a common difference h, then

$$2f(x)f(h)=f(x+h)+f(x-h).$$

By this formula we may interpolate an ordinate between any two (except the last two), the formula for bisection being

$$f(x)=\frac{\frac{1}{2}\{f(x+h)+f(x-h)\}}{f(h)}.$$

It will be best to use the tabular amplitudes. Thus, to interpolate an ordinate between $f(2.60)$ and $f(2.65)$, the difference of whose amplitudes is .05, we have $f(h)=f(.05)=1.00125$, $f(x-h)=f(2.60)=6.76901$, $f(x+h)=f(2.65)=7.11234$: these numbers substituted in the formula give

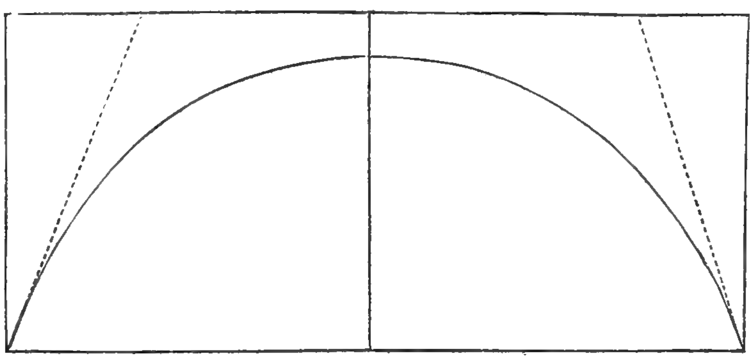
$$f(x)=f(2.625)=\frac{3.55617+3.38455}{1.00125}=6.93203.$$

These numbers reduced to feet, give

$$x=2.625\times 18.343585=48.152 \text{ feet}, \quad y=6.93203\times 6=41.592 \text{ feet}.$$

It has been found that the angle ϕ, which lines touching the curve at the extreme ordinates of the catenary make with the horizontal axis, is $82^\circ\ 30'\ 19''$; and, ϕ' denoting the like angle in the arch, we have $a:a'=\tan\phi:\tan\phi'$ (art. 37), therefore $\phi'=68^\circ\ 5'\ 22''$. Professor Robison, in his Essay on Arch in the *Encyclopædia Britannica*, has taken this arch as an example from Hutton's *Essay on Bridges;* and he says, "It is by no means deficient in gracefulness, and is abundantly roomy for the passage of craft; so that no objection can be offered against its being adopted on account of its mechanical excellency." The reader may, however, form his own opinion as to these qualities from the subjoined diagram, which represents a vertical section of the arch along its road-way, constructed by a scale from the table.

Fig. 10.



I believe enough has been done in this memoir to enable engineers properly instructed in mathematics, to construct arches having the form of equilibrated curves. The requisite tables now follow.

Table of Corresponding Values of x, the *Abcissa*, or *Amplitude*, $f(x)$; the *Ordinate*; F (x), the *Arc* of a Catenary; and φ, the *Angle* which a tangent to the curve at the top of the ordinate makes with the horizontal axis: The Parameter being the unit of the numbers by which they are expressed.

TABLE I.—The Amplitude between $x = 0$, and $x = .01$.

Amp. x	Ordinate $f(x)$	Arc F (x)	Angle φ
.0000	1.00000 00000	0.00000 00000	0° 0′ 0″
.0001	1.00000 00050	0.00010 00000	0 0 21
.0002	1.00000 00200	0.00020 00000	0 0 41
.0003	1.00000 00450	0.00030 00000	0 1 2
.0004	1.00000 00800	0.00040 00000	0 1 23
.0005	1.00000 01250	0.00050 00000	0 1 43
.0006	1.00000 01800	0.00060 00000	0 2 4
.0007	1.00000 02450	0.00070 00001	0 2 24
.0008	1.00000 03200	0.00080 00001	0 2 45
.0009	1.00000 04050	0.00090 00001	0 3 6
.0010	1.00000 05000	0.00100 00002	0 3 26
.0011	1.00000 06050	0.00110 00002	0 3 47
.0012	1.00000 07200	0.00120 00003	0 4 8
.0013	1.00000 08450	0.00130 00004	0 4 28
.0014	1.00000 09800	0.00140 00005	0 4 49
.0015	1.00000 11250	0.00150 00006	0 5 9
.0016	1.00000 12800	0.00160 00007	0 5 30
.0017	1.00000 14450	0.00170 00008	0 5 51
.0018	1.00000 16200	0.00180 00010	0 6 11
.0019	1.00000 18050	0.00190 00011	0 6 32
.0020	1.00000 20000	0.00200 00013	0 6 53
.0021	1.00000 22050	0.00210 00015	0 7 13
.0022	1.00000 24200	0.00220 00018	0 7 34
.0023	1.00000 26450	0.00230 00020	0 7 54
.0024	1.00000 28800	0.00240 00023	0 8 15
.0025	1.00000 31250	0.00250 00026	0 8 36
.0026	1.00000 33800	0.00260 00029	0 8 56
.0027	1.00000 36450	0.00270 00033	0 9 17
.0028	1.00000 39200	0.00280 00037	0 9 38
.0029	1.00000 42050	0.00290 00041	0 9 58
.0030	1.00000 45000	0.00300 00045	0 10 19
.0031	1.00000 48050	0.00310 00050	0 10 39
.0032	1.00000 51200	0.00320 00055	0 11 0
.0033	1.00000 54450	0.00330 00060	0 11 21
.0034	1.00000 57800	0.00340 00066	0 11 41
.0035	1.00000 61250	0.00350 00071	0 12 2
.0036	1.00000 64800	0.00360 00078	0 12 23
.0037	1.00000 68450	0.00370 00084	0 12 43
.0038	1.00000 72200	0.00380 00091	0 13 4
.0039	1.00000 76050	0.00390 00099	0 13 25
.0040	1.00000 80000	0.00400 00107	0 13 45
.0041	1.00000 84050	0.00410 00115	0 14 6
.0042	1.00000 88200	0.00420 00123	0 14 26
.0043	1.00000 92450	0.00430 00133	0 14 47
.0044	1.00000 96800	0.00440 00142	0 15 8

TABLE I.—continued.

Amp. x	Ordinate $f(x)$	Arc $F(x)$	Angle ϕ
.0045	1.00001 01250	0.00450 00152	0° 15′ 28″
.0046	1.00001 05800	0.00460 00162	0 15 49
.0047	1.00001 10450	0.00470 00173	0 16 10
0048	1.00001 15200	0.00480 00184	0 16 30
.0049	1.00001 20050	0.00490 00196	0 16 51
.0050	1.00001 25000	0.00500 00208	0 17 11
.0051	1.00001 30050	0.00510 00221	0 17 32
.0052	1.00001 35200	0.00520 00234	0 17 53
.0053	1.00001 40450	0.00530 00248	0 18 13
.0054	1.00001 45800	0.00540 00262	0 18 34
.0055	1.00001 51250	0.00550 00277	0 18 55
.0056	1.00001 56800	0.00560 00293	0 19 15
.0057	1.00001 62450	0.00570 00309	0 19 36
.0058	1.00001 68200	0.00580 00325	0 19 56
.0059	1.00001 74050	0.00590 00342	0 20 17
.0060	1.00001 80001	0.00600 00360	0 20 38
.0061	1.00001 86051	0.00610 00378	0 20 58
.0062	1.00001 92201	0.00620 00397	0 21 19
.0063	1.00001 98451	0.00630 00417	0 21 40
.0064	1.00002 04801	0.00640 00437	0 22 0
.0065	1.00002 11251	0.00650 00458	0 22 21
.0066	1.00002 17801	0.00660 00479	0 22 41
.0067	1.00002 24451	0.00670 00501	0 23 2
.0068	1.00002 31201	0.00680 00524	0 23 23
.0069	1.00002 38051	0.00690 00548	0 23 43
.0070	1.00002 45001	0.00700 00572	0 24 4
.0071	1.00002 52051	0.00710 00597	0 24 25
.0072	1.00002 59201	0.00720 00622	0 24 45
.0073	1.00002 66451	0.00730 00649	0 25 6
.0074	1.00002 73801	0.00740 00676	0 25 26
.0075	1.00002 81251	0.00750 00703	0 25 47
.0076	1.00002 88801	0.00760 00732	0 26 8
.0077	1.00002 96451	0.00770 00761	0 26 28
.0078	1.00003 04201	0.00780 00791	0 26 49
.0079	1.00003 12052	0.00790 00822	0 27 10
.0080	1.00003 20002	0.00800 00853	0 27 30
.0081	1.00003 28052	0.00810 00886	0 27 51
.0082	1.00003 36202	0.00820 00919	0 28 11
.0083	1.00003 44452	0.00830 00953	0 28 32
.0084	1.00003 52802	0.00840 00988	0 28 53
.0085	1.00003 61252	0.00850 01024	0 29 13
.0086	1.00003 69802	0.00860 01060	0 29 34
.0087	1.00003 78452	0.00870 01097	0 29 55
.0088	1.00003 87202	0.00880 01136	0 30 15
.0089	1.00003 96052	0.00890 01175	0 30 36
.0090	1.00004 05003	0.00900 01215	0 30 56
.0091	1.00004 14053	0.00910 01256	0 31 17
.0092	1.00004 23203	0.00920 01298	0 31 38
.0093	1.00004 32453	0.00930 01341	0 31 58
.0094	1.00004 41803	0.00940 01384	0 32 19
.0095	1.00004 51253	0.00950 01429	0 32 40
.0096	1.00004 60804	0.00960 01475	0 33 0
.0097	1.00004 70454	0.00970 01521	0 33 21
.0098	1.00004 80204	0.00980 01569	0 33 41
.0099	1.00004 90054	0.00990 01617	0 34 2
.0100	1.00005 00004	0.01000 01667	0 34 23

TABLE II.—The Amplitude between $x = 0$, and $x = 1 \ldots 5$.

Amp. x	Ordinate $f(x)$	Arc $F(x)$	Angle ϕ
.00	1.00000 00000	0.00000 00000	0° 0′ 0″
.01	1.00005 00004	0.01000 01667	0 34 23
.02	1.00020 00067	0.02000 13334	1 8 45
.03	1.00045 00338	0.03000 45002	1 43 7
.04	1.00080 01067	0.04001 06675	2 17 28
.05	1.00125 02604	0.05002 08359	2 51 49
.06	1.00180 05401	0.06003 60065	3 26 9
.07	1.00245 10006	0.07005 71807	4 0 27
.08	1.00320 17070	0.08008 53606	4 34 44
.09	1.00405 27345	0.09012 15492	5 8 59
.10	1.00500 41681	0.10016 67500	5 43 12
.11	1.00605 61029	0.11022 19676	6 17 24
.12	1.00720 86441	0.12028 82074	6 51 33
.13	1.00846 19071	0.13036 64762	7 25 39
.14	1.00981 60171	0.14045 77817	7 59 43
.15	1.01127 11096	0.15056 31332	8 33 44
.16	1.01282 73300	0.16068 35410	9 7 43
.17	1.01448 33955	0.17082 00174	9 41 37
.18	1.01624 37873	0.18097 35759	10 15 29
.19	1.01810 43658	0.19114 52319	10 49 17
.20	1.02006 67556	0.20133 60025	11 23 1
.21	1.02213 11530	0.21154 69070	11 56 41
.22	1.02429 77643	0.22177 89663	12 30 17
.23	1.02656 68062	0.23203 32037	13 3 48
.24	1.02893 85057	0.24231 06446	13 37 15
.25	1.03141 30999	0.25261 23168	14 10 38
.26	1.03399 08362	0.26293 92504	14 43 55
.27	1.03667 19725	0.27329 24782	15 17 7
.28	1.03945 67769	0.28367 30354	15 50 14
.29	1.04234 55278	0.29408 19602	16 23 16
.30	1.04533 85141	0.30452 02934	16 56 12
.31	1.04843 60352	0.31498 90790	17 29 2
.32	1.05163 84007	0.32548 93636	18 1 46
.33	1.05494 59309	0.33602 21975	18 34 25
.34	1.05835 89567	0.34658 86339	19 6 57
.35	1.06187 78192	0.35718 97294	19 39 22
.36	1.06550 28703	0.36782 65442	20 11 42
.37	1.06923 44727	0.37850 01420	20 43 54
.38	1.07307 29993	0.38921 15901	21 16 0
.39	1.07701 88342	0.39996 19597	21 47 58
.40	1.08107 23718	0.41075 23258	22 19 50
.41	1.08523 40176	0.42158 37675	22 51 34
.42	1.08950 41877	0.43245 73679	23 23 11
.43	1.09388 33091	0.44337 42144	23 54 41
.44	1.09837 18198	0.45433 53987	24 26 2
.45	1.10297 01686	0.46534 20169	24 57 16
.46	1.10767 88152	0.47639 51697	25 28 23
.47	1.11249 82307	0.48749 59625	25 59 21
.48	1.11742 86970	0.49864 55052	26 30 11
.49	1.12247 13071	0.50984 49129	27 0 52
.50	1.12762 59652	0.52109 53055	27 31 26
.51	1.13289 33869	0.53239 78081	28 1 51
.52	1.13827 40988	0.54375 35509	28 32 7
.53	1.14376 86391	0.55516 36695	29 2 15
.54	1.14937 75573	0.56662 93049	29 32 4

TABLE II.—continued.

Amp. x	Ordinate $f(x)$	Arc $F(x)$	Angle φ
.55	1.15510 14141	0.57815 16037	30° 2′ 4″
.56	1.16094 07821	0.58973 17182	30 31 45
.57	1.16689 62451	0.60137 08064	31 1 17
.58	1.17296 83987	0.61307 00321	31 30 40
.59	1.17915 78501	0.62483 05653	31 59 54
.60	1.18546 52182	0.63665 35821	32 28 59
.61	1.19189 11339	0.64854 02649	32 57 55
.62	1.19843 62397	0.66049 18021	33 26 40
.63	1.20510 11901	0.67250 93891	33 55 16
.64	1.21188 66517	0.68459 42276	34 23 43
.65	1.21879 33029	0.69674 75261	34 52 0
.66	1.22582 18344	0.70897 04999	35 20 8
.67	1.23297 29492	0.72126 43714	35 48 6
.68	1.24024 73623	0.73363 03699	36 15 54
.69	1.24764 58012	0.74606 97321	36 43 32
.70	1.25516 90056	0.75858 37018	37 11 0
.71	1.26281 77281	0.77117 35306	37 38 18
.72	1.27059 27333	0.78384 04773	38 5 27
.73	1.27849 47989	0.79658 58088	38 32 25
.74	1.28652 47150	0.80941 07995	38 59 14
.75	1.29468 32847	0.82231 67319	39 25 52
.76	1.30297 13238	0.83530 48967	39 52 20
.77	1.31138 96610	0.84837 65927	40 18 38
.78	1.31993 91384	0.86153 31271	40 44 46
.79	1.32862 06108	0.87477 58155	41 10 43
.80	1.33743 49463	0.88810 59822	41 36 31
.81	1.34638 30265	0.90152 49602	42 2 8
.82	1.35546 57460	0.91503 40915	42 27 35
.83	1.36468 40133	0.92863 47270	42 52 51
.84	1.37403 87501	0.94232 82267	43 17 57
.85	1.38353 08919	0.95611 59600	43 42 53
.86	1.39316 13880	0.96999 93057	44 7 39
.87	1.40293 12014	0.98397 96521	44 32 15
.88	1.41284 13091	0.99805 83974	44 56 40
.89	1.42289 27020	1.01223 69493	45 20 54
.90	1.43308 63854	1.02651 67257	45 44 59
.91	1.44342 33787	1.04089 91547	46 8 53
.92	1.45390 47155	1.05538 56744	46 32 37
.93	1.46453 14440	1.06997 77336	46 56 10
.94	1.47530 46268	1.08467 67915	47 19 34
.95	1.48622 53414	1.09948 43179	47 42 47
.96	1.49729 46797	1.11440 17937	48 5 49
.97	1.50851 37487	1.12943 07106	48 28 42
.98	1.51988 36704	1.14457 25715	48 51 24
.99	1.53140 55817	1.15982 88906	49 13 56
1.00	1.54308 06348	1.17520 11936	49 36 18
1.05	1.60379 44336	1.25385 66845	51 25 35
1.10	1.66851 85538	1.33564 74701	53 10 40
1.15	1.73741 48395	1.42077 80702	54 51 39
1.20	1.81065 55673	1.50946 13554	56 28 34
1.25	1.88842 38772	1.60191 90803	58 1 32
1.30	1.97091 42303	1.69838 24373	59 30 38
1.35	2.05833 28957	1.79909 26350	60 55 59
1.40	2.15089 84654	1.90430 15014	62 17 41
1.45	2.24884 24016	2.01427 21135	63 35 51

Table. II.—continued.

Amp. x	Ordinate $f(x)$	Arc $F(x)$	Angle ϕ
1.50	2.35240 96152	2.12927 94551	64° 50′ 36″
1.55	2.46185 90782	2.24961 11044	66 2 2
1.60	2.57746 44712	2.37556 79532	67 10 17
1.65	2.69951 48679	2.50746 49593	68 15 27
1.70	2.82831 54579	2.64563 19338	69 17 40
1.75	2.96418 83097	2.79041 43663	70 17 2
1.80	3.10747 31763	2.94217 42881	71 13 40
1.85	3.25852 83445	3.10129 11781	72 7 42
1.90	3.41773 15308	3.26816 29115	72 59 13
1.95	3.58548 08261	3.44320 67545	73 48 19
2.00	3.76219 56911	3.62686 04078	74 35 8
2.05	3.94831 80049	3.81958 31014	75 19 44
2.10	4.14431 31704	4.02185 67422	76 2 14
2.15	4.35067 12775	4.23418 71197	76 42 42
2.20	4.56790 83289	4.45710 51705	77 21 16
2.25	4.79656 75305	4.69116 83059	77 58 0
2.30	5.03722 06493	4.93696 18055	78 32 58
2.35	5.29046 94435	5.19510 02813	79 6 16
2.40	5.55694 71670	5.46622 92137	79 37 58
2.45	5.83732 01529	5.75102 65664	80 8 9
2.50	6.13228 94797	6.05020 44810	80 36 53
2.55	6.44259 27243	6.36451 10583	81 4 14
2.60	6.76900 58066	6.69473 22284	81 30 16
2.65	7.11234 49292	7.04169 37162	81 55 2
2.70	7.47346 86188	7.40626 31061	82 18 37
2.75	7.85327 98727	7.78935 20115	82 41 4
2.80	8.25272 84169	8.19191 83542	83 2 25
2.85	8.67281 30807	8.61496 87599	83 22 44
2.90	9.11458 42947	9.05956 10747	83 42 4
2.95	9.57914 67171	9.52680 70112	84 0 28
3.00	10.06766 19958	10.01787 49274	84 17 58
3.05	10.58135 16735	10.53399 27491	84 34 38
3.10	11.12150 02419	11.07645 10395	84 50 28
3.15	11.68945 83539	11.64660 62270	85 5 33
3.20	12.28664 62005	12.24588 39966	85 19 54
3.25	12.91455 70625	12.87578 28547	85 33 32
3.30	13.57476 10440	13.53787 78766	85 46 32
3.35	14.26890 89989	14.23382 46448	85 58 53
3.40	14.99873 66587	14.96536 33887	86 10 38
3.45	15.76606 89726	15.73432 33362	86 21 48
3.50	16.57282 46711	16.54262 72876	86 32 26
3.55	17.42102 10636	17.39229 64240	86 42 33
3.60	18.31277 90831	18.28545 53606	86 52 11
3.65	19.25032 85889	19.22433 74601	87 1 20
3.70	20.23601 39433	20.21129 04168	87 10 3
3.75	21.27229 98730	21.24878 21271	87 18 20
3.80	22.36177 76326	22.33940 68607	87 26 13
3.85	23.50717 14840	23.48589 17476	87 33 43
3.90	24.71134 55085	24.69110 35970	87 40 51
3.95	25.97731 07683	25.95805 60665	87 47 38
4.00	27.30823 28360	27.28991 71971	87 54 5
4.05	28.70743 97100	28.69001 73354	88 0 13
4.10	30.17843 01364	30.16185 74610	88 6 4
4.15	31.72488 23573	31.70911 79408	88 11 37
4.20	33.35066 33089	33.33566 77321	88 16 54

Table II.—continued.

Amp. x	Ordinate $f(x)$	Arc $F(x)$	Angle φ
4.25	35.05983 82903	35.04557 40564	88° 21′ 56″
4.30	36.85668 11293	36.84311 25703	88 26 43
4.35	38.74568 48689	38.73277 80563	88 31 16
4.40	40.73157 30024	40.71929 56625	88 35 36
4.45	42.81931 12846	42.80763 27176	88 39 42
4.50	45.01412 01485	45.00301 11520	88 43 37
4.55	47.32148 77597	47.31092 05553	88 47 20
4.60	49.74718 37388	49.73713 19031	88 50 53
4.65	52.29727 35895	52.28771 19876	88 54 16
4.70	54.97813 38646	54.96903 85875	88 57 27
4.75	57.79646 81112	57.78781 64160	89 0 31
4.80	60.75932 36329	60.75109 38858	89 3 25
4.85	63.87410 91118	63.86628 07342	89 6 11
4.90	67.14861 31340	67.14116 65509	89 8 48
4.95	70.59102 36652	70.58394 02563	89 11 18
5.00	74.20994 85248	74.20321 05778	89 13 41

Table III.

Angle φ	Amplitude x	Ordinate $f(x)$	Arc $F(x)$
0° 00′	0.00000 00000	1.00000 00000	0.00000 00000
0 30	0.00872 67570	1.00003 80784	0.00872 68678
1 00	0.01745 41787	1.00015 23280	0.01745 50649
1 30	0.02618 29299	1.00034 27925	0.02618 59216
2 00	0.03491 36760	1.00060 95443	0.03492 07695
2 30	0.04364 70831	1.00095 26852	0.04366 09429
3 00	0.05238 38186	1.00137 23460	0.05240 77793
3 30	0.06112 45507	1.00186 86871	0.06116 26202
4 00	0.06986 99494	1.00244 18981	0.06992 68119
4 30	0.07862 06866	1.00309 21985	0.07870 17068
5 00	0.08737 74360	1.00381 98375	0.08748 86635
5 30	0.09614 08736	1.00462 50947	0.09628 90482
6 00	0.10491 16783	1.00550 82796	0.10510 42353
6 30	0.11369 05314	1.00646 97327	0.11393 56083
7 00	0.12247 81177	1.00750 98255	0.12278 45609
7 30	0.13127 51251	1.00862 89606	0.13165 24976
8 00	0.14008 22452	1.00982 75725	0.14054 08347
8 30	0.14890 01736	1.01110 61279	0.14945 10013
9 00	0.15772 96102	1.01246 51258	0.15838 44403
9 30	0.16657 12592	1.01390 50985	0.16734 26091
10 00	0.17542 58297	1.01542 66119	0.17632 69807
10 30	0.18429 40358	1.01703 02658	0.18533 90449
11 00	0.19317 65972	1.01871 66950	0.19438 03091
11 30	0.20207 42390	1.02048 65693	0.20345 22994
12 00	0.21098 76926	1.02234 05949	0.21255 65617
12 30	0.21991 76954	1.02427 95143	0.22169 46626
13 00	0.22886 49917	1.02630 41078	0.23086 81911
13 30	0.23783 03328	1.02841 51937	0.24007 87591
14 00	0.24681 44770	1.03061 36293	0.24932 80028
14 30	0.25581 81906	1.03290 03122	0.25861 75844

TABLE III.—continued.

Angle ϕ	Amplitude. x	Ordinate $f(x)$	Arc $F(x)$
15° 00′	0.26484 22478	1.03527 61804	0.26794 91924
15 30	0.27388 74309	1.03774 22140	0.27732 45441
16 00	0.28295 45314	1.04029 94359	0.28674 53857
16 30	0.29204 43497	1.04294 89127	0.29621 34950
17 00	0.30115 76955	1.04569 17565	0.30573 06815
17 30	0.31029 53887	1.04852 91251	0.31529 87889
18 00	0.31945 82595	1 05146 22242	0.32491 96962
18 30	0.32864 71486	1.05449 23081	0.33459 53195
19 00	0.33786 29081	1.05762 06812	0.34432 76133
19 30	0.34710 64016	1.06084 86996	0 35411 85725
20 00	0.35637 85047	1.06417 77725	0.36397 02343
20 30	0.36568 01057	1.06760 93637	0.37388 46795
21 00	0.37501 21059	1.07114 49936	0.38386 40350
21 30	0.38437 54199	1.07478 62405	0.39391 04756
22 00	0.39377 09765	1.07853 47427	0.40402 62258
22 30	0.40319 97192	1.08239 22003	0.41421 35624
23 00	0.41266 26063	1.08636 03774	0.42447 48162
23 30	0.42216 06120	1.09044 11041	0.43481 23750
24 00	0.43169 47267	1.09463 62785	0.44522 86853
24 30	0.44126 59578	1.09894 78695	0.45572 62555
25 00	0.45087 53300	1.10337 79190	0.46630 76582
25 30	0.46052 38861	1.10792 85441	0.47697 55327
26 00	0.47021 26880	1.11260 19405	0.48773 25886
26 30	0.47994 28170	1.11740 03848	0.49858 16081
27 00	0.48971 53744	1.12232 62376	0.50952 54495
27 30	0.49953 14828	1.12738 19469	0.52056 70506
28 00	0.50939 22864	1.13257 00507	0.53170 94317
28 30	0.51929 89520	1.13789 31812	0.54295 56996
29 00	0.52925 26697	1.14335 40679	0.55430 90515
29 30	0.53925 46539	1.14895 55416	0.56577 27782
30 00	0.54930 61443	1.15470 05384	0.57735 02692
30 30	0.55940 84066	1.16059 21038	0.58904 50164
31 00	0.56956 27333	1.16663 33972	0.60086 06190
31 30	0.57977 04456	1.17282 76966	0.61280 07881
32 00	0.59003 28932	1.17917 84034	0.62486 93519
32 30	0.60035 14564	1.18568 90474	0.63707 02608
33 00	0.61072 75468	1.19236 32928	0.64940 75932
33 30	0.62116 26087	1.19920 49433	0.66188 55612
34 00	0.63165 81199	1.20621 79485	0.67450 85168
34 30	0.64221 55937	1.21340 64101	0.68728 09586
35 00	0.65283 65797	1.22077 45888	0.70020 75382
35 30	0.66352 26654	1.22832 69112	0.71329 30679
36 00	0.67427 54776	1.23606 79775	0.72654 25280
36 30	0.68509 66843	1.24400 25694	0.73996 10750
37 00	0.69598 79958	1.25213 56582	0.75355 40501
37 30	0.70695 11666	1.26047 24140	0.76732 69880
38 00	0.71798 79976	1.26901 82151	0.78128 56265
38 30	0.72910 03371	1.27777 86575	0.79543 59167
39 00	0.74029 00835	1.28675 95659	0.80978 40332
39 30	0.75155 91871	1.29596 70046	0.82433 63858
40 00	0.76290 96521	1.30540 72893	0.83909 96312
40 30	0.77434 35388	1.31508 69999	0.85408 06855
41 00	0.78586 29665	1.32501 29933	0.86928 67378
41 30	0.79747 01154	1.33519 24182	0.88472 52646
42 00	0.80916 72292	1.34563 27296	0.90040 40443
42 30	0.82095 66185	1.35634 17049	0.91633 11740
43 00	0.83284 06629	1.36732 74611	0.93251 50861

TABLE III—continued.

Angle ϕ	Amplitude x	Ordinate $f(x)$	Arc $F(x)$
43° 30′	0.84482 18144	1.37859 84727	0.94896 45667
44 00	0.85690 26008	1.39016 35910	0.96568 87748
44 30	0.86908 56286	1.40203 20649	0.98269 72631
45 00	0.88137 35870	1.41421 35624	1.00000 00000
45 30	0.89376 92515	1.42671 81944	1.01760 73930
46 00	0.90627 54877	1.43955 65396	1.03553 03138
46 30	0.91889 52558	1.45273 96713	1.05378 01253
47 00	0.93163 16148	1.46627 91856	1.07236 87100
47 30	0.94448 77273	1.48018 72329	1.09130 85011
48 00	0.95746 68645	1.49447 65499	1.11061 25148
48 30	0.97057 24115	1.50916 04951	1.13029 43864
49 00	0.98380 78727	1.52425 30867	1.15036 84072
49 30	0.99717 68780	1.53976 90432	1.17084 95661
50 00	1.01068 31887	1.55572 38269	1.19175 35926
50 30	1.02433 07047	1.57213 36907	1.21309 70041
51 00	1.03812 34713	1.58901 57291	1.23489 71565
51 30	1.05206 56868	1.60638 79323	1.25717 22989
52 00	1.06616 17106	1.62426 92455	1.27994 16322
52 30	1.08041 60719	1.64267 96317	1.30322 53728
53 00	1.09483 34789	1.66164 01411	1.32704 48216
53 30	1.10941 88281	1.68117 29851	1.35142 24379
54 00	1.12417 72157	1.70130 16167	1.37638 19205
54 30	1.13911 39479	1.72205 08182	1.40194 82945
55 00	1.15423 45536	1.74344 67956	1.42814 80067
55 30	1.16954 47968	1.76551 72821	1.45500 90287
56 00	1.18505 06905	1.78829 16500	1.48526 09685
56 30	1.20075 85119	1.81180 10327	1.51083 51936
57 00	1.21667 48179	1.83607 84588	1.53986 49638
57 30	1.23280 64623	1.86115 89967	1.56968 55771
58 00	1.24916 06146	1.88707 99148	1.60033 45290
58 30	1.26574 47797	1.91388 08554	1.63185 16871
59 00	1.28256 68194	1.94160 40264	1.66427 94824
59 30	1.29963 49759	1.97029 44112	1.69766 31193
60 00	1.31695 78969	2.00000 00000	1.73205 08076
60 30	1.33454 46628	2.03077 20447	1.76749 40162
61 00	1.35240 48167	2.06266 53396	1.80404 77553
61 30	1.37054 83962	2.09573 85325	1.84177 08860
62 00	1.38898 59689	2.13005 44682	1.88072 64653
62 30	1.40772 86705	2.16568 05702	1.92098 21270
63 00	1.42678 82466	2.20268 92646	1.96261 05055
63 30	1.44617 70984	2.24115 84517	2.00568 97083
64 00	1.46590 83325	2.28117 20327	2.05030 38416
64 30	1.48599 58162	2.32282 04973	2.09654 35991
65 00	1.50645 42373	2.36620 15832	2.14450 69205
65 30	1.52729 91712	2.41142 10147	2.19429 97312
66 00	1.54854 71535	2.45859 33356	2.24603 67739
66 30	1.57021 57612	2.50784 28464	2.29984 25472
67 00	1.59232 37024	2.55930 46652	2.35585 23658
67 30	1.61489 09162	2.61312 59298	2.41421 35624
68 00	1.63793 86825	2.66946 71626	2.47508 68534
68 30	1.66148 97465	2.72850 38278	2.53864 78957
69 00	1.68556 84559	2.79042 81096	2.60508 90647
69 30	1.71020 09159	2.85545 09514	2.67462 14939
70 00	1.73541 51627	2.92380 44002	2.74747 74195
70 30	1.76124 13593	2.99574 43124	2.82391 28856
71 00	1.78771 20167	3.07155 34868	2.90421 08777
71 30	1.81486 22443	3.15154 53045	2.98868 49627

Table III.—continued.

Angle φ	Amplitude x	Ordinate $f(x)$	Arc $F(x)$
72° 00′	1.84273 00347	3.23606 79775	3.07768 35372
72 30	1.87135 65893	3.32550 95234	3.17159 48024
73 00	1.90078 66900	3.42030 36198	3.27085 26185
73 30	1.93106 91274	3.52093 65221	3.37594 34226
74 00	1.96225 71940	3.62795 52785	3.48741 44438
74 30	1.99440 92565	3.74197 75358	3.60588 35088
75 00	2.02758 94218	3.86370 33052	3.73205 08076
75 30	2.06186 83153	3.99392 91629	3.86671 30949
76 00	2.09732 39967	4.13356 54944	4.01078 09335
76 30	2 13404 30420	4.28365 75697	4.16529 97701
77 00	2.17212 18296	4.44541 14826	4.33147 58743
77 30	2.21166 80792	4.62022 63153	4.51070 85037
78 00	2.25280 27044	4.80973 43447	4.70463 01095
78 30	2.29566 20607	5.01585 17363	4.91515 70311
79 00	2.34040 06925	5.24084 30642	5.14455 40160
79 30	2.38719 47201	5.48740 42660	5.39551 71743
80 00	2.43624 60537	5.75877 04831	5.67128 18196
80 30	2.48778 76890	6.05885 79567	5.97576 43644
81 00	2.54209 04361	6.39245 32215	6.31375 15147
81 30	2.59947 15731	6.76546 90751	6.69115 62383
82 00	2.66030 61276	7.18529 65343	7.11536 97224
82 30	2.72504 18020	7.66129 75755	7.59575 41127
83 00	2.79421 90579	8.20550 90481	8.14434 64280
83 30	2.86849 86556	8.83367 14720	8.77688 73569
84 00	2.94870 02391	9.56677 22335	9.51436 44542
84 30	3.03585 77506	10.43343 05246	10.38539 70801
85 00	3.13130 13316	11.47371 32457	11.43005 23028
85 30	3.23678 25219	12.74549 48432	12.70620 47362
86 00	3.35467 35124	14.33558 70262	14.30066 62567
86 30	3.48830 01458	16.38040 82394	16.34985 54761
87 00	3.64253 33573	19.10732 26093	19.08113 66877
87 30	3.82492 47412	22.92558 56261	22.90376 55484
88 00	4.04812 54187	28.65370 83478	28.63625 32829
88 30	4.33585 19194	38.20155 00141	38.18845 92970
89 00	4.74134 87604	57.29868 84986	57.28996 16308
89 30	5.43451 49799	114.59301 34801	114.58865 01293
90 00	Infinite.	Infinite.	Infinite.

Esquisse très rapide et très imparfaite d'une Ca[rte] destinée à indiquer la position des Dykes de trap, et celle des Montagnes d'une portion de l'Ile D'ARRAN.

L. A. Nr. Fe[c]
Edimbourg, 15 Avr[il]

Explication des Lettres.

A *Tornidneon*
B *Chaistel Abhal cime de Ceim-na-Caillich*
C *Suidh*
D *Grande Fente de Ceim-na-Caillich*
E *Col ou Head de Glen Rosa*
F *Kidvoe*
G *Goatfield*
H *Cich-na-Nighean*
K *Cir-mhor*
L *Meal Doon*
M *Benach Clivan*
N *Ben Talshan*
O *Bealach-nid-voe*
P *Ben-huish*
Q *Glen Shant Rock*
R *Prise d'eau du Moulin (Mill-dam)*
S *Moul-Gaobh ou Wind Mill*
T *Torninjerk*
U *Ploverfield, colline de Granite à grain fin, découverte & ainsi nommée par moi.*
V *Skian-Rhein*
X *Craig-an-fiach*
X' *Craig-an-Ulrach*
Y *Les trois Sheens, ou Fairies' Hills*
W *Dun Dow*
Z *Dun Feune*

Les dykes sont figurés par une barre — alongée dans le sens de leur direction; ils sont chacun accompagnés d'un numéro correspondant à celui qu'ils portent dans le catalogue. Vu la petitesse de l'échelle, il a été impossible de les placer toujours à leur place exacte, et de conserver leur distances respectives.

XXXII.—*Documents sur les Dykes de Trap d'une partie de l'Ile d'Arran.* Par Mons. L. A. Necker. (Pl. XXIII.)

(Presented 20th April 1840.)

Principaux Résultats de l'examen des Dykes de Trap de la partie orientale et centrale d'Arran, entre Loch Ranza et King's Cross Point.

Avant de parler des dykes contenus dans la portion d'Arran au midi de la baye de Lamlash, portion que, vu l'augmentation du nombre des dykes et le peu de temps que j'avois à donner à leur étude, je n'ai pu examiner que très rapidement et incomplètement, je vais rassembler ici en une sorte de résumé, les résultats généraux de l'examen des faits consignés dans les tableaux suivants.

1°. Les dykes se prolongent presque toujours dans une direction rectiligne, à l'exception des dykes 8 et 9 qui se ramifient, des Nos. 35, 37, 38, 101, et 102, qui se courbent ou ont une direction ondoyante, et des Nos. 2 et 40, qui se courbent à angle droit de leur première direction.

2°. Les dykes ont généralement leurs deux côtés parallèles, à l'exception aussi des Nos. 8 et 9, qui ressemblent à de simples veines.

3°. Ils ne sont séparés par aucune lisière des roches qu'ils traversent, excepté encore les 8 et 9, qui ont par place des lisières d'argile rouge.

4°. Ils sont en général formés d'une seule espèce de roche, sauf les dykes composés comme celui de Kidvoe, 55, qui est au milieu de pechstein, bordé des deux côtés d'argilolite, et à la partie extérieure de trap également des deux côtés; et comme le dyke très compliqué, si tant est qu'on doive le regarder comme un seul dyke, celui qui est dans le petit ruisseau à la croisée des routes de Lamlash et des Corygills, formé de lits verticaux alternativement de grunstein et de porphyre argilolitique, énumérés entre les Nos. 84 à 89 inclusivement.

5°. Tous les dykes sont verticaux, ou se devient au plus de 20° de la verticale.

6°. Il y en a plusieurs qui sont très rapprochés et parallèles, tels que 15 et 16; 17 et 18; 19 et 20; 3 et 4, qui le sont aussi avec 5 et 6; 44 et 45; 47 et 48; 54 et 55; 113 et 114; 128 et 129; 131 et 132.

7°. Il est de vrais dykes qui pourtant sont parallèles au couches traversées, mais ils sont fort rares, et ne se voyent que dans des couches inclinées au moins de 50°, et seulement dans les grès rouges les plus anciens. Je n'en connois d'exemples que les Nos. 69 et 70, et peut-être 12? qui est douteux.

8o. Je n'ai trouvé qu'un seul dyke qui procédat évidemment d'une couche de trap supérieure, le No. 29: cependant, d'après les nombreuses observations que j'ai faites à cet égard, particulièrement à Skye, il me paroît indubitable que tous les dykes d'Arran ont dû une fois aboutir à des couches semblables, mais ces couches auront dû être détruites en tout ou en très grande partie, par l'action des éléments, dans l'immense intervalle de temps qui s'est écoulé entre l'époque actuelle et celle où se déposoient les grès houillers et les grès rouges nouveaux, dont les vraies couches de trap dans l'île d'Arran paroissent contemporaines.

9o. Les seuls dykes de pechstein que j'ai vus sont les Nos. 58 et 59, et celui qui fait partie du dyke composé de Kidvoe, No. 55. Ils sont tous dans le granite.

10o. Les dykes d'argilolite (claystone) sont aussi très peu abondants; je n'ai remarqué que celui du sommet de Goatfield 43, ceux du dyke composé de Kidvoe 55 (dans le granite), ceux de la côte entre Lamlash et King's Cross Point 136, 144, et 149; enfin ceux qui sont entre les dykes de grunstein de 84 à 89, à la croisée des routes de Lamlash et des Corygills vers Brodick, si tant est pourtant que ces masses de porphyre argilolitique soient reéllement des dykes. Tous ces derniers sont dans le grès rouge nouveau. Malgré la rareté des dykes de pechstein et d'argilolite, on voit un grand nombre de fragments épars de ces deux roches sur les pentes de plusieurs montagnes granitiques, où il est impossible de les trouver en place, ce qui doit faire présumer que l'action des élémens a dû détruire non seulement les filons dont ces fragments faisoient partie, mais aussi les masses de granite que traversoient ces filons.

11o. Les dykes auprès desquels j'ai observé de l'endurcissement dans les couches traversées sont, les Nos. 22, 29, 40, 90, 93, 95, 99, 104, 105, 112, 115, 130, 133, 142, 143, 147.

12o. Ceux auprès desquels j'ai positivement remarqué qu'il n'y avoit pas d'endurcissement, sont les Nos. 29, 30, 35, 39.

13o. Je n'ai vu aucun dyke dans le granite, ni dans le mica-schiste ou talc-schiste, altérer ni endurcir ces roches à la jonction. Le Dr MacCulloch avoit déjà fait la même remarque pour le granite. (*Western Islands*, t. ii. p. 413.)

Malgré ce qui vient d'être dit dans les deux derniers paragraphes, on ne feroit pas une juste appréciation de l'effet des dykes sur les roches traversées, si l'on ne tenoit compte que des cas dans lesquels l'endurcissement est patent et remarquable. Dans les dykes creux la seule circonstance que les murs de grès ou de granite qui forment les parois de la fente, conservent leurs surfaces planes et verticales comme les murs des canaux, prouve par ce fait seul un léger endurcissement, sans lequel ces portions là auroient, après la destruction du trap, éprouvé les mêmes sillonements, les mêmes arrondissements, produits de l'action des éléments ou des vagues de la mer, que les parties voisines des mêmes masses de grès ou de granite.

14°. Les dykes auprès desquels j'ai observé que le grès rouge traversé étoit devenu blanc sont les Nos. 37, 40, 82, 83, 104, 112, 115, 133.

15°. Ceux auprès desquels j'ai positivement remarqué qu'il n'y avoit eu aucun changement dans la couleur du grès rouge, sont les Nos. 29, 35, 39, 72, 75, 124.

16°. Les très larges dykes ne se trouvent que sur les côtes; ceux de l'intérieur (sans celui de 50 pieds dans le Glen Cloy Burn, No. 71), et surtout ceux du groupe granitique, sont peu épais.

17°. Les plus longs dykes courent en général N. et S. presque vrai, tels que le grand dyke 116, et son correspondant de l'autre côté de la baye 34; tel que le long dyke 122 sur la côte entre Corygills et Clachland Point; tel enfin que les dykes 49, 51, 52, et 53, qui se montrent dans les prolongemens les uns des autres dans le fond de la vallée ou Glen Rosa dans toute son étendue, puis sur le col qui sépare ce glen du Glen Sannox, et puis se remontrent encore au-delà de ce dernier glen, dans la remarquable et singulière fissure par laquelle la crête orientale de Ceim-na-Caillich * est traversée de part en part. Il me paroit indubitable que ces divers dykes ne sont que des portions séparées d'un seul et même dyke, long de plusieurs milles, et dont la décomposition a bien pu déterminer l'existence du Glen Rosa, comme elle a déterminé la fente de Ceim-na-Caillich.

18°. Depuis Corygills à Lamlash les dykes manifestent une tendance toujours plus marquée à devenir saillants, et entre Clachland Point et Lamlash ils le sont tous plus ou moins, ainsi que dans toute la côte plus au sud. Or c'est surtout au sud de Corygills que le grès rouge nouveau et ses marnes dominent. Partout ailleurs les dykes sont en général plutôt creux que saillants.

19°. On remarquera qu'en général il y a partout deux séries distinctes de dykes, dont l'une est perpendiculaire à l'autre ou à peu près telle, et que sur les rivages l'une de ces séries est à peu près parallèle, tandis que l'autre est à peu près perpendiculaire à la côte.

20°. Les dykes sont plus nombreux sur les côtes que dans l'intérieur des terres; ils sont aussi plus nombreux vers le midi que vers le nord de l'île. Cela

* Cette curieuse fente a attiré dès les plus anciens temps l'attention des habitans d'Arran, et c'est à elle qu'a été donné le nom de *Ceim-na-Caillich* (Cime de la Sorcière), qu'on a plus tard transféré à la sommité culminante plus à l'ouest. Mr Headrick à rappelé l'histoire de cette sorcière telle que l' a transmise la tradition, pour expliquer l'origine de cette fissure. On a peine à reconnoitre sous cette ignoble image, les nobles et poétiques idées dont les Chants d'Ossian ont imbu ses descendants les bergers de ces sauvages montagnes. Mais si nous pouvions croire que sous l'emblême da la sorcière, on ait voulu désigner les nuages qui reposent presque sans cesse sur ces monts et les torrents d'eau qui les échappent, l'explication des anciens Celtes d'Arran seroit alors la plus conforme aux doctrines de la géologie la plus moderne et la plus avancée.

vient probablement dans les deux cas, de ce que c'est dans les terrains les plus récents que les dykes sont les plus nombreux; or la portion la plus moderne du terrain houiller et le grès rouge nouveau, abondent plus sur les côtes que dans l'intérieur, et plus au midi qu'au nord. Le nombre des dykes augmente ainsi avec la nouveauté des terrains. Le granite en contient plus que les schistes micaces et talqueux, où ils sont en général fort rares. Je n'en ai pas vu un seul dans la longue côte entre Sannox (north) et Corrie, côte toute occupée par les couches du grès rouge ancien et les plus anciennes du terrain houiller. Je n'en ai observé qu'un ou deux dans les grès rouges anciens des environs de Brodick. Et l'on en trouve infiniment plus dans les grès rouges nouveaux que dans le terrain houiller. C'est cependant peut-être moins l'ancienneté elle-même des terrains, que les circonstances dépendantes de leur structure et de leur consolidation qui paroissent avoir eu la plus grande influence sur la quantité des dykes.

Si, comme une comparaison attentive des faits que renferme cette notice, avec les belles et remarquables observations de Mr Milne dans le bassin-houiller d'Edimbourg, et les details curieux donnés par Mr Landale sur celui du comté de Fife, sembleroit le faire penser, les dykes ne sont que des failles occupées et agrandies par le trap en fusion, qui couloit dans ces fentes, on pourroit aisément concevoir comment dans certains terrains, d'une nature et d'une consistance particulière, et dans certaines circonstances, il ait pu se produire plus de failles que dans d'autres.

Vue générale des Dykes de la portion méridionale d'Arran.

Ici, vu la rapidité de ma marche, le nombre beaucoup plus considérable de dykes et souvent leur grande complication, je suis forcé pour le présent à me borner à un simple et très général aperçu, produit d'une reconnoissance superficielle.

Au midi du King's Cross Point commence le Whiting Bay; les rivages sont toujours formés des grès et des marnes du grès rouge nouveau; plus de 40 dykes de basalte et de grunstein les traversent, dans plusieurs directions, dont la principale m'a paru comme ailleurs en général presque perpendiculaire à la côte. J'y ai remarqué entre autres un groupe de quatre ou cinq longs dykes qui convergent et se réunissent en un point vers le SE., tandis que peu avant cette réunion ils sont tous traversés par un long dyke dirigé à peu près N. vrai. Plus loin deux dykes, de 6 à 8 pieds de large, forment par leur intersection un croix de St André, l'un étant dirigé à peu près NO. l'autre à peu près NE.

Plus au midi la point de Largiebeg est encore de grès rouge nouveau, coupé de nombreux dykes. Immédiatement après paroit une très épaisse couche de grunstein ou syenite à gros grains, qui descend des montagnes vers la mer et forme les Dipping Rocks. Elle repose en stratification parallèle sur des marnes et des grès rouges, qui pourroient bien appartenir au terrain houiller. Ces grès recouvrent eux-mêmes une seconde couche inférieure de grunstein à grain fin, ou de klingstein, qui forme le plateau élevé de Kildonan. Plus bas que ce plateau régnent sur le rivage des grès rouges et des marnes, traversées par des dykes larges de 10 à 25 pieds. Plusieurs sous Kildonan sont dirigés N. et N. 10° E. magnétique. On peut compter plus à l'est, jusque vers le hameau de Portlick, au moins 9 à 10 dykes, de 10 à 20 pieds de large, séparés par des espaces de grès rouge, et dirigés entre le NE. et la N. 20° O. magn. Deux d'entreux dirigés NE. magn. et trois N. 20° O. magn. s'avancent au loin dans la mer comme de longues jetées. Plusieurs dykes traversent aussi la masse ou couche de klingstein de Kildonan. Le rivage sous Portlick est de marne violette foncée, traversée par environ 7 dykes, seulement de 1 à 3 pieds de large, verticaux et courants N 20° O. Trois ou quatre entre eux s'avancent dans la mer. L'un d'eux est coupé par un dyke dirigé E. vrai.

En regardant d'un autre côté depuis le château de Kildonan, on voit sur le rivage environ 10 autres dykes, tous dirigés N. magn. ou N. 10° O. magn., tous dans le grès rouge, tous s'avançant dans la mer, et tous très larges. Quelques uns sont saillants. Dans quelques uns le grès est blanc et dur à la jonction des deux côtés du dyke. Ils sont croisés par un dyke dirigé N. vrai. Ces dykes finissent avec la falaise trapéenne de Kildonan. Le dernier dyke saillant de ce groupe se trouve précisément avoir la direction de la ligne qui passe par le sommet d'Ailsa Craig et le fanal de Pladda, et est en quelque sorte le prolongement de cette droite.

On aperçoit au loin vers le sud-ouest, depuis cette extrémité du plateau de Kildonan, un autre groupe composé de quelques autres dykes qui ont la même direction que ce dernier.

Entre ce point et le petit port de Drumlaborach, je n'ai pas vu de dykes; mais entre ce port et Bennan Head j'ai compté, sur le rivage, 23 dykes s'avançant dans la mer, dirigés tous N. 10° O. Il y en a un en particulier, très large et saillant à une hauteur considérable, qui forme une jetée naturelle au port de Drumlaborach. Les deux dykes les plus rapprochés de Bennan Head sont les seuls minces de ce long groupe de dykes; le plus à l'est est dirigé N. 45° E.; et le plus à l'ouest N. 45° O. toujours magnétique.

La haute et longue colline nommée Bennan Head est une énorme masse, et probablement une couche, d'un beau porphyre à base d'argilolite, ou de feldspath compacte.

Les Struey Rocks, que je n'ai vu que de loin, paroissent également une grande masse de grunstein ou de basalte, probablement aussi une couche.

Entre Kilbride Point ou les Struey Rocks et le rivage ou plage de sable sous Lagg, sont deux longs promontoires, formés de dykes dirigés N. 10° à 20° E. magn. Plus à l'ouest il y a un promontoire plus court, qui ne contient qu' un ou deux dykes, ayant la même direction que les précédents, Tous ces dykes sont verticaux.

Plus à l'ouest encore on arrive à un grand port, nommé South End Harbour. Ce port, un des objets les plus curieux qu'on puisse voir sur cette côte, déjà si remarquable, est entièrement naturel, l'art n'est entré pour rien dans sa construction. Deux dykes très longs, et l'un d'eux très épais, dirigés, l'un N. magn. l'autre N. 10° E. magn. environ, forment les deux murs qui limitent ce port à l'est et à l'ouest; un mince dyke transversal, qui s'avance de l'E. à l'O. magn., dans l'intervalle, entre ces deux dykes, forme comme une jetée qui abrite l'intérieur du port de la fureur de lames, laissant à l'ouest une large entrée pour les bâtiments, qui peuvent dans ce beau et régulier bassin, sorti tel quel des mains de la Nature, demeurer à l'ancre en toute sureté par tous les vents. Enfin, comme pour compléter l'architecture de ce bassin, un assez large dyke dirigé comme la jetée précédente E. magn. limite le port au nord du côté de terre, et forme comme un quay au rivage de sable. Dans l'intérieur du port se voyent les couches de grès rouge et de marne verdâtre que traversent les dykes; elles sont dirigées au N. 45° O. magn. et plongent de 10° au SO. magn.

D'autres dykes appartiennent encore au même groupe. Celui qui forme en partie l'entrée du port comme une jetée, est une portion d'un éventail de cinq dykes divergents, dirigés entre le N. 45° E. et l'E. magn., et qui se réunissent en un point au N. 70° E. magn. environ. Un peu plus à l'est est un autre mince dyke dirigé N. 45° E., qui vers son extrémité sud se divise en deux branches. Enfin le long dyke qui forme le mur occidental du port est accompagné à l'ouest d'un dyke de 30 pieds de largeur, qui lui est parallèle, étant comme lui dirigé N. magn.

A l'ouest du Southend Harbour est un long groupe de dykes, nommé Claitschi-more Point, ou Pointe des grands Dykes: il se compose de six, dont deux seuls sont longs et larges. Tous sont dirigés N. magn. Entre le Claitschi-more Point et la Montagne de Corryravie à l'ouest, on ne voit plus que deux seuls promontoires, de dykes tous courts, perpendiculaires à la côte.

Dans l'intérieur des terres, au village de Lagg, le ruisseau a exposé les dykes suivants. En allant du nord au sud, on les voit paroitre successivement dans ses berges, dans l'ordre suivant. 1°. Dyke de grunstein de 30 pieds de large, variable dans sa direction, mais en général dirigé N. magn., courbe, vertical. 2°. Dyke de grunstein de 10 pieds de large, dirigé N. 45° E. et plongeant de 80° à l'O. 3°. Un dyke en forme de coin, la pointe en haut dirigée E. Sous le pont de Lagg est un dyke, de 3 à 4 pouces de large, de grunstein décomposé à grain fin, ou de trap vert-d'asperge, dirigé N. 10° E. et plongeant au N. 80° O. magn. Enfin, à l'est de l'auberge de Lagg et de la distillerie voisine, un dyke de 25 à 30 pieds de large, dirigé N. magn. et plongeant à l'E. 70° environ, traverse le ruisseau et sa berge septentrionale.

Plus à l'ouest la Montagne de Corryravie, toute formée d'un beau porphyre, à base d'argilolite (claystone), qui s'étend jusqu' à la pointe SO. de l'île, et delà suit au nord ses rivages jusqu' à Blackwater, paraît avoir intercepté tous les dykes, du moins je n'en ai apperçu aucun dans toute cette extrémité sud-ouest d'Arran.

Au nord du Blackwater, les dykes recommencent au promontoire de Drumodoon. Je n'ai pas eu occasion, dans ce dernier voyage, de visiter toute cette portion de la côte occidentale d'Arran qui s'étend entre Blackwater et Glen Catacol près du Loch Ranza, mais je parcourus toute cette côte en 1809, et j'ai donné le résumé des observations que j'ai faites à cette époque dans mon *Voyage en Ecosse et aux Iles Hébrides,* tom. 2. Excepté les intéressans dykes composés, si nombreux à Tormore, où le pechstein, le grunstein, le basalte, l'argilolite et le klingstein, se voyent souvent associés dans le même dyke, et excepté quelques dykes de grunstein, sur la côté au nord de l'Irsa et du Machrywater, qui traversent le mica-schiste, les dykes sont en général très peu nombreux sur cette côte. M. le Professeur Jameson a signalé tous ces dykes, et décrit en grand détail et figuré ceux de Tormore, dans sa *Mineralogy of the Scottish Isles,* avec une exactitude telle qu'il devient inutile de répéter de nouveau ce qui a été déjà si bien dit par lui. Et à cette occasion qu'il me soit permis ici de rendre hommage à la précision et à la vérité des observations contenues dans cet ouvrage, qui, quoique écrit maintenant il y a 40 ans, doit certainement être toujours considéré comme un des guides les meilleurs et les plus utiles que puisse prendre le géologue qui veut étudier Arran avec quelque soin.

Je vais maintenant essayer, en terminant, d'évaluer s'il est possible en gros, et très approximativement seulement, quelle peut être la quantité de dykes de trap contenue dans la totalité de l'Ile d'Arran.

J'observerai d'abord, que dans mon tableau, quoique fait avec toute l'attention que je pouvois y donner, j'ai dû nécessairement omettre une certaine quantité de dykes, qui par mille causes ont pu échapper à mon observation. J'ai même des preuves certaines de pareilles omissions en voyant quelques dykes signalés par M. Jameson dans des lieux où j'ai passé sans en voir. Il y a plus, dans deux ou trois circonstances j'ai essayé en vain de retrouver des dykes et de petites masses de trap que j'avois étudiés, dessinées et décrites dans mon *Voyage,* et dont j'ai encore des échantillons dans ma collection. Telles sont en particulier les dykes qui traversent l'argilolite du Windmill, et dont l'un se bifurque en haut; ces dykes sont cités dans la *Min. of the Sc. Isles,* t. i, p. 27. J'ai aussi plusieurs fois cherché sans succès le trap qui accompagne les couches de pechstein du bois de Brodick et de l'ancienne route de Lamlash: ce trap, dont j'ai aussi des échantillons, me paroît d'après mes observations de 1807, faites avec soin et détails sur cet objet, devoir être considéré moins comme un dyke que comme une couche de klingstein associée au pechstein. Les échantillons que je recueillis alors, et que je conserve, établissent clairement la différence de cette roche avec celle des dykes et ses passages au pechstein même, ainsi que je l'ai enoncé dans mon *Voyage.*

Il est donc clair, que je n'ai pas signalé ici la totalité des dykes qui se trouvent dans les lieux que j'ai étudié avec attention ; mais je ne puis croire que le nombre des dykes omis, en y comprenant ceux mentionnés par M. Jameson, *Min. of the Sc. Isles*, v. i. pages 27, 30. 38, and 39, 70 (ceux-ci sont peut-être les couches de trap que j'ai remarquées au nord de Corrie), 73, (l'un d'eux probablement la grande fente de Ceim-na-Caillich), 77, 81, 84, 110, 111 ; ceux signalés par MacCulloch, *Western Islands*, t. ii. p. 412 and 413 ; et enfin ceux que M. Headrick a cité, surpasse le tiers de ceux que j'ai mentionné dans le tableau, et cela d'après l'expérience que j'ai faite de mon aptitude à en découvrir de nouveaux dans des lieux que j'avois déjà examinés à plusieurs reprises. Cela étant, le nombre des dykes de la partie NE. de l'ile, entre Loch Ranza et King Cross Point, dont je donne ici la description détaillée en la carte, devoit être porté à 200

Entre King's Cross Point et le Mont Corryravie, j'évalue le nombre des dykes que j'ai pu voir sur la côte sud de l'île, à environ 144

Total, . . 344

Mais l'évaluation précédente ne comprend que les dykes de la surface d'une moitié environ de l'île ; tout l'intérieur de la partie meridionale n'y est pas compris, non plus que la côte NO., ni le groupe granitique de Ben Vearan entre cette côte et la rivière Irsa ; et quoiqu'il soit connu que l'intérieur des terres renferme toujours moins de dykes que les côtes, et que la côte NO. est en général très dépourvue de dykes, quoiqu' enfin cette moitié de l'île soit bien plus petite que celle que j'ai parcourue ; omettant ces circonstances, je porterai pour elle un nombre égal à la première,—soit 344

Formant un total de . . 688

ou, en nombre rond, de 700 Dykes de Trap dans la totalité de l'île d'Arran. Doublant même encore ce nombre si l'on vouloit, pour y comprendre tous les dykes cachés par les bruyères vastes et étendues dans l'intérieur, par les grèves de sable sur les rivages, ou placés dans des recoins inaccéssibles des montagnes, on n'arriveroit pas encore au nombre de 1500 ; et pourtant en parlant de telle ou telle côte, de telle localité d'Arran, il est souvent échappé à ceux des géologues qui ont décrit Arran, à moi-même peut-être tout le premier, de dire qu'on y voyoit des *innombrables* dykes de trap. Or, je crois avoir maintenant montré que, loin de ne pouvoir être comptés, on peut à présent concevoir l'espérance de voir chacun des dykes de cette île individuellement étudié, numéroté, décrit et enregistré dans un catalogue descriptif et raisonné, analogue à celui que j'ai aujourd' hui l'honneur de mettre sous les yeux de la Société Royale.

Topographie de quelques parties de l'Ile d'Arran.

Pour l'intelligence de la liste des dykes d'Arran et de la carte qui l'accompagne, et aussi pour celle des hauteurs qui va suivre, quelque détails topographiques sur les montagnes du groupe granitique et des Glens Cloy et Sherrig sont nécessaires, vu que plusieurs de ces montagnes ne se trouvent sur aucune carte de cette île.

Groupe granitique oriental.—L'existence des deux vallées principales, Glen Sannox et Glen Rosa, qui traversent le groupe granitique, l'une à peu près de l'Ouest a l'Est et l'autre du Nord au Sud, a déterminé la configuration intérieure de ce groupe, et conjointement avec le petit vallon du Garbhock, qui court aussi en grande partie du nord au sud, la forme et la disposition des diverses cimes dont se compose ce groupe.

Le Glen Sannox prend son origine à l'ouest, dans le bas d'une arrête qui joint le Chaistel Abhal (apellé ordinairement Ceim-na-Caillich) au nord avec le Mont Kidvoe au midi. Les sommités les plus remarquables qui terminent sa berge gauche ou septentrionale sont, en allant de l'ouest à l'est et en employant les lettres qui désignent ces cimes sur l'esquisse de carte, 1^o. Chaistel Abhal (B), la plus haute de ce petit chainon et même du l'île entière après Goatfield. 2^o. Ceim-na-Caillich) (D), ou la remarquable fissure entre deux aiguilles de roches dont il a été parlé dans le catalogue des dykes et dans son résumé. 3^o. Suidh (*the Seat*) (C), après laquelle l'arrête descend rapidement vers la mer.

La berge droite ou sud de Glen Sannox, commence à la sommité pyramidale ou aiguille de Kidvoe (F), après laquelle son niveau s'abaisse considérablement, et forme le col qui sert de communication entre Glen Rosa et Glen Sannox, et qui est nommé *the head of Glen Rosa* (E). Immediatement après le faîte de cette berge se releve à peu près à la hauteur de Kidvoe, sur la montagne qui forme l'extrémité septentrionale de l'arrète de Goatfield, et conserve la même élévation à peu près jusqu'à l'entrée du Glen Sannox, où elle forme une sommité conique très élégante, nommée Cich-na-Nighean (H).

Dans une direction presque à angle droit des deux arrêtes décrites ci-dessus, et courant du nord au sud, s'élévent au midi du Glen Sannox trois chainons parallèles, séparés entre eux par le long et profond Glen Rosa et par le petit Glen Gharbhock, bien plus court et plus elevé.

Le plus oriental des trois chainons, et aussi le plus considérable et le plus elevé, est celui au milieu duquel est Goatfield (G), sommité culminante de l'île. Ce chainon, qui s'élève abruptement comme une muraille, et sans aucun contrefort notable à l'est du Glen Rosa, présente dù côté oriental vers la côte, quelques arrêtes ou contre-forts, dont les plus remarquables sont Cich-na-Nighean (H) dejà cité, à l'entrée du Glen Sannox, au midi duquel et séparé par un profond entonnoir (*corrie*) s'éleve celui nommé Cir-mhor (K). Le plus méridional de ces contre-forts est celui qui porte le nom de Meal-Doon (L). A partir de là et de la cime de Goatfield, le chainon s'abaisse tout à coup, et descend au niveau de la mer vers Brodick. Cette base de Goatfield, formée de schiste-talqueux, prend, au NO. et à l'Ouest de Brodick, le nom de Glen Shant Rock (Q) ; c'est autour d'elle que le Rosa Burn, après avoir longtemps coulé du nord au sud, se contourne et prend avec la vallée une direction de l'ouest à l'est environ. Cette partie inférieure du Glen Rosa est souvent apelée le Glen Shant ; mais ce n'est pas là la manière dont les habitants du lieu employent ce mot. Ils nomment Glen Rosa, comme appartenant au Glen Rosa Farm, toute la rive droite de Rosa Burn, et Glen Shant, comme dépendant du Glen Shant Farm, toute la rive gauche, depuis l'origine de la vallée du Rosa jusqu'à sa terminaison vers la mer.

Le second chainon, parallèle à celui de Goatfield, et qui forme la rive droite ou ouest du Glen Rosa, se compose de deux aiguilles de granite, dont la plus septentrionale (M), un peu au midi de Kidvoe, se nomme Ben-ach-Clivan, et la seconde la plus au sud s'apelle Ben Talshan (N) ; ce chainon intermédiaire est beaucoup moins élevé que le premier, celui de Goatfield, et que là troisième, celui de Ben-huish. Au midi de Ben Talshan régne une croupe de montagne arrondie, beaucoup plus basse que cetté dernière aiguille, et remarquable par les grands basses de granite, semblables à d'épaisses couches, dont elle est formée. Elle se nomme Ben-breach, et s'étend jusqu' au Garbhock Burn.

Enfin, le troisième chainon parallèle se compose d'une longue arrête de rochers escarpés, terminée au nord et au sud par deux hautes cimes. Celle du nord (O), nommée par erreur par quelques guides et habitans de Brodick, Ben-huish, tandis que ce dernier nom est celui de la cime du midi ; celle du nord (O), dis-je, se nomme Bealach-nid-voe. La cime la plus méridionale (P) s'apelle Ben-huish, et quoique la moins élevée donne en général son nom à tout le chainon. C'est par erreur qu' elle est quelquefois apellée Ben Talshan, nom qui, comme je l'ai dit, est celui d'une des deux aiguilles du second chainon,

Je tiens la nomenclature, que j'ai adoptée ici pour toutes ces montagnes, d'un vieux guide natif du Glen Rosa Farm, et qui a été longtemps berger dans ces sauvages regions. Ce sont les noms qui ont été transmis de père en fils des les temps les plus reculés.

Glen Sherrig et Glen Cloy.—L'arrête qui sépare Glen Rosa de Glen Sherrig, n'offre aucune cime remarquable, et s'étend en plateau depuis le tournant, ou angle SO. de Glen Rosa, jusqu' au pied de Ben-huish. L'arrête du côté sud du Glen Sherrig, celle qui sépare ce glen du Glen Cloy, prend, dans sa partie inférieure et orientale vers la mer, le nom de Stromach. Vers l'ouest une sommité d'argilolite assez haute s'élève au-dessus de l'arrête, et se nomme Mont Gaobh ou Windmill (S).

En remontant le Glen Cloy, on voit qu'il se divise, à peu près vers le milieu de sa longueur, en deux glens distincts, dont les ruisseaux se réunissent pour former le Glen Cloy Burn. Le plus septentrional de ces deux glens, qui est aussi le plus petit, est le Glen Ormidale: il s'ouvre à la base méridionale du Mont Gaobh ou Windmill, et est entouré de toutes parts de rochers hauts et escarpés. Un promontoire de montagnes, également élevées et escarpées, le sépare du glen plus méridional et plus grand, nommé Glen Dhu, toute environné de rochers, taillés presqu' à pic. La sommité la plus élevée du promontoire, qui sépare les deux glens, est la cime conique du Torninjerk (T), a son origine à l'ouest. Au dessous de cette cime, et plus à l'est, est le Skian Bhein (V) (White Knife), séparé du Torninjerk (Heap of Berries), par une large fente, qui n'est autre chose que le dyke creux 73 du catalogue et de la carte. A l'ouest du Torninjerk s'élève seulement de quelques mêtres au dessus de lui, une tête de colline arrondie et presque toute couverte de bruyère (U), qui sembleroit n'avoir aucun titre à attirer l'attention: c'est cependant là où j'ai eu la bonne fortune, le 29 Mai 1839, de découvrir une masse en place de granite, à grain très fin, qui n'avoit encore été apperçue par aucun observateur, et qui se trouve fort au sud des limites jusqu' àlors assignées au granite dans l'île d'Arran. En effet jusqu' à ce moment on n'avoit jamais cru que le granite s'avançat plus au midi que les bases de Goatfield et de Benhuish. Mais voici à présent, dans la partie de l'île au sud de la baye de Brodick, une nouvelle masse de granite, qui montre au jour sa tête, toute environnée d'un côté de syénite, dans laquelle le granite envoye des filons, d'un autre de claystone ou argilolite, qui semble passer à une sorte de protogine; voici enfin une masse de granite en contact presqu' immediat avec les grès rouges et verts, changés là au Torninjerk en une véritable roche de quartz compacte et très dure, et qui au midi de la masse de granite, également altérés et endurcis, paroissent l'appuyer sur elle immediatement et sans aucun intermédiaire visible. Comme cette découverte me paroit d'une haute importance, non seulement pour la géographie minéralogique d'Arran, mais plus encore pour la theorie du granite et de son soulevement, j'ai cru devoir la signaler ici quoiqu' étrangère au but principal de ce travail, me reservant de décrire plus tard, dans tous ses détails, cette intéréssante localité. Comme cette petite colline n'avoit aucun nom, j'ai cru pouvoir lui donner celui de *Ploverfield*, vû les nombreux Pluviers dorés (*Charadrius pluvialis*) qui l'habitent, et font rétentir de leurs sifflements cette solitude reculée, et comme exprimant le contraste entre ce modeste petit champ arrondi de granite et la haute et fière cime granitique de Goatfield, qui semble, comme son nom l'indique, n'être accessible que pour des chèvres.

Continuant delà à suivre la crête de l'arrête qui entoure le Glen Dhu, ou Glen Cloy proprement dit, nous trouvons d'abord en X, les rochers nommés Craig-an-Ulrach (*Eagle's Rock*); puis ceux dits Craig-an-Fiach (*Corbie's Rock*), qui sont le bord d'un grand plateau à l'angle SO. du glen (X.) Enfin à l'est de ces derniers rochers de grès, s'élèvent les trois cimes distinctes et rapprochées, toutes trois de trap, apellées Shee-ens ou Fairies'-hills. Et avec elles se termine notre apperçu topographique, le reste des montagnes de la partie de l'île qui nous a occupé étant bien connu et se trouvant tracé sur plusieurs cartes.

Hauteurs de quelques lieux dans l'Ile d'Arran au-dessus du niveau moyen de la mer, déterminées avec le Baromètre, dans les mois de Mai, Juin, et Juillet 1839, *et calculées en mètres d'après la méthode d'Olmanns.*

Les lettres majuscules après le nom des sommités correspondent à celles de la carte et de l'apperçu topographique précédent.

	Mètres.
Haut du bois du Chateau de Brodick, sous Meal-Doon (grès rouge ancien) . .	243.8
Sommet de Meal-Doon (L) (grès rouge nouveau)	411.5
Plateau à l'Angle SO. de Glen Rosa (mica-schiste)	304.3
Sommet de Ben-huish (granite à gros grain) (P)	832.8
Sommet de Bealach-nid-voe (O) (granite à gros grain)	884.6
Point culminant de la route de Brodick à Lamlash (grès nouveau) . . .	118.9
Sommité la plus haute de la crête entre les bayes de Brodick et de Lamlash, au SO. ou S. des Corygills (trap syenitique)	257.3
Sommet de Dundow (W) (Argilolite porphyrique en prismes)	215.7
Sommet de Dunfeune (Z) (trap syenitique) estimée à 190$^{m.}$	
Sommet de Goatfield (G) (granite à gros grain)	902.0
Prise d'eau du Moulin de Brodick (Mill-dam R) (schiste et granite) . . .	329.0
Hameau le plus haut des Corygills (grès rouge nouveau)	100.25
Sommet du Moul Gaobh ou Windmill (S) (argilolite)	400.5
Sommet de Ploverfield (U) (granit à grain fin)	456.57
Sommet du Torninjerk (T) (grès changé en roche de quartz) estimée à 448$^{m.}$	
Fond du Glen Dhu de Glen Cloy (grès à poudingue)	113.5
Sommet de la Colline du Milieu des 3 Schee-ens (Y) (trap)	395.0
Sommet de la plus occidentale des Schee-ens (Y) (trap) estimée à 378$^{m.}$	
Cette colline de trap s'élève d'environ 8 mêtres au dessus du plateau de grès nouveau sur lequel elle repose, ce qui porte la hauteur estimée de ce plateau à 370$^{m.}$	
Sommet de la plus orientale et la plus haute des Schee-ens (Y) (trap) . . .	406.4
Plateau au sommet des rochers Craig-an-Fiach (X) (grès et poudingue) . .	407.8
Sommet du Chaistel Abhal ou Ceim-na-Caillich (B) (granite)	889.4
Col du Glen Rosa ou *Head of Glen Rosa* (E) (granite à gros grain) . . .	470.6
Sommet de Kidvoe (F) (granite à gros grain)	828.6
Sommet de la coupure faite par le ruisseau Eis-na-birach au pied du Tornidneon et du plateau qui suit au midi cette coupure (mica-schiste)	98.0
Le fond du Glen Eis-na-birach et le lit du ruisseau sont à environ 25 mêtres au dessous du plateau et du haut de la coupure, leur hauteur absolue estimée donc de 73$^{m.}$	
Sommet du Tornidneon (A) (Jonction du mica-shiste et du granite) . . .	315.8
Point culminant de la route de Loch Ranza aux carrières d'ardoise . . .	305.0
(Cette route est dans le schiste talqueux et l'ardoise).	

	Mètres.
Hauteur estimée des carrières d'ardoise 320$^{m.}$, des sommités de schiste talqueux qui les dominent 355$^{m.}$	
Sommité NO. la plus basse de l'île de Lamlash dite *Wie top* (argilolite) . .	269.9
Sommité SE. la plus haute de l'île de Lamlash (argilolite)	299.5
Plateau de Kildonan au plein pied de l'auberge (klingstein)	35.9
Plateau de haut de l'île de Pladda à pied du phare (grunstein ou klingstein) . .	13.3
Sommet des Dipping Rocks (trap syénitique)	71.9
Sommet du Bennan-head (porphyre à base d'argilolite)	154.5
Lagg, plein pied de l'auberge	10.2

Les trois derniers nombres, et surtout les deux derniers, sont probablement un peu trop hauts, le baromètre ayant baissé dans la journée où ces observations ont été faites, d'une quantité qu'il ne m' a pas été possible de déterminer.

EDIMBOURG, 15 *Avril* 1840.

DYKES DE TRAP DANS L'ILE D'ARRAN.

No.	Localités.	Nature.	Etat.	Largeur.	Direction magn.	Inclinaison ou plonge de—à.	Couches traversées. Nature.	Couches traversées. Direction.	Couches traversées. Inclinaison.	Remarques.
A. Je n'ai apperçu aucun dyke de trap ni dans le Tornidneon, ni dans les montagnes de schiste micacé et talqueux qui bordent à l'Est et à l'Ouest la vallée du Loch Ranza, ni sur la rive ouest de ce lac, ni sur la côte à l'Ouest entre Loch Ranza et Glen Catacol, ni même plus loin fort au-delà de ce glen, ni enfin sur la route de Loch Ranza à Scriteen et Salt Pans par les carrières d'ardoise.										
B. Sur la côte orientale du Loch Ranza, dirigée N. 18° O. magn.										
1	Vis à vis du chateau	Grunstein vert	Plein	4 pieds	N. 10° E.	Vertical	Talc schiste et ardoises	Contournés	Contournés	C'est peut-être le prolongement du No. 62.
C. Sur la côte entre Loch Ranza et Newton Farm dirigée N. 57° E. magn. en allant de l'Ouest à l'Est.										
2	Entrée E du Loch Ranza	Basalte ou grunstein fin	Creux	4 pieds	N. 25° O. puis N. 50° E.	Vertical, puis plongeant de 47° au NO.	Talc schiste	N. 70° E.	68° au S. 20° E.	Les schistes prennent ensuite une direction N. 45° E. et plongent de 47° au N. 45° O. Le dyke se courbe et s'enfonce sous eux.
3	Ibid. plus à l'est	idem	id.	5 pieds	N. 20° O.	Vertical	id.	N. 45° E.	47°N. 45°O.	
4	Ibid. un peu plus à l'est	idem	id.	2 pieds	N. 20° O.	id.	id.	id.	id.	
5	Ibid. plus à l'est	Trap	id.	8 pieds	N. 20° O.	Id. ou à peu près	id.	N. 80° E.	50°S. 10E.	
6	Ibid. id. à la fin du dépôt de grès et sous Newton Farm	id.	id.	15 pieds	N. 20° O.	id. id.	id.	id.	id.	Peut-être penètre t'il aussi dans les grès dirigés N. 50° E. et plongeant de 25° à 30° au N. 40° O. qui recouvrent les schistes.
D. Sur la côte entre un point au Nord du Coq d'Arran et les Salt Pans, dirigée N. 23° à N. 33° O.										
7	Au nord du Coq d'Arran	Grunstein porphyrique	En partie saillant	7 pieds	N. 45° E.	70° au S. 45° E.	Grès rouge nouveau	Variable	Variable	Ce dyke sur le rivage est saillant comme un mur et ailleurs appliqué comme une haute muraille contre la montagne.
8	Au Coq d'Arran même	Grunstein	Plein	1 pied	N. 10° E.	60° à 70° S. 80° E.	id.	id.	id.	Se divise et se ramifie au Sud en 3 branches, et se courbe vers le N. dans sa portion septentrionale.
9	Ibid. tout près du No. 8.	idem	id.	1 pied	N. 10° E.	...	id.	id.	id.	Très court. Se divise au S. en 2 branches. De même que le No. 8, il est en plusieurs endroits bordé d'une lisière argilo-ferrugineuse d'un rouge vif.
10	Plus au sud vers un petit port	Basalte	Creux	5 pieds	N. 45° E.	Vertical	id.	id.	id.	Sur le rivage vertical. Dans la montagne plongeant de 75° à 80° au S. 45° E. Creux dans les deux endroits.
11	Au sud du petit port	Basalte noir	Creux ou plein	4 pieds	N. 10° O.	80° au N. 80° E.	id.	id.	id.	Ce dyke éprouve quelques inflexions legères dans sa direction, En partie creux en partie plein. Se prolonge sur une longueur de plus de 200 mètres jusqu' au petit port.
E. Sur la côte entre les Salt Pans et Sannox, dirigée N. 18° O.										
12	Immediatement au sud des Salt Pans	Grunstein porphyrique	Creux	?	E.	?	Couches houillères	E.	N.	On no voit que de foibles traces de ce dyke, qui paroit un lit parallèle aux couches houillères. Son toit est un grès houiller gris blanc et son lit un schiste houiller noir, à empreintes végétales et tiges de *Stigmaria?* &c.
13	Plus au sud entre un calcaire noir à *Productus* et un calcaire rouge avec marnes vertes	Grunstein	...	1½ pied	N. 10° O.	70° au N. 10° O.	id.	N. 70° O.	50°? N. 20°E.	

N.B.—Plus au sud viennent de vraies couches de trap et trap-tuff, parallèles à celles des conglomerats, grès et marnes.

14	Au sud du grand éboulement et au sud de la limite entre les fermes de Laggan et Laggantwine	Trap	...	Quelques pouces	N. 10° E.	Vertical	Poudingues violatres avec argiles blanches et violettes	N. 20° O.	50°N. 70°E.	
15	Plus au sud	Trap vert	Creux	4 pieds	E. magn.	id.	id.	id.	id.	Ces 2 dykes très rapprochés sont parallèles.
16	A 11 pieds plus au sud du No. 15.	idem	id.	5 pieds	E. magn.	id.	id.	id.	id.	
17	Plus au sud	Basalte	...	8 pieds	Environ N. 45° O.	id.	id.	id.	id.	idem idem idem.
18	Au sud et tout près du 17	idem	...	18 pouces	Environ N. 45° O.	id.	id.	id.	id.	
19	Plus au sud, mais au nord du mur limite entre les fermes de Laggantwine et de N. Sannox	Grunstein	...	Large	E. magn.	?	id.	id.	id.	idem idem idem
20	Ibid.	id.	...	id.	E. magn.	?	id.	id.	id.	Je n'ai point vu de dykes entre le No. 20, et la carrière de Corrie.

N.B.—Immédiatement au Nord de Corrie il y a deux couches distinctes de trap et de trap-tuff, intercallées aux couches houillères et parallèles à ces couches.

F. Sur la côte de Corrie à Screeb, dirigée N. 3° O.

21	Intérieur de la grande carrière de Corrie.	Grunstein	Plein	3 pieds et 1 pouce	N. 10° à 20° O.	48° au S. 80° à 70° O.	Calcaire à *productus*	N. 20° E.	20° au S. 20° E.	
22	Ibid. en contact avec le 21	id.	id.	4½ pieds	N. 20° O.	Vertical	id.	id.	id.	Ce dyke paroit double: les couches calcaires à l'ouest du dyke sont soulevées de 3 pieds. Un coin de couches calcaires épais, en haut de 6 pieds, sépare les dykes 21 et 22.
23	Ibid. à 30 mètres plus à l'ouest	id.	id.	3 pieds et 1 pouce	N. 10° à 20° O.	48° au S. 80° à 70° O.	id.	id.	id.	A durci, marbré et teint en vert le calcaire à la jonction.
24	Extrémité sud du village de Corrie	id.	...	3 à 4 pieds	N. 20° O.	Vertical	Grès rouge nouveau	O.	20° à 30° au S.	Le grès des deux côtés du dyke est très rouge et non altéré.
25	Plus au sud	id.	...	2 à 3 pieds	E. vrai	...	id.	ENE.	SSE.	
26	Plus au sud	id.	...	Très mince	N. vrai	...	id.	id.	id.	
27	Immédiatement au sud d'un petit port au sud de Corrie	id.	...	1 pied	E. un peu S. vrai	Vertical	id.	id.	id.	

G. Dans l'intérieur des terres, entre Corrie et Meal Doon sur Screeb.

N.B.—A la petite carrière de calcaire entre Corrie et la carrière de Screeb est une masse de klingstein resemblant au trap du Bass Rock, et qui paroit une couche, parallèle aux couches du calcaire à *Productus* qui la recouvrent.

28	Vers la jonction du granite et du schiste dans le Corrie Burn	Grunstein vert clair	Saillant	2 pieds	N. 30° O.	70° au N. 60° E.	Grès rouge et verdâtre ancien.	N. 35° E.	60° au S. 55° E.	Le grès rouge ancien recouvre là, en stratification parallèle, le schiste talqueux superposé au granite.
29	Dan le lit du Water of Man ou White Water, sur la hauteur entre Corrie et Meal Doon	Basalte noir porphyrique	Tantôt saillant tantôt creux	3 pieds	N. 10° à 20° O. ou N.	Vertical	Grès houiller	N. 70° à 60° E.	25° au S. 20° à 30° E.	Aboutissant dans le haut à une couche de basalte de même nature, superposée au grès houiller, qui reste rouge même au contact du basalte; là il est parfois durci et parfois non.
30	Ibid. un peu plus haut dans le lit du ruisseau	Trap compacte vert clair	Saillant de plusieurs pieds	2½ pieds	idem	idem	idem	id.	id.	Ce dyke n'aboutit pas seulement comme le 29 à la couche de basalte supérieure, mais il la traverse. N'altère pas les grès à la jonction. Il paroit être le prolongement d'un des deux suivants, Nos. 31 ou 32, de la carrière de Screeb.
31	Carrière de Screeb	Grunstein	Creux au milieu	10 pieds	N.	80° envir. à l'O.	Grès houiller et calcaire à *productus*	ENE.	SSE.	
32	ibid.	id.	Plein	4 pieds	N.	85° à l'O.	id.	id.	id.	
33	Sommet de Meal Doon	id.	à demi-vide	Large	N. vrai	85° à l'O. vrai	Grès rouge nouveau	Variable	Variable	L'inclinaison générale des grès de Meal Doun est au SE.

No.	Localités.	Nature.	Etat.	Largeur.	Direction. magn.	Inclinaison ou plonge de—à.	Couches traversées. Nature.	Couches traversées. Direction.	Couches traversées. Inclinaison.	Remarques.
	H. Sur la côte de Screeb à la pointe de rochers la plus près de Brodick, dirigée N. 47° à 57° E.									
34	A 100 pas au sud de la porte au commencement du bois du chateau	Grunstein	Creux	3 pieds	N. 25° E.	80° au S. 65° E.	Grès rouge nouveau	N. ou NO.	O. ou SO.	Ce dyke est précisement dans la direction du Grand Dyke près de Corygills No 116, et paroit en être le prolongement.
35	Plus au sud	id.	id.	4 pieds	N. 70° puis 50° E.	Vertical	id.	id.	id.	N'altère pas le grès au contact. Sa direction est un peu courbe.
36	Plus au sud	id.	id.	5¾ pieds	N. 45° E.	idem	id.	N. 66° E.	Variable	
37	Plus au sud	id.	id.	15 pieds	N. 10° puis 20° O.	Presque vertical	id.	id.	id.	Sur le côté Est du dyke la direction du grès devient momentanement N. 20° O. Le grès est très blanc d'abord jusqu'à environ 25 pieds de la jonction des deux côtés, puis plus loin il n'est blanc que près de la jonction.
38	Plus au sud	id.	id.	1 pied	N. 30° puis 25° O.	Vertical	id.	id.	id.	
39	Tout près du 38	id.	id.	1 à 1½ pied	N. 70° E.	idem	id.	N. 80° E. et N. 45° E.	70° au S. 10° E. et au S. 45° E.	Ce dyke n'altère pas les grès qui ici, comme sur toute cette côte, offrent deux sens de direction et d'inclinaison différents.
40	Tout près du 39	id.	id.	3 pieds	N. 22° O. puis N. 45° E.	80° env. au S. 68° O. puis au S. 45° O.	id.	id.	id.	Le grès du toit et d'une partie du lit est blanc et durci. Ce dyke se courbe et devient parallèle à la direction des couches.

I. Dans le groupe central de montagnes, principalement granitiques, entre Brodick et Loch Ranza.

N.B.—Je n'ai apperçu aucun dyke de trap dans la montagne de schiste micacé qui forme la base meridionale de Goatfield, ni dans le Glen Shant Rock qui en est le prolongement à l'Ouest, ni dans le schiste micacé de la berge opposée du Glen Shant ou extrémité sud de Glen Rosa.

No.	Localités.	Nature.	Etat.	Largeur.	Direction. magn.	Inclinaison ou plonge de—à.	Couches traversées. Nature.	Couches traversées. Direction.	Couches traversées. Inclinaison.	Remarques.
41	Vers la prise d'eau (Mill Dam) du moulin de Brodick	Traces d'un dyke de grunstein à grains moyens, à la jonction du granite et du schiste micacé ou plutôt dans le granite ; sa direction est indéterminable.								
42	Entre cette prise d'eau (Mill Dam) et le bas de l'arrète de Goatfield	Grunstein vert très decomposé	Plein ?	?	N. ?	?	Granite à gros grain			On ne voit de ce dyke que des affleurements de distance en distance, alignés du S. au N., ce qui paroit être sa direction.
43	Dans les rochers de la crête immediatement au N. ou NO. de sommet de Goatfield	Argilolite (clay-stone)	Plein	?	N. 45° O.	Vertical	Granite à gros grain			
	Glen Rosa.									
44	Lit du Rosa Burn au dessous du confluent du Garbhock Burn	Trap verdâtre	Plein	5 pieds	N. 25° E.	?	Schiste talqueux	E. un peu S.	Ondulés	Les feuillets du schiste sont irréguliers. Comme on ne voit que la surface des dykes, on ne peut determiner l'inclinaison.
45	Ibid. tout près du 44	id.	id.	Environ 2 pieds	N. 25° E.	?	idem	id.	id.	Id. id. Peut-être n'est ce qu' une branche du 44.
46	Ibid. près des précédents	Trap bleu foncé, à grain fin	id.	1½ pied	N. magn.	?	id.	id.	id.	
47	Lit du Garbhock Burn vers son confluent et un peu au dessus du pont	Basalte noir à grain fin	id.	1 pied	E. vrai	Vertical	Granite à gros grain			Ces deux dykes 47 et 48 sont très rapprochés et parallèles.
48	Ibid. à quelques pieds plus au sud	idem.	id.	6 pouces	id.	id.	id.	id.		
49	Glen Rosa, sous le milieu de l'arrête qui joint Glen-shant Rock à Goatfield	Trap vert grisâtre porphyrique	Partie plein partie vide	6 pieds	N. 10° E.	70° à 80° au S. 80° E.	id.	id.		
50	Face sud de la 2de aiguille de Glen Rosa dite Ben-ach-Clivan	?	Vide	Enorme	N. magn.	80° à l'O.	id.	id.		Ce dyke forme une coupure où large fente à bords parallèles, du haut en bas du rocher. Ne l'ayant vu que de loin, je n'ai pu déterminer sa nature ni sa largeur.
51	Dans le lit du Rosa Burn au dessous du pied NO. de la pyramide de Goatfield	Trap vert grisâtre porphyrique	Creux et plein	6 pieds	N. 10° E.	70° à 80° au S. 80° E.	id.	id.		Très probablement le prolongement du 49. Idem.

52	Au Col (ou *Head*) du Glen Rosa	Trap vert grisâtre porphyrique	Plein	6 pieds ?	N. 10° E.	70° à 80° au S. 80° E.	Granite à gros grains.			Très probablement le prolongement du 49.
53	Fente de la base Est de Ceim-na-Caillich	idem ?	Creux	idem ?	N. 10° E.	75° au S. 80° E.	id.	id.		Je n'ai vu qu' à distance. Ce dyke 53 qui forme la remarquable fissure de Ceim-na-Caillich et qui, etant précisement dans la direction des 49, 51, et 52, ne forme probablement avec eux qu'un seul et même dyke.
54	Pied de Kidvoe sur le col du Glen Rosa	Trap noir	En partie creux	3 pieds	N. 65° O.	85° au N. 25° E. en haut et vertical en bas	id.	id.		
55	Ibid. un peu au sud du précédent	Composé de Pechstein vert au milieu ; Argilolite de chaque côté, et trap à chaque salbande	Creux	8½ pieds	N. 65° O.	Environ 80° au N. 25° E. ou vertical	id.	id.		Ce remarquable dyke, formé de 3 sortes de roches en 5 bandes, est parallèle au 54. L'épaisseur du pechstein au centre varie entre ½ pouce et 1½ pied. L'argilolite de chaque côté a 2 pieds, et la bande de trap aussi de chaque côté 1½ pied. Le dyke forme un profond ravin, qui remonte presque au sommet de Kidvoe. En haut l'argilolite disparoit; le pechstein a 5 pieds et chaque salbande de trap a 4 pieds d'épaisseur. Celle au Nord est decomposée en boules.
56	Sommet de Kidvoe sous le rocher supérieur du côté de l'est	Trap	Plein	Environ 2 pieds	NO. vrai	...	id.	id.		
57	Au même endroit du côté de l'ouest, et près du pied de Ceim-na-Caillich	idem	Plein	idem	idem	...	id.	id.		
58	Sur le revers SO. de Chaistel Abhal, sommité de Ceim-na-Caillich	Pechstein vert porphyrique	id.	10 pieds	N. 10° E.	Vertical ?	id.	id.		Ces deux dykes de pechstein, pleins et même un peu saillants, sont parfaitement parallèles et très rapprochés. C'est peut-être un seul large dyke renfermant une longue masse de granite.
59	Même endroit, mais un peu plus à l'est et plus bas	idem	id.	2 pieds	N. 10° E.	idem	id.	id.		
60	Sommet de Ceim-na-Caillich ou Chaistel Abhal (Ptarmigan Castle)	Trap.	?	?	?	?	id.	id.		Ce ne sont que de simples indices.
61	Base nord de Ceim-na-Caillich, là où commence le fond plat du Glen Eis-na-birach	Trap.	Creux	4 ou 5 pieds ?	N. 10° E.	Vertical ou plongeant de 80° à l'O.	id.	id.		Ce dyke forme pendant quelque temps le lit du ruisseau d'Eis-na-birach. Le dyke No. 1, à l'entrée du Loch Ranza, paroit être le prolongement de celui-ci.
62	Dans le lit du Garbhock, vers la base de Ben-huish	Trap vert	En grande partie creux	Environ 9 pieds	N. 20° E.	Vertical	id.	id.		Renferme comme une veine de granite dirigée comme le dyke. Les salbandes droites et parallèles de ce dyke forment longtemps les berges, de 30 à 40 pieds de haut, du ruisseau.
63	Dans la montagne qui borde à l'est le petit Glen du Garbhock	Grunstein	Plein	5 à 6 pieds	N. 45° O.	Environ 80° au S. 45° O.	id.	id.		
64	A l'ouest du même Glen, au nord du sommet méridional de Benhuish	id.	id.	id.	id.	id.	id.	id.		Le dyke 64 se trouve presque vis à vis, et seulement un peu plus au sud, du précédent No. 63, dont il a l'épaisseur, la direction, et l'inclinaison.
65	Bealach-nid-voe, cime culminante de la chaine de Benhuish au nord	Grunstein	Creux	?	N. 45° O.	80° environ au S. 45° O.	id.	id.		Les dykes dans le granite sont presque toujours creux et jamais saillants. Les murs de granite qui les renferment sont parallèles, verticaux ou à peu près, coupés net, et jamais endurcis ni altérés dans leur grain.
66	Ben-huish, sommité au sud de la même chaine, nommée quelquefois pas erreur Ben Talchan	Trap verdâtre	Creux	?	N. vrai	Vertical	id.	id.		

K. Dans les terrains sécondaires aux environs de Brodick, dans l'intérieur des terres ; y compris les routes de Springbank à Lamlash et à Corrygills.

N.B.—Une petite couche de pechstein porphyrique est subordonnée aux grès rouges vers le bois au sud de Brodick.

67	Dans le Glen Shant au-dessus de Glen Rosa Farm	Grunstein décomposé	En partie creux	?	N. ou à peu près	Vertical ?	Grès schisteux micacé	N.	Environ 80° à l'E.	Donne passage à une source d'eau.
68	Ibidem	id.	id	?	id.	id.	id.	id.	id.	Idem.

No.	Localités.	Nature.	Etat.	Largeur.	Direction magn.	Inclinaison ou plonge de—à.	Couches traversées. Nature.	Couches traversées. Direction.	Couches traversées. Inclinaison.	Remarques.
69	A la base NE. du Windmill (*Moul-Gaobh*) au haut d'un ruisseau descendant dans le Glen Sherrig	Grunstein vert foncé à petit grain	En partie creux	1 pied	N. 60° E.	70° à 80° au SE.	Grès rouge à mica	N. 60° E.	70° à 80° au SE.	Malgré de légères differences dans la direction, l'inclinaison, et l'épaisseur, les deux dykes 69 et 70 paroissent être le prolongement l'un de l'autre. Tous deux, quoique parallèles à la stratification du grès rouge, me paroissent, vu leur aspect, leur peu d'épaisseur, et la finesse de leur grain, devoir être rangés, non parmi les couches, mais parmi les filons.
70	A l'entrée du Glen Sherrig sous l'église de Brodick	idem	Plein	6 à 7 pieds	N. 70° E.	Vertical	idem	N. 70° E.	Verticales	
71	Dans le Glen Cloy, formant le lit d'un petit tributaire du Glen Cloy Burn	Grunstein vert clair compacte en boules	Creux	50 pieds	N.	Vertical	Grès rouge et poudingues	NE.	10° à 15° au SO.	
72	Dans le Glen Cloy Burn, immédiatement au-dessus du confluent du Glen Dhu Burn	idem	Plein	13 pieds	N. 10° O.	Vertical	idem	id.	id.	Ne change pas au contact la couleur rouge des grès et poudingues.
73	Dans la berge sud du Glen Du à l'origine du Mont Skian Bhein	Grunstein	Creux	10 à 12 pieds	N. vrai	Environ 70° à l'E. vrai	Grès blanc et poudingues	...	..	Forme la large fente qui sépare le Mont Skian Bhein du Mont Torninjerk.
74	Immédiatement à l'est du No. 73	idem	un peu creux	18 pouces	NO. vrai	85° au NE. vrai	id.	...	...	
75	A la base SE. du Mont Torninjerk	Grunstein compacte	...	6 pieds	N. 80° E.	80° à 85° au N. 10° O.	Grès rouge et poudingues	NE.	10° à 15° au SO.	Ce dyke, qui ne change pas au contact la couleur rouge des grès et poudingues, traverse le Glen Cloy Burn.
76	Plateau au SO. du Glen Cloy.	Grunstein	...	...	N. un peu O.	85° à l'E.	Grès blanc et poudingues	...	...	

N.B.—Une couche mince de syénite à petit grain, et à feldspath globulaire, traverse le Glen Cloy Burn immédiatement sous le jardin de Kilmichael.

No.	Localités.	Nature.	Etat.	Largeur.	Direction magn.	Inclinaison ou plonge de—à.	Couches traversées. Nature.	Couches traversées. Direction.	Couches traversées. Inclinaison.	Remarques.
77	Au-dessus de la forge de Springbank, sur le vieux chemin de Lamlash	Grunstein en boules	Plein	...	N. 10° E.	Vertical	Grès rouge nouveau	Variable	Variable	

N.B.—Vers le haut de la vieille route de Lamlash, le chemin traverse une couche de pechstein vert, accompagné de klingstein.

No.	Localités.	Nature.	Etat.	Largeur.	Direction magn.	Inclinaison ou plonge de—à.	Couches traversées. Nature.	Couches traversées. Direction.	Couches traversées. Inclinaison.	Remarques.
78	Nouvelle route de Lamlash à partir du pont de Springbank, immédiatement au sud du pont	Grunstein vert clair ou wacke	Plein	?	E.	?	Conglomérat du grès rouge nouveau	Affleurements		On ne voit de ce dyke que des traces sur le bord du chemin.
79	Même route plus au sud	Grunstein à grain très fin	idem	3 à 5 pieds	E. vrai	Vertical ?	Grès rouge nouveau	Variable	Variable	Le grunstein de ce dyke est très décomposé.
80	Tout près du No. 79	idem	idem	id.	id.	id.	id.	id.	id.	Idem.
81	Plus au sud	Grunstein vert à grain fin	idem	?	N. 10° E.	Vertical	id.	N. 45° O.	10° à 15° au S. 45° O.	Il n'y a que des traces très obscures de ce dyke.
82	Même route plus au sud; au nord du 5me pont, y compris celui de Springbank.	Grunstein	idem	3 pieds	N. 40° O.	Vertical	id.	id.	id.	Les grès sont blancs et très durcis, des deux côtés du dyke, jusqu' à un pied au moins de la jonction.

N.B.—Dans le haut de la descente de la route sur Lamlash, passe une grande et épaisse couche de syénite augitique et zéolitique, passant dans le bas à un grunstein, puis à un klingstein. Cette couche commence à l'ouest aux Sheens ou Fairies Hills, au haut de la berge sud du Glen Cloy; avec quelques interruptions elle forme la haute crête qui sépare les deux bayes de Brodick et de Lamlash, la colline de Dunfeune, et descend dans la mer à Clachland Point. Les passages de la syénite au grunstein et au klingstein, se voyent à l'ouest de la route de Lamlash, et aussi, mais moins clairement, aux Sheens ou Fairies Hills.

83 Vers le haut de la descente de la route de Lamlash, sur le chemin, est un amas de boules de grunstein, de 4 pouces à 2 pieds de diamètre, indiquant un dyke, et suivi au sud de grès gris, blanc, compacte et dur.

No.	Localités.	Nature.	Etat.	Largeur.	Direction magn.	Inclinaison ou plonge de—à.	Couches traversées. Nature.	Couches traversées. Direction.	Couches traversées. Inclinaison.	Remarques.
84	Route de Springbank à Corygills à la croisée de la nouvelle route de Lamlash	Grunstein ou syenite	Plein	7 à 8 pieds	N. 30° E.	?	Grès rouge nouveau	N.	5° à l'E.	Le dyke No. 84, n'est bordé que du côté ouest par le grès rouge nouveau, avec ses poudingues et ses argiles bigarrées; du côté Est 1 à 2 pieds de porphyre argilolitique le séparent du dyke No. 85.

85	Ibid. plus à l'est	Grunstein ou syénite	Plein	1 à 2 pieds	N. 30° E.	?	Porphyre argilolitique.			Les singuliers dykes de No. 84 à No. 89 paroissent former par leur ensemble un très large dyke, composé de bandes alternatives de grunstein et de porphyre argilolitique. Les grès, poudingues et marnes rouges qu'il traverse, ne se voyent qu'à l'ouest du No. 84, et à l'est du No. 89. Il forme le lit d'un petit ruisseau, parallèle à la route.
86	Ibid. id.	Grunstein en boules	id.	6 pieds	id.	?	idem	idem		
87	Ibid. id.	Grunstein	id.	2 pieds	id.	?	idem	idem		
88	Ibid. id.	id.	id.	?	id. ?	?	idem	idem		
89	Ibid. id.	id.	id.	?	id. ?	?	D'un côté, idem l'autre grès rouge	N.	idem de 5° à 10° à l'E.	
90	Ibid. id.	Grunstein en boules	id.	27 pieds	N. 30° E.	?	Grès dur	N.	5° à 10° à l'E.	Le grès à l'est du dyke est dur, et marbré de rouge et de gris.
91	Traversant la route des Corygills au-delà des dykes précédents	Grunstein à gros grains	id.	?	?	?	?	?	?	Ce dyke, sur lequel passe le chemin, est souvent couvert des remblais, gravier, pierres, &c. de la route, qui le cachent.
92	Traversant la route à la première maison du plus haut hameau des Corygills	Grunstein à gros grains porphyrique	id.	Très large	N. 45° E.	?	?	?	?	On ne peut déterminer exactement la largeur de ce grand dyke, ni son inclinaison, non plus que la nature et la position des couches qu'il traverse, probablement les grès rouges nouveaux.

L. Sur la côte entre les bayes de Brodick et de Lamlash, de Springbank par Corygills et Clachland Point à Lamlash.

1ère partie. De Springbank à la première masse d'argilolite, direction de la côte N. 43° O.

* De Springbank au Pont où la route neuve de Lamlash quitte la côte.

93	A Springbank dans les premiers rochers de la côte	Grunstein à grain moyen	Creux	30 pieds	N. 20° O.	Vertical	Conglomérat blanc	N. 70° E.	20° au S. 20° O.	Le conglomérat est très endurci à la jonction; le grain du grunstein n'est pas plus fin dans les côtés que dans le milieu.
94	Ibid.	Idem brunâtre	id.	6 pieds	N.	idem	idem	id.	id.	Se bifurque et la principale branche tourne au SE. Les deux branches se terminent en pointe.
95	Ibid.	Grunstein	id.	6 pieds	N. 20° E.	idem	id.	id.	id.	
96	Ibid.	Trap à grain fin feldspathique et amygdalin gris de perle.	id.	3 pieds	N. 10° O.	Idem ou plongeant de 80° à 85° à N. 80° E.	id.	id.	id.	Le grain du dyke 96 est fin au milieu comme sur les côtés. Il y a un peu d'endurcissement à la jonction, dans le conglomérat.
97	Ibid.	Grunstein	id.	1 pied	N. 35° E.	Vertical ou plongeant de 75° à 80° S. 55° E.	id. et grès rouge	id.	id.	Ce dyke est tout près de la maison Stoddart et du pont.

** A l'Est du Pont du Springbank.

98	Sur la côte à l'est du pont en allant vers l'est ou SE.	Grunstein	En partie creux	Epais	N. un peu E. vrai	Presque vertical	Grès et conglomérats rouges	NO. ou NNO.	SO. au OSO.	
99	Ibid.	id.	id.	id.	id.	id.	id.	id.	id.	Les 2 murs du dyke No. 99, sont de grès blanc et dur jusqu'à 6 à 9 pouces de distance du dyke.
100	Ibid.	id.	id.	id.	id.	id.	id.	id.	id.	
101	Ibid.	id.	Creux	8 pieds	N. 35° E.	Vertical	Grès rouge	N. 70° E.	20° au S. 20° O.	Ce dyke se courbe dans sa direction, court au N. et plonge un peu à l'E.
102	Ibid.	id.	...	12 à 15 pieds	N.	id.	id.	id.	id.	Se courbe et court au N. 20° E. Il est tout près du précédent.
103	Ibid.	Trap feldspathique	...	1 pied	N. 35° E.	id.	id.	id.	id.	
104	Ibid.	Grunstein	...	6 pieds	N. 45° E.	80° à l'E.	id.	id.	id.	Le dyke 104 correspond au 1er ravin creux dans la falaise. Le grès est blanc, très dur et cristallin à quelque distance de la jonction, surtout sur le côté E.
105	Ibid.	id.	...	5 pieds	N. 45° E.	70° à 75° au S. 45° E.	id.	id.	id.	A la jonction du dyke 105 des deux côtés le grès est très blanc jusqu' à 3 ou 4 pieds de distance, au-delà le blanc se fond insensiblement dans le rouge qui domine depuis le pont.
106 à 111	Ibid.	Cinq ou six très gros dykes de grunstein	...	20 à 30 pieds	N. ou NE. ou N. un peu O. vrais	Verticaux ou à peu près	Conglomérat rouge	NE. vrai	20° au SO. vrai	
112	Ibid.	Grunstein	Saillant	8 pieds	N. vrai	Vertical	Grès rouge	...	...	Ce dyke 112 a des deux côtés des salbandes, de 3 à 4 pieds d'épaisseur de grès blanc et dur, ce qui cause la saillie du dyke.

No.	Localités.	Nature.	Etat.	Largeur.	Direction magn.	Inclinaison ou plonge de—à.	Couches traversées. Nature.	Couches traversées. Direction.	Couches traversées. Inclinaison.	Remarques.
113	Sur la côte à l'Est du pont en allant vers l'Est ou SE.	Grunstein	...	5 à 6 pieds	NO. vrai	Vertical ?	Grès rouge	...	...	Ces deux dykes rapprochés sont parallèles.
114	Ibid.	id.	...	id.	id.	id.	id.	...	...	

Ici vient une épaisse masse d'argilolite, partie prismatique et partie schistoide, qui paroit une couche parallèle à celles du grès et qui change la direction de la côte ; à extrémité orientale de cette masse est le dyke,

No.	Localités.	Nature.	Etat.	Largeur.	Direction magn.	Inclinaison ou plonge de—à.	Couches traversées. Nature.	Couches traversées. Direction.	Couches traversées. Inclinaison.	Remarques.
115	A l'extrémité Est de la masse d'argilolite	Grunstein	Plein dans la falaise creux sur la côte	2 pieds	NE. vrai	Vertical	Conglomérat	NE. vrai	20° au SO. vrai	Dans la falaise le dyke 115 est plein ; dans le haut il est tout dans le conglomérat, mais dans le bas, est bordé d'un côté par le l'argilolite, de l'autre par le conglomérat. Sur la côte il reparoit en creux et a des deux côtés des salbandes de grès blanc saillantes, et d'un pied d'épaisseur chacune.

2eme partie. De la masse d'argilolite à l'extrémité SE. de la baye de Brodick, direction de la côte N. 107° E.

No.	Localités.	Nature.	Etat.	Largeur.	Direction magn.	Inclinaison ou plonge de—à.	Couches traversées. Nature.	Couches traversées. Direction.	Couches traversées. Inclinaison.	Remarques.
116	A l'est et près du 115	Basalte compacte à grain fin	Creux	12 pieds	N. 25° E.	80° au S. 65° E.	Conglomérat rouge nouveau	NO. vrai	20° au SO. vrai	Le No. 116 est le grand dyke cité par Playfair. Hutt. Theor. pour sa largeur ; c'est dans la falaise qu'il est le plus remarquable, par la grande fissure qu'il produit dans le rocher. Mais il reparoit sur la côte toujours creux, seulement il est là rejeté de quelques pieds à l'O. de sa première direction. Il est absolument dans l'alignement du No. 34 de l'autre côté de la baye.
117	Plus à l'est	Grunstein ou basalte	Creux ?	Large	N. 25° E.	80° au S. 65° E.	id.	id.	id.	
118	Idem.	Grunstein vert-noirâtre	...	1 à 2 pieds	E. vrai	70° au N. vrai	id.	id.	id.	Parallèle à la côte ou à peu près.

3eme partie. De l'extrémité SE. de la baye de Brodick à Clachland Point, direction de la côte N. 18° O.

No.	Localités.	Nature.	Etat.	Largeur.	Direction magn.	Inclinaison ou plonge de—à.	Couches traversées. Nature.	Couches traversées. Direction.	Couches traversées. Inclinaison.	Remarques.
119	Sur le rivage immédiatement sous le 1ere maison à l'est des Corygills inférieurs	Grunstein	?	6 à 7 pieds	NE.	?	Grès rouge nouveau	Variable, mais en général S. plus au moins E.	45° ou 30° au SO. ou à l'OSO.	Ici la côte est en entier de grès rouge, à direction et inclinaison très variables. La falaise est de grès rouge et de conglomérat, plongeant de 25° au SO. vrai.
120	Plus au SE.	id.	Creux	1 pied	NO. vrai	Vertical	id.	id.	id.	

Ici vient une deuxième masse ou couche d'argilolite prismatique et schistoïde, moins épaisse que la précédente, mais accompagnée sur sa lisière nord par une masse, en forme de coin, d'un pechstein en grand partie rempli de boules de sphérulite d'une à 3 lignes de diamêtre.

No.	Localités.	Nature.	Etat.	Largeur.	Direction magn.	Inclinaison ou plonge de—à.	Couches traversées. Nature.	Couches traversées. Direction.	Couches traversées. Inclinaison.	Remarques.
121	Au sud de l'argilolite	Grunstein	Creux	1 pied	NO. vrai	Vertical	id.	id.	id.	
122	Le long du rivage au nord, et au sud de la masse d'argilolite	Grunstein	En partie creux et en partie plein et saillant	6 pieds	Presque N. vrai	70° à l'E. vrai	id.	id.	id.	A très peu près parallèle à la côte que ce très long dyke suit pendant un grand espace. Il éprouve par fois des rejets, surtout vers la masse d'argilolite, où il est rejeté vers l'E. A basse marée on voit d'autres très longs dykes creux, parallèles à celui-ci et à la côte. Je ne les ai aperçu que du haut des collines et ne puis les détailler. N.B.—Soit que la direction de la boussole ait été affectée dans mon observation, soit que la position de la côte sur la carte soit fautive, je n'ai pu y faire observer le parallélisme du dyke 122 et de la côte, qui est très frappant dans la nature.
123	Sous Dunfeune et vers l'origine nord du lit de pechstein ci-dessous	idem	Saillant	Très large et haut	A peu près E. vrai	Vertical	Grès rouge nouveau	...	...	

Dans la falaise sous Dunfeune est un vrai lit de pechstein prismatique, fort épais, parallèle aux couches du grès rouge nouveau et des marnes jaunes qui l'accompagnent. Cette couche de pechstein paroit correspondre à celles qu'on voit plus haut vers Dundow et probablement à celle qui, bien plus à l'ouest, traverse l'ancienne route de Brodick à Lamlash, signalée entre les Nos. 77 et 78.

				Largeur.	magn.	plonge de—à.	Nature.	Direction.	Inclinaison.	Remarques.
124	Un peu au N. de Clachland Point	Grunstein	Creux	6 pieds	N. 45° E.	80° au S. 45° E.	Grès rouge nouveau	...	...	Le grès rouge n'est pas décoloré à la jonction avec le dyke 124.

La couche de syénite augitique et zéolitique de Dunfeune descend jusqu' à la mer et se perd sous ses eaux à Clachland Point. C'est la même couche signalée entre les Nos. 82 et 83. A Clachland Point elle est recouverte par des couches parallèles d'un grès fort durci et décoloré.

4ème partie. De Clachland Point à Lamlash. Direction de la côte N. 72° E.

125	1er. Cap à l'O. de Clachland Point.	Grunstein	Saillant de 3 à 4 pieds	10 à 12 pieds	N. 30° O.	Vertical	Grès rouge nouveau	N. 80° E.	10° au S. 10° E.	
126	Plus à l'ouest	id.	Saillant de 6 à 7 pieds	6 à 7 pieds	N. 10° O.	id.	id.	id.	id.	
127	Idem	id.	Saillant	6 à 7 pieds	N. 10° O.	id.	id.	id.	id.	Le dyke 127 saille en mur de 10 à 12 pieds de haut, 6 à 7 d'épaisseur et 150 à 250 de longueur. Il est divisé en prismes horizontaux, d'environ 2 pieds de diamètre. Sa hauteur s'abaisse par fois.
128 et 129	Idem et à l'est des chaumières de Clachland	2 dykes de trap	Saillant de 6 à 8 pieds	Le 128, 8 à 10 pieds, le 129, 16 à 20 pieds.	N. 20° O.	Verticaux	id.	id.	id.	Le grès rouge et surtout jaunâtre reparoit entre les 2 dykes 128 et 129, que sans cela on eut pris pour un seul dyke extrêmement large. La distance entre 128 et 129 est de 63 à 75 pieds.
130	Plus à l'O. et à l'ouest des dites chaumières	Grunstein verdâtre à grain fin	Un peu saillant	6 pieds	N. 20° O.	70° à 80° au N. 70° E.	id.	id.	id.	Le grès est gris blanc et très dur sous le dyke 130 du côté ouest, et il est jaunâtre avec rognons d'argile rouge sur le dyke à l'est.
131	Au bas de l'avenue d'arbres, route de Brodick	Grunstein noir à grain fin	Saillant de 5 à 6 pouces	12 pieds	N. 20° O.	Vertical	id.	id.	id.	Les deux dykes 131 et 132 parallèles sont éloignés l'un de l'autre de 50 pieds. Le 131 a des salbandes verdâtres terreuses de 3 à 4 pouces, et du grès jaunâtre et blanchâtre un peu durci, sur les côtés.
132	Ibid.	idem	id.	8 pieds	N. 20° O.	id.	id.	id.	id.	
133	Sur le rivage en avant de l'angle SO. et dans la falaise derrière l'angle NO. de l'auberge Bannatyne à Lamlash	Grunstein décomposé	Plein	3½ pieds	N. 45° O.	80° au N. 45° E.	Idem, tendre avec argile schisteuse noire	ENE.	5° au SSE.	Dans la falaise derrière l'auberge on voit que, à 2 pieds du côté SO., et du côté NE. à 3 pieds de la jonction, le grès est devenu très dur, gris blanc, quartzeux et cristallin. Ce dyke 133 se prolonge sur le rivage à 60 pieds au NE. du dyke suivant 134.
134	Sur le rivage, à 25 pieds à l'ouest de l'auberge Bannatyne à Lamlash	Grunstein	Un peu saillant	12 pieds	N. 45° O.	80° au N. 45° E.	Grès rouge nouveau, et marnes rouges et jaunâtres	id.	id.	Le dyke No. 134 reparoit aussi dans la falaise.
135	A 250 pieds au SO. du precédent (No. 134)	idem	id.	12 pieds	N. 45° O.	80° au N. 45° E.	id.	id.	id.	

M. Sur la côte de Lamlash à King's Cross Point, dirigée N. 53° O.

136	A Coardon au sud de Lamlash sur la colline et sur la route du sud	Argilolite (claystone)	...	?	E. vrai	Vertical	Grès rouge nouveau presque horizontal			
137	Ibid.	Grunstein décomposé en boules	...	?	id.	id.	id.	id.	id.	
138	Rivage à Gorrton-Allister	Grunstein	...	1 pied	N. 45° O.	id.	id.	id.	id.	Ce dyke 138 est croisé par le suivant 139.
139	Ibid.	id.	...	1 pied	NNE. vrai	id.	id.	id.	id.	Le dyke 139 croise le 138.
140	Ibid. plus au SE.	id.	...	12 pieds	NNE. vrai	80° au SSE. vrai	id.	id.	id.	
141	Ibid. plus au SE.	id.	...	1 pied	N. 45° O.	Vertical	id.	id.	id.	
142	Rivage dit King's Cross Shore	id.	A demi-saillant	5 pieds	E.	Vertical	id.	id.	id.	Le dyke 142 est bordé, des deux côtés, de murs de grès dur, saillants aussi.
143	Ibid. plus au SE.	id.	Creux	3 pieds	N. 75° E.	80° au S. 15° O.	id.	id.	id.	Des murs de grès saillants bordent le dyke 143.

No.	Localités.	Nature.	Etat.	Largeur.	Direction magn.	Inclinaison ou plonge de—à.	Couches traversées.			Remarques.
							Nature.	Direction.	Inclinaison.	
144	Rivage dit King's Cross Shore plus au SE.	Argilolite violâtre (claystone)	Plein	9 pieds	N. 10° O.	Vertical ?	Grès rouge nouveau presque horizontal			
145	Ibid. id.	?	Creux	5 pieds	N. 10° O.	idem	id.	id.	id.	On ne voit que le creux formé par le dyke, qui lui-même est invisible, en sorte qu'on ne peut en connoitre la nature. Il est croisé par le dyke 146.
146	Ibid. vis à vis le milieu de l'île de Lamlash	Grunstein	Un peu saillant	?	N. 10° E.	80° au S. 80° E.	id.	id.	id.	Le dyke 146 croise le 145.
147	Ibid. plus au SE.	Grunstein ?	Creux	1 pied	E. vrai	70° au N. vrai	id.	id.	id.	Les parois de grès du dyke 147 sont saillantes et minces.
148	Ibid. id.	id. ?	Saillant	$1\frac{1}{2}$ pied	N. 50° E.	80° ? au S. 40° E.	id.	id.	id.	
149	Ibid. id.	Argilolite blanche (claystone)	Plein	12 pieds	N. 45° O.	?	id.	id.	id.	

Le Promontoire nommé King's Cross Point est formé par une couche très puissante de syénite ou grunstein, à gros grains (en boules), augitique et zéolitique, semblable à celle qui forme l'extrémité sud de l'île de Lamlash. Sur le revers SO. de ce promontoire, qui sépare les bayes de Lamlash et de Whiting, et à l'entrée du Whiting Bay, ce grunstein à gros grains est interrompu sur des espaces considérables par des masses plates de grunstein à grain fin, en espèces de prismes verticaux, ou en compartiments semblables à des pavés. Ce sont peut-être de très larges dykes.

Dans la liste des filons ou dykes trapéens d'Arran qui précède, je n'ai fait que relever avec exactitude mes observations telles qu'elles ont été écrites sur les lieux mêmes dans mes notes, pendant des excursions nombreuses, et pour certains points fréquemment répétées, entre le milieu d'Avril et le milieu de Juillet 1839. Quoique toutes ces observations ne portent pas le même degré de précision et d'exactitude, j'ai cru ne devoir rien changer à l' expression adoptée au moment même de l'observation. Le plus ou moins de difficulté dans l'examen de chaque localité, de temps employé à cet examen, de clarté dans les phénomènes eux-mêmes, a dû nécéssairement avoir de l'influence sur la précision du résultat. Cependant même dans les cas les plus défavorables, lorsque les observations ont été faites avec le plus de promptitude, dans des courses rapides, et lorsque des phénomènes géologiques plus importants requerroient la plus grande part de l'attention, j'ai toujours cherché à mettre dans l'observation des dykes toute l'exactitude que permettoient les circonstances. La nature minéralogique des dykes n'a été minutieusement étudiée que dans un très petit nombre de cas ; une pareille étude, avec la récolte, l'étiquettement et l'enrégistrement d'échantillons qu'elle eut nécéssité, en exigeant un temps très considérable, eut rendu impossible le travail actuel. Je me suis donc en général contenté d'une inspection rapide de la nature de la roche, du dyke et de celles qu'il traverse : Nommant *basalte*, le trap noir à grain fin ; *grunstein*, le trap vert ou bleu ou brun à grains plus gros et plus distincts ; et enfin *trap*, une roche presque compacte à grain très fin, d'aspect plus ou moins terreux. J'ai en général noté avec soin l'etat actual du dyke et signalé si il est plus ou moins creux, plein ou saillant, car cette circonstance a de l'intéret pour comparer le degré plus ou moins grand de résistance qu'offrent à la décomposition soit le trap de divers genres des dykes soit les roches qu'ils traversent ; et l'on peut trouver là un moyen d'apprécier dans ces dernières des degrés d'endurcissement si peu considérables qu'il seroit impossible de les reconnoitre autrement. La largeur, épaisseur ou puissance des dykes, a été très généralement simplement appréciée et estimée à l'œil, de même que quelques autres mesures de distances et de longeur : souvent j'ai compté le nombre de mes pas, mais au lieu de faire entrer ce genre de mesure dans ces colonnes, j'ai dévié pour cet objet de la règle que je m'étois imposée de donner mes notes sans changements, et j'ai reduit les pas en pieds en les multipliant par $2\frac{1}{2}$, ce qui s'accorde avec l'expérience que j'ai faite de la longeur de mes pas. La détermination de la direction et de l'inclinaison tant des dykes que des couches traversées, m'a paru l'objet le plus essentiel de tous, aussi est-ce celui auquel j'ai consacré le plus de soin et d'attention. Lorsque, dans la colonne de la direction des dykes et des couches, on voit figurer un azimuth suivi du mot *magn.*, ou indiqué en degrés comptés à partir du N., l'observation a été faite avec précision à l'aide de la boussole, et le résultat donne l'azimuth magnétique. Dans les autres cas la direction a été seulement estimée avec plus ou moins d'exactitude, souvent à l'aide de l'ombre et du soleil, l'heure etant connue ; j'ai alors ajouté le mot *vrai* à la suite du rhumb qui indique cette direction ; les résultats de cette espèce doivent inspirer plus de confiance que ceux qui ne portent que la lettre indiquant le rhumb sans addition de degrés, et qui ont été estimés purement et simplement en azimuth vrai. Malgré toute la peine que j'ai pris pour déterminer avec la boussole d'une manière satisfaisante la direction des dykes, je ne puis répondre que dans tous les cas l'aiguille aimantée n'ait pas été sensiblement affectée par le dyke lui-même ; j'ai du même mentionner cette crainte au long dyke No. 122. L'azimuth de l'inclinaison étant une dépendance immédiate de celui de la direction, tout ce qui vient d'être dit s'y rapporte également. Quant à l'angle d'inclinaison, il a été en général mesuré, soit avec le petit fil à plomb de Saussure, soit avec le clinomètre de M. Adie, souvent avec tous les deux ; dans les deux cas, cet angle est indiqué avec précision ; lorsqu'il est donné entre certaines limites, ou avec doute, cet angle a été seulement estimé à vue.

XXXIII.—*An Account of the Iron Mines of Caradogh, near Tabreez in Persia, and of the Method there practised of producing Malleable-Iron by a single process directly from the Ore. By* James Robertson, *Civil and Mining Engineer, Major Persian Service, and late Director of the Shah's Ordnance Works, Persia; Cor. M. W.S., and Cor. F.A.SS.*

(Read 2d March 1840.)

The ancient Greeks have laid claim to the earliest discovery of the method of manufacturing iron, but it will appear that the art was known in Persia at least as early as among the Greeks. The method of producing malleable-iron by a single process directly from the ore, is not indeed quite unknown at the present day, but it is believed to be altogether disused in Great Britain and throughout Europe; but there is no doubt that, in Britain, particularly at Castle Cough, Glamorganshire, and at Furness, near Ulverston, in Lancashire, as well as elsewhere, malleable-iron must have been known long before the discovery of cast-iron. In the 17th century, malleable-iron appears to have been made directly from the ore, in preference to the method now practised. In the Philosophical Transactions (for 1693, vol. xvii. p. 695), there is the following short notice by Mr Sturdy, of the method as then practised at Milthorpe-forge in Lancashire. "The forge is like a common blacksmith's, with a hearth made of sow-iron, in which they make a charcoal fire, *and put in ore*, first broken into pieces like a pigeon's egg; it is melted by the blast, leaving the iron in a lump, which is never in a perfect fusion; this is taken out and beaten under great hammers, played with water, and, after several heatings in the same furnace, it is brought into bars. They get about one hundredweight of metal at one melting, being the produce of about three times as much *ore;* no limestone or any other flux is used." It has been doubted by an intelligent author (Farey on the Steam-Engine, p. 271), whether, by the process here described, the iron was really made directly from the ore, or only from pig metal. The existence, however, of a similar process at the present day in Persia, evidently the same which has been practised in that country from a very remote period, will make it appear not the least improbable that iron may have been thus produced from the rich hematite or fibrous red iron-ore of Lancashire.

The writer of this paper having resided for more than two years in the neighbourhood of the Persian mines, and having been during that time engaged in superintending the manufacture of cast-iron, trusts that the following short account of the mines, and of the very primitive process of the iron manufacture, which came constantly under his observation, may be found interesting, if it be not also of some practical advantage, even where the manufacture is conducted with all the refinements of modern scientific improvements.

We have no historical record from which to ascertain the period at which the iron mines in the district of Caradogh were first wrought. But there is every reason to suppose that they were resorted to from the remotest antiquity. The district itself is very secluded, and is of a wild, forbidding aspect; it has, without almost any interval, formed part of the Median, and latterly of the Persian, empire; and, under the rule of native princes, has all along been free from the revolutions which have so frequently convulsed Western Asia. The iron mines themselves also bear evident marks of antiquity. They form large quarry-like excavations, thickly surrounded by immense tumuli of iron-sand and small pieces of ore, thrown out in the course of working. Upon a rough calculation, founded on the size of the excavated hollow which it exhibits, one only of the numerous iron mines which abound in the district, was estimated by the writer of this notice to have now afforded above 4,000,000 cubic feet of iron-ore. Taking the specific gravity of the ore at 5, a cubic foot would weigh about 300 lb., and consequently seven cubic feet would weigh about a ton; and 4,000,000 cubic feet, the total quantity excavated from that mine, would weigh 571,428 tons. Now, at the present day, 2000 horse loads is a full allowance for the yearly quantity carried away, and as each horse carries about 2 cwt., we have a total of 200 tons per annum as the exported produce at present. It may be reasonably assumed, that this quantity has, upon an average, never been exceeded during the many ages in which the mines have been wrought. Indeed, this estimate certainly exceeds the actual average yearly produce; for although a considerable quantity of Russian iron is now imported, to supply the increasing wants of the inhabitants, it cannot be imagined that, in periods of their early history, the natives would require nearly so much iron as they now do. Upon that assumption, and without taking into account the other neighbouring mines, it would follow that 2857 years have passed since the soil was first removed from the surface of the mine alluded to. Were the other neighbouring mines taken into account, the antiquity of the whole would be proportionally increased. The writer has not by any means stated these as calculations, or as at all approximating to accuracy, but still he thinks that, from such data, fanciful as they may in some measure appear, an estimate may legitimately be formed on the very great antiquity of the Persian mines.

The native smiths are dispersed in small hamlets, situated in the woods which clothe the sides of the ravines, through which the mountain torrents flow into the river Arras (the ancient Araxes). The iron which is produced, although soft, is extremely tough. It is much superior to the Russian iron, with which the greater part of Asia is now supplied, and is manufactured chiefly into horse-shoes, and horse-shoe nails, for which there is a great demand in Tabreez and the surrounding districts, and among the Koords or Nomadic tribes who frequent the mountain pastures in summer. The trade in it is shared between the Mahomedans and the native Armenians; and although by no means extensive or deserving the name of the "Persian iron trade," it gives employment to a considerable part of the population, in quarrying the ore, burning the charcoal, and transporting these articles to the forge.

There are numerous mines in Caradogh, affording iron-ore of the most valuable description, and of various kinds; but those held in the highest estimation are the Jewant, Koordkandy, and Marzooly ores.

The Jewant mine is situated in an immense vein of red iron-ore. This ore, on its fracture, often exhibits streaks of prismatic colours, as if at one time it had been subjected to the action of heat; quantities of iron-sand are dispersed in the interstices of the vein.

The Koordkandy mine, situated on the summit of a very steep mountain, produces rich magnetic iron-ore, from a vein of great dimensions. The Marzooly mine also affords excellent magnetic iron-ore in great abundance. The vein in which the last is situated runs across several hills, and is in most parts 100 feet in width.

In working these mines, the richest pieces only of the ore are carried away, the remainder is thrown aside. They are worked very irregularly, and without concert, as there is no restriction imposed as to the mode of mining by the Government. A few individuals sink a shaft through the rubbish, and excavate as much as they require; another party soon after arrive, and fill the first hollow up in the course of sinking another shaft; and in this way the rubbish is repeatedly turned over, and gradually subsides and is consolidated into a mass as the ore is removed from beneath, thus forming a serious obstacle to any one who might attempt to work the vein in a more regular manner. The ore is carried to the villages only during the summer, as the depth of the snow in winter renders the mountain paths impassable. It is there retailed to the smiths, who purchase a horse-load of 2 cwt. for about 1s. Sterling, or 10s. per ton.

The ores above described, when smelted singly, produce that kind of iron which by English workmen is called *hot-short*, and by the Persians *salt-iron.* The smiths, however, by means of a mixture, produce iron of an excellent quality, which they term *sweet-iron.* The most common mixture is two parts Jewant ore to one of Koordkandy, and two parts of Koordkandy to one of Marzooly.

Materials for smelting the ore are found in an extensive natural forest which occupies the central parts of the district of Caradogh. This forest covers the flat bottoms between the mountains, and spreads to a considerable height up their sheltered sides, dwindling into dwarf trees and bushes in the elevated and more exposed situations. It consists chiefly of coppice oak, which springs from the roots of trees cut and recut during a long succession of years. This jungle is partitioned among the villages situated on its confines, the inhabitants of which earn a livelihood by supplying the city of Tabreez and adjoining towns with fuel.

The charcoal is made in the following manner: A rectangular hollow is dug in the earth, about twelve feet long, six feet wide, and four feet deep. The sides are formed of the natural ground, or common alluvial cover; a small sloping doorway is cut at one end, and at the other a chimney is built rising to the height of about six feet. The pit is filled up to the level of the ground with cut branches of all dimensions, placed horizontally and lengthways in the hollow, and are covered over with earth, and secured effectually against the admission of air, excepting by a small hole in the built-up door-way, which is left open to produce a current; the heap is kindled through the small opening in the door-way, and after it has burned for two or three days the covering is removed, and the charcoal thus produced is then stored for sale. One of these hearths will produce about one ton of charcoal, which sells at thirteen shillings sterling.

The charcoal thus produced, however, is seldom used in the manufacture of iron, the smiths preferring that prepared in the following manner: The cut branches are merely laid horizontally on the surface of the ground, and piled up to a considerable height; having been lighted from beneath, they are allowed to burn in the manner of an open fire, till the smoke and flame have nearly ceased; the fire is then quenched with water, when there remains a charcoal which is very light, and is found to reduce the ores of iron in a much less time than the heavier charcoal produced by the first method.

As the iron is manufactured on a very small scale, a very simple forge answers the purpose. It consists merely of a hollow hearth dug out of the clay floor of the hut, about fourteen inches square in the bottom, and nine inches deep, for receiving the ore and fuel; and of another hearth immediately thereto adjoining, intended to receive the slag, and consisting of a larger excavation, about three inches deeper than the former, and situated betwixt it and the wall at the other extremity in which the chimney is constructed. A wall is built on each of the two sides, two or three feet high, and the whole is covered over with large stones capable of resisting the action of the fire. The whole of the first or iron-hearth into which the blast is introduced is left open above and at the sides; but a low wall is built next the bellows to prevent the heat from injuring them. The whole is afterwards plastered over wi clay and chopped straw, in order to maintain the draught of the chimney entire. The chimney is carried up through the wall of the hut, and seldom rises higher than its roof.

The construction and dimensions of these hearths, and of the different implements required in working them, will be best explained by the accompanying drawings.

Fig. 9.

Fig. 1.

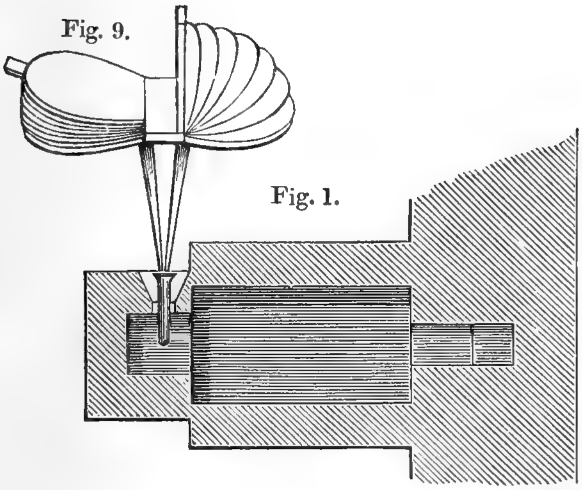


Fig 2.

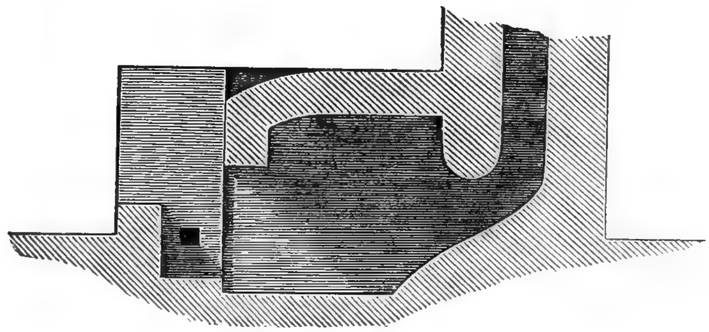


Fig. 3.

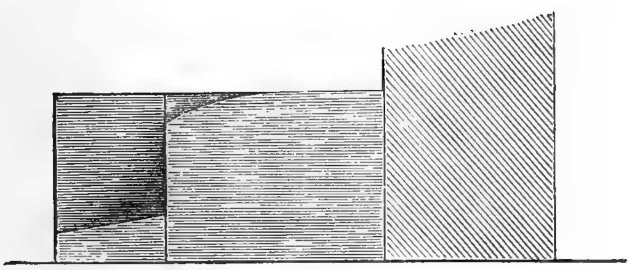


Fig. 4.

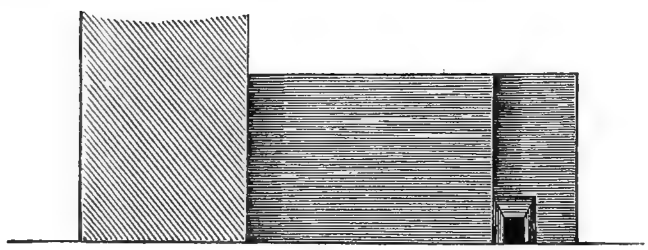


Fig. 9.

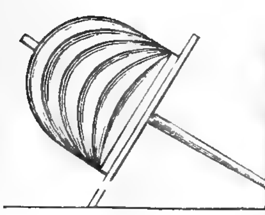


Fig. 5.

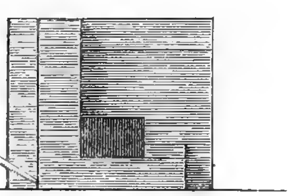
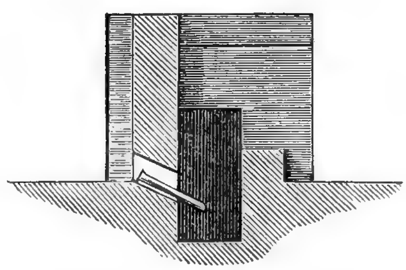
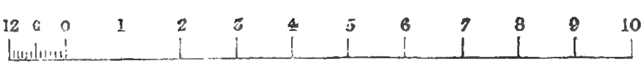


Fig. 6.



Scale.

Inches.



Feet.

Fig. 7.

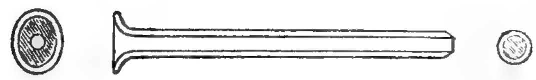


Fig. 8.

Fig. 10.

Fig. 11.

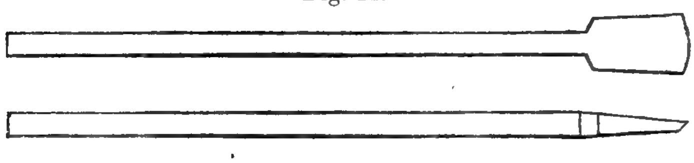


Fig. 12.

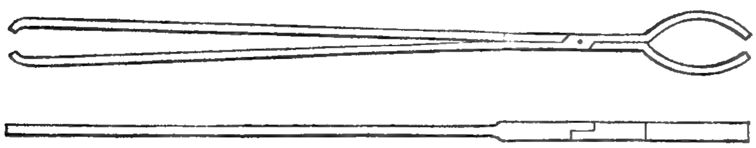


Fig. 13.

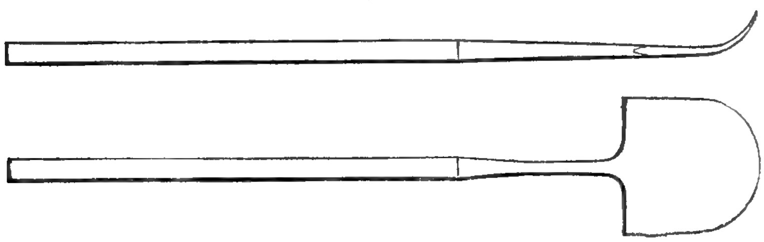


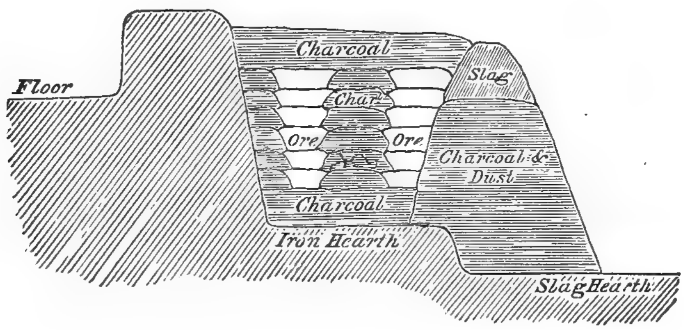
Fig. 14.

Scale.

DESCRIPTION OF FIGURES.

Fig. 1. Ground-plan of forge.
Fig. 9. A pair of double bellows.
Fig. 2. Vertical section of the forge and chimney.
Fig. 3. Side view of forge.
Fig. 4. Side view of forge.
Fig. 9. Side view of bellows.
Fig. 5. End view of forge.
Fig. 6. Section of ore-hearth through the centre.
Figs. 7, 8, 10, 11, 12, 13, 14, Tools employed in the manufacture.

The operator having carefully selected charcoal of a small size and light weight, proceeds to clear it from dust and sand with a small meshed riddle, removing all the heavy pieces of charcoal or stones that may be accidentally mixed with it. The raw ore being next selected and mixed, and being broken into small pieces about the size of a hazel-nut, is thoroughly moistened with water. A dam is then made between the iron and slag hearths, composed of charcoal and charcoal dust well rammed down, and the top is coped with iron-slag from a former smelting. The following sketch will shew this arrangement:



The Twyére pipe (Fig. 7), which is made of white clay, and bears a violent heat for a long time without melting, is then inserted through the small hole in the side wall of the first iron hearth. The point of the pipe is made to reach half-way across the iron hearth, and within six inches of the bottom, as shewn in Fig. 6. A layer of charcoal, of three inches thick, is then spread over the bottom of the iron hearth, and upon this two other layers laid across, one directly under the Twyére pipe of about six inches in breadth and three inches deep, and the other at the front of the hearth of the same thickness, to correspond with the overlying part of the dam. The two trenches which are thus formed are filled up with the moistened ore, well rammed down. A second layer of charcoal, in a state of ignition, is thereafter laid over the former under the twyére pipe, and other successive layers of charcoal and ore are filled in, corresponding with those in the bottom. When the hearth has been nearly filled up in this way, a covering of charcoal is spread over the surface of the whole on a level with the top of the dam. The bellows are then blown, and a workman, who stands at the side of the hearth, keeps constantly pushing down the charcoal in the middle with an iron rod (Fig. 8), and from time to time throws small quantities into the centre of the fire as it gradually subsides. At the commencement, one man at a time is sufficient to blow the bellows, but, towards the close, two are required, the one standing behind the other. The bellows, of the form shewn in the figure marked 9, are in general use all over Persia. After blowing for an hour or an hour and a half, part of the twyére pipe having melted from the violence of the heat, the blast is stopped for a moment, for the purpose of pushing the twyére pipe farther in towards the centre of the hearth. It is then again continued, and in about three hours, or three and a half hours from the commencement, the ore becomes consolidated, but not fused. The blast is then again stopped until that half of the bloom which is next to the slag hearth is turned over with an iron bar (Fig. 10), and pushed on the top of the dam, while the other half is turned round to the centre of the fire. The blast is then immediately recommenced, and the metal of the half bloom in the centre of the fire speedily falls to the bottom. The remaining half of the bloom is then drawn into the centre, and treated in a similar manner, very little charcoal being placed on the top of the fire during this part of the process. When the metal has entirely disappeared by sinking to the bottom of the hearth, the whole semifluid mass is stirred about for a quarter of an hour longer with an iron rod (Fig. 10). The blast being then stopped, the twyére pipe is withdrawn, and the operator taking his shovel (Fig. 13), pushes the burning charcoal together with the dam into the lower hearth; the slag immediately runs off, and exposes the glowing iron lying in the bottom of the upper hearth; the metal is then beaten with the back of the shovel into a more solid state, and after being dexterously cut with an iron chisel bar (Fig. 11), from the sides of the hearth, and forced from the bottom, it is removed to the floor of the hut with a large pair of tongs (Fig. 12). The iron is next beaten with large hammers as it lies on the ground, in order to expel the slag and other impurities from its pores; and after being in this way formed into a rough mass, it is lifted to the anvil, when it is again hammered into a more regular shape. It is next cut into two pieces with large hammers (Fig. 14), and is then fit for being drawn into bars of the dimensions required.

At a single smelting, one hearth generally affords about 30 lb. of malleable iron, to produce which there is only required about double that quantity of ore, and three times the weight of charcoal. One smith with his assistants will make about three or four smeltings in one day, or 1 cwt.

It must strike every one acquainted with the iron manufacture, that this *yield* is in a high proportion to the materials used. In England, about four tons of raw ore and eight tons of coal are required to produce one ton of bar-iron; while, by the process above described, the same quantity of iron, of a much superior quality, is produced in Persia from less than half of these materials. The greater productiveness is no doubt to be attributed in a great measure to the superior richness of the Persian ores, and the use of charcoal; but the simplicity of the process must also have a considerable share in diminishing the waste of materials; for the roasting, smelting, refining, puddling, shingling, balling, and drawing-out, or something very similar, is all there effected, as it may be said, at one heat, and in a very few hours.

The rich iron-ores of Cumberland and Lancashire, and many others in Britain, particularly the blackband ironstone of Scotland, which has so recently attracted the attention of iron-masters, if manufactured in the same manner, would undoubtedly produce similar results, and thus create a great saving in time, labour, and capital, as well as diminish the waste of materials.

In conclusion, the writer would beg once more to draw attention to the fact that malleable-iron can be readily made directly from the ore, contrary to what he believes to be the prevalent opinion in this country.

Since writing the preceding pages, the writer has had an opportunity of becoming acquainted with a similar process to the one already described, now successfully practised near the town of Malatia on the Syrian frontier, in the central parts of Asia Minor. The iron-ores in this district are of the richest description, and were examined by the writer at the command of the Turkish government, with the view of establishing iron-works on the scale of British iron-works, for the supply of the Turkish ordnance. The method there pursued is, if possible, still more simple than that of the Persians, as the furnaces are in the form of a small cupola, and the fuel is simply dry wood.

PROCEEDINGS

OF THE

EXTRAORDINARY GENERAL MEETINGS,

AND

LIST OF MEMBERS ELECTED AT THE ORDINARY MEETINGS,

SINCE MAY 2. 1836.

PROCEEDINGS, &c.

November 28. 1836.

At a General Meeting held this day, Dr HOPE, V. P., in the Chair, the following Office-bearers were elected for the ensuing year:—

Sir T. MAKDOUGAL BRISBANE, Bart., K. C. B., President.

The Hon. Lord GLENLEE,
Dr HOPE,
Sir D. BREWSTER,
Right Hon. Lord GREENOCK,
Rev. Dr CHALMERS,
Dr ABERCROMBIE, } Vice-Presidents.

JOHN ROBISON, Esq., General Secretary.

Dr CHRISTISON,
Professor FORBES, } Secretaries to the Ordinary Meetings.

CHARLES FORBES, Esq., Treasurer.

Dr TRAILL, Curator.

JOHN STARK, Esq., Assistant Curator.

COUNSELLORS.

Sir GEORGE BALLINGALL, M. D.
J. T. GIBSON-CRAIG, Esq.
Hon. Lord MEADOWBANK.
THOMAS THOMSON, Esq.
Venerable Archdeacon WILLIAMS.
Professor HENDERSON.
GEORGE FORBES, Esq.
Rev. Dr WELSH.
Sir H. JARDINE.
Sir CHARLES BELL.
DAVID MILNE, Esq.
JAMES SMITH, Esq.

The following Fellows, in terms of Law XXI., were appointed a Committee to audit the Treasurer's accounts:—

SIR HENRY JARDINE.
CLAUD RUSSELL, Esq.
JOHN STARK, Esq.

January 2. 1837.

MEMBERS ELECTED.

ORDINARY.

John Archibald Campbell, Esq. W. S.

January 16. 1837.

John Scott Russell, Esq.
Charles Maclaren, Esq.
A. Smith, Esq., B. A. F. T. C., Cambridge.

February 20. 1837.

Richard Parnell, M. D.
P. D. Handyside, M. D.

At an Extraordinary General Meeting, held on Wednesday 12th July 1837, the Right Hon. Lord Greenock, V. P., in the Chair, it was resolved unanimously to present an Address of Condolence and Congratulation to Her Majesty the Queen, on the occasion of her accession.

A draft of an Address having been read and approved of, it was moved by Mr G. J. Bell, and seconded by Dr Traill, that it be adopted; which, on being put from the Chair, was unanimously agreed to.

On the motion of Dr Borthwick, seconded by Mr T. Thomson, it was resolved to present an Address of Condolence to Her Majesty the Queen-Dowager; and it was remitted to the Council to prepare it, and to forward the two Addresses for presentation.

The Addresses here follow:—

TO THE QUEEN.

May it please your Majesty,

We, the President and Fellows of the Royal Society of Edinburgh, established under the patronage of your Majesty's Royal Grandsire, for the promotion of Letters and Useful Science, most humbly approach your Majesty with the loyal tender of our duty and homage.

Whilst we join in the general condolence of our fellow-subjects on the lamented demise of a Sovereign, endeared by many great and good qualities to the nation over which he had ruled with a truly paternal benignity and care, we hail with auspicious hopes, and most ardent prayers, the opening prospects of a reign which, even in the first acts of your Majesty's sovereignty, has commanded the admiration and secured the affection of all your Majesty's loyal subjects.

Among the great public virtues which have distinguished your Majesty's Royal Predecessors, and by which their reigns have been pre-eminently illustrated and adorned, it is more peculiarly our duty, in this Society, to commemorate their truly noble and patriotic efforts for the promotion of Literature, of Science, and of the Useful and the Elegant Arts; and, in the firm and loyal confidence that these peaceful glories of the past age will suffer no diminution under your Majesty's gracious influence, it shall be our province and duty to contribute our humble endeavours for the farther advancement of the proper objects of our social institution, and to merit the continuance of that Royal patronage and support by which our labours have been hitherto encouraged and upheld.

That your Majesty may long continue to sway the sceptre of your ancestors over a loyal and happy people, is the earnest prayer of,

May it please your Majesty,

Your Majesty's most dutiful subjects and servants,

THE PRESIDENT AND FELLOWS OF THE ROYAL SOCIETY OF EDINBURGH.

Signed in name and by appointment of the Society,

THOMAS MAKDOUGAL BRISBANE, *P.*
JOHN ROBISON, *Sec.*

TO THE QUEEN-DOWAGER.

MAY IT PLEASE YOUR MAJESTY,

We, the President and Fellows of the Royal Society of Edinburgh, beg leave to be permitted to approach your Majesty with the loyal expression of our sincere and deep-felt sorrow at the demise of our late most Gracious Sovereign. Under the severe pressure of an event so calamitous, it must be soothing to the best feelings of your Majesty's heart, that the reign of your Majesty's beloved and lamented Consort, alas too short! had gained for him the imperishable glory of having well approved himself one of the best of Kings,—the beneficent and venerated Father and Protector of his people.

In retiring from the more burthensome cares and duties of Royalty, your Majesty will carry with you, and be blessed in the consciousness of having well earned, the gratitude and loyal affection of a great and enlightened nation; and whilst, in these sentiments, we cordially join in the universal consent of all the faithful subjects of his late Majesty, we cannot omit to commemorate our sense of the peculiar obligations under which this Society (established for the promotion of Letters and useful Science), has been laid by his late Majesty's princely munificence; and by means of which we may hope the more effectually to advance the important objects of our social institution.

That your Majesty may be speedily restored to serenity and peace of mind, and may continue to reap the inestimable fruits of a life well spent in the faithful discharge of all the high duties which it has been your Majesty's lot to sustain, is the earnest prayer of,

May it please your Majesty,

Your Majesty's most loyal and devoted servants,

THE PRESIDENT AND FELLOWS OF THE ROYAL SOCIETY OF EDINBURGH.

Signed in name and by appointment of the Society,

THOMAS MAKDOUGAL BRISBANE, *P.*
JOHN ROBISON, *Sec.*

Edinburgh, 14th July 1839.

Memorandum.—On application having been made by the Secretary at the Home Office, it was ascertained that the Address to the QUEEN could be presented by a Deputation *only at* *a Levee;* and as no Levee was to be held in the interval between the time at which the Address was drawn up, and the month of January 1838, the Secretary thought himself justified in transmitting the Address through the Secretary of State for the Home Department, which was accordingly done, and an acknowledgment of it having been laid before Her Majesty, and having been very graciously received, was subsequently received from Lord John Russell.

The Address to the Queen-Dowager was transmitted through Her Majesty's Chamberlain, Earl Howe, and a similar acknowledgment was received through the same channel.

Letter from Lord John Russell.

Sir, *Whitehall, Nov.* 20. 1837.

I have had the honour to lay before the Queen the loyal and dutiful Address on the occasion of Her Majesty's accession to the Throne, from the President and Fellows of the Royal Society of Edinburgh, and have to inform you that the same was very graciously received by Her Majesty.

I have the honour to be, Sir, your obedient servant,

(Signed) J. Russell.

To the President of the Royal Society of Edinburgh.

Letter from Earl Howe, Lord Chamberlain to Her Majesty Queen Adelaide.

Sir, *St Leonards*, 16*th Nov.* 1837.

I have not failed to submit the Address of kind condolence from the President and Fellows of the Royal Society of Edinburgh to Queen Adelaide, and am honoured by Her Majesty's commands to express how consolatory has been to the Queen-Dowager's feelings this proof of attachment to herself, and of respect for the memory of the late King.

I have the honour to be, Sir, your obedient humble servant,

(Signed) Howe.

To Sir T. M. Brisbane, Bart.

November 27. 1837.

At a General Meeting held this day, Dr Hope, V. P., in the Chair, the following Office-bearers were elected for the ensuing year:—

Lieut.-Gen. Sir Thomas Makdougal Brisbane, Bart., G. C. B., President.

The Right Hon. Lord Glenlee,
Dr Hope,
Sir David Brewster, K. H.
Right Hon. Lord Greenock,
Rev. Dr Chalmers,
Dr Abercrombie,
} Vice-Presidents.

Sir John Robison, K. H., General Secretary.

Dr Christison,
Professor Forbes,
} Secretaries to the Ordinary Meetings.

Charles Forbes, Esq., Treasurer.

Dr Traill, Curator of the Museum.

John Stark, Esq., Assistant Curator.

COUNSELLORS.

Venerable Archdeacon WILLIAMS.
Professor HENDERSON.
GEORGE FORBES, Esq.
Rev. Dr WELSH.
Sir H. JARDINE.
Sir CHARLES BELL, K. H.
DAVID MILNE, Esq.
JAMES SMITH, Esq.
Dr GREVILLE.
T. JAMESON TORRIE, Esq.
Sir JAMES MILES RIDDELL, Bart.
Professor DUNBAR.

The following Committee was appointed to audit the Treasurer's accounts :—

Sir HENRY JARDINE.
CLAUD RUSSELL, Esq.
JOHN STARK, Esq.

MEMBERS ELECTED.

December 4. 1837.

ORDINARY.

JOHN CLARK, M. D., K. H.

January 1. 1838.

WILLIAM NICOL, Esq.

HONORARY.

March 5. 1838.

Professor TIEDEMANN, Heidelberg.
Professor MULLER, Gottingen.

ORDINARY.

May 7. 1838.

WILLIAM SCOT, Esq. H. E. I. C. Service.
THOMAS MANSFIELD, Esq. Accountant.
ALAN STEVENSON, Esq. Civil Engineer.

November 26. 1838.

At a General Meeting held this day, Dr HOPE, V.P., in the Chair, the following Office-bearers were elected for the ensuing year :—

Sir T. MAKDOUGAL BRISBANE, Bart., G.C.B., President.

The Hon. LORD GLENLEE,
Dr HOPE,
Sir DAVID BREWSTER, K.H,
Right Hon. LORD GREENOCK,
Rev. Dr CHALMERS,
Dr ABERCROMBIE,
} Vice-Presidents.

Sir JOHN ROBISON, K.H., General Secretary.

Dr CHRISTISON,
Professor FORBES,
} Secretaries to the Ordinary Meetings.

JOHN RUSSELL, Esq., Treasurer.
Dr TRAILL, Curator.
JOHN STARK, Esq., Assistant Curator.

COUNSELLORS.

Sir H. Jardine.
Sir Charles Bell, K.H.
David Milne, Esq.
James Smith, Esq.
Dr Greville.
T. Jameson Torrie, Esq.
Sir James Miles Riddell, Bart.
Professor Dunbar.
Thomas Thomson, Esq.
Rev. John Sinclair.
Dr Graham.
J. Scott Russell, Esq.

The following Fellows were appointed a Committee to audit the Treasurer's accounts:—

John Mackean, Esq.
J. T. Gibson-Craig, Esq.
William Paul, Esq.
W. A. Cadell, Esq.

MEMBERS ELECTED.

ORDINARY.

January 7. 1839.

James Auchinleck Cheyne, Esq. of Kilmaron. David Smith, Esq., W. S.

January 21. 1839.

Adam Hunter, M.D. Henry Marshall, Dep. Insp. Gen. of Army Hospitals.
Rev. Philip Kelland, A.M., Professor of Mathematics.

March 4. 1839.

William Ferguson, Esq., Surgeon.

March 18. 1839.

William Alexander, Esq., W.S. F. Brown Douglas, Esq., Advocate.

April 1. 1839.

Lieutenant-Colonel Swinburne.

November 25. 1839.

At a General Meeting held this day, Dr Hope, V.P., in the Chair, the following Office-bearers were elected for the ensuing year:—

Sir T. Makdougal Brisbane, Bart., G.C.B., G.C.H., President.

The Hon. Lord Glenlee,
Dr Hope,
Sir David Brewster,
Right Hon. Lord Greenock, K.C.B.
Rev. Dr Chalmers,
Dr Abercrombie,
} Vice-Presidents.

Sir John Robison, K.H., General Secretary.

Dr Christison,
Professor Forbes,
} Secretaries to the Ordinary Meetings.

John Russell, Esq., Treasurer.
Dr Traill, Curator of Library and Instruments.
John Stark, Esq., Curator of Museum.

COUNSELLORS.

Dr GREVILLE.	Dr GRAHAM.
T. JAMESON TORRIE, Esq.	Dr ALISON.
Sir JAMES MILES RIDDELL, Bart.	Sir HENRY JARDINE.
Professor DUNBAR.	JOHN SHANK MORE, Esq.
THOMAS THOMSON, Esq.	Professor HENDERSON.
J. T. GIBSON-CRAIG, Esq.	Professor KELLAND.

The following Committee was named to audit the Treasurer's accounts:—

J. T. GIBSON-CRAIG, Esq.	WILLIAM PAUL, Esq.
W. A. CADELL, Esq.	

MEMBERS ELECTED.

ORDINARY.

January 6. 1840.

ALAN A. MACONOCHIE, Esq., Advocate. MARTYN J. ROBERTS, Esq.
ROBERT DAUN, M.D., Dep. Insp. Gen. of Army Hospitals.

February 3. 1840.

ROBERT CHAMBERS, Esq. Sir JOHN MACNEILL, G.C.B.
JAMES FORSYTH, Esq.

Memorandum.—3d February 1840.—At the ordinary meeting of this date, it was proposed to the Society by the Council, that an Address should be presented to the Queen on the occasion of her Marriage. The meeting having agreed to this measure, the following draft of an Address was read, and a remit was made to the Council to prepare the Address for transmission, and to take the necessary steps for having it presented to Her Majesty.

TO THE QUEEN.

MAY IT PLEASE YOUR MAJESTY,

We, the President and Fellows of the Royal Society of Edinburgh, established for the promotion of Letters and Science, feel it to be our bounden duty to join with all your Majesty's faithful and loyal subjects in offering our sincere and cordial congratulations on the auspicious union of your Majesty with an illustrious Prince, who has become the fortunate object of your Majesty's choice, in circumstances which afford to your Majesty's devoted subjects the fullest assurance that, under a gracious Providence, it cannot fail to contribute to your Majesty's domestic happiness, and to alleviate the burden of those cares, and of those high and arduous duties to which, for the happiness of those realms, your Majesty has been called.

That your Majesty and your Royal Consort may long live in the undisturbed enjoyment of public and private prosperity, is the earnest prayer of,

May it please your Majesty,
Your Majesty's most dutiful subjects and servants,
THE PRESIDENT AND FELLOWS OF THE ROYAL SOCIETY OF EDINBURGH.

Signed in name and by appointment of the Society,

THOMAS MAKDOUGAL BRISBANE, *P.*
JOHN ROBISON, *Sec.*

At the ordinary General Meeting held on the 2d March 1840, the following letter was read:

My Dear Sir, *Kensington Palace, February* 1840.

At the last Levee I had the honour of presenting to Her Majesty the loyal Address of the President and Fellows of the Royal Society of Edinburgh, which the Queen was pleased to receive most graciously.

I remain, with consideration, dear Sir,

(Signed) Augustus.

To Sir Th. M. Brisbane, Bart. G. C. B.
Pres. R. S. Ed.

February 17. 1840.

John Cockburn, Esq.
Sir William Scott, Bart.
Rev. C. H. Terrot.

March 2. 1840.

Rev. R. Traill, D.D.
Robert Bryson, Esq.
Edward J. Jackson, Esq.

March 16. 1840.

John Shedden Patrick, Esq.
John Learmonth, Esq.

April 6. 1840.

G. A. Stuart, Esq.
Right Hon. T. B. Macaulay, M.P.

April 20. 1840.

Gilbert Laurie Finlay, Esq.
John Mackenzie, Esq.
John Thomson, Esq.

LIST OF THE PRESENT ORDINARY MEMBERS IN THE ORDER OF THEIR ELECTION.

Major-Gen. Sir THOMAS M. BRISBANE, Bart., G.C.B., &c., F.R.S. Lond.,
PRESIDENT.

Date of Election.

Sir William Miller, Baronet, Lord Glenlee.
The above Gentleman is the only surviving member of the Edinburgh Philosophical Society.

THE FOLLOWING MEMBERS WERE REGULARLY ELECTED.

1787 James Home, M.D. *Professor of the Practice of Physic.*
1788 Thomas Charles Hope, M.D., F.R.S.Lond. *Professor of Chemistry.*
Right Honourable Charles Hope, *Lord President of the Court of Session.*
1798 Alexander Monro, M.D. *Professor of Anatomy, &c.*
1799 Sir George Stuart Mackenzie, Baronet, F.R.S. Lond.
Robert Jameson, Esq. *Professor of Natural History.*
1802 Colonel D. Robertson Macdonald.
1804 William Wallace, LL.D. *Emeritus Professor of Mathematics.*
1805 Thomas Thomson, M.D., F.R.S. Lond. *Professor of Chemistry, Glasgow.*
1806 Robert Ferguson, Esq. *of Raith*, F.R.S.Lond.
George Dunbar, Esq. *Professor of Greek.*
1807 John Campbell, Esq. *of Carbrook.*
Thomas Thomson, Esq. *Advocate.*
1808 James Wardrop, Esq. *Surgeon Extraordinary to his Majesty.*
Sir David Brewster, K.H., LL.D., F.R.S.Lond.
1811 Sir Charles Bell, K.H., F.R.S.Lond. *Professor of Surgery.*
David Ritchie, D.D. *Emeritus Professor of Logic.*
Major-General Sir Thomas Makdougal Brisbane, Bt., G.C.B., G.C.H., F.R.S.Lond.
John Thomson, M.D. *Professor of General Pathology, Edinburgh.*
James Jardine, Esq. *Civil Engineer.*
Captain Basil Hall, R.N., F.R.S.Lond.
J. G. Children, Esq. F.R.S.Lond.
Alexander Gillespie, Esq. *Surgeon, Edinburgh.*
W. A. Cadell, Esq. F.R.S. Lond.
Macvey Napier, Esq. F.R.S. Lond. *Professor of Conveyancing.*
James Pillans, Esq. *Professor of Humanity.*

Date of Election.	
1812	Sir George Clerk, Bart. F.R.S. Lond.
	Daniel Ellis, Esq. *Edinburgh.*
1813	William Somerville, M.D., F.R.S. *London.*
	J. Henry Davidson, M.D. *Edinburgh.*
1814	Sir Henry Jardine, *King's Remembrancer in Exchequer.*
	Patrick Neill, LL. D. *Secretary to the Wernerian and Horticultural Societies.*
	Right Honourable Lord Viscount Arbuthnot.
	John Fleming, D. D., *Professor of Natural Philosophy, King's Coll., Aberdeen.*
	Alexander Brunton, D. D. *Professor of Oriental Languages.*
	Professor George Glennie, *Marischal College, Aberdeen.*
1815	Robert Stevenson, Esq. *Civil Engineer.*
	Sir Thomas Dick Lauder, Baronet, *of Fountainhall.*
	Henry Home Drummond, Esq. *of Blair-Drummond.*
	Sir Charles Granville Stuart Menteath, Bart. *of Closeburn.*
	William Thomas Brande, Esq. F. R. S. Lond., and *Professor of Chemistry in the Royal Institution.*
1816	Colonel Thomas Colby, F. R. S. Lond. *Royal Engineers.*
	Leonard Horner, Esq. F. R. S. Lond.
	Henry Colebrooke, Esq. *Director of the Asiatic Society of Great Britain.*
	George Cooke, D. D. *Professor of Moral Philosophy, St Andrews.*
	Honourable Lord Fullerton.
	Sir John Robison, K. H. *Edinburgh.*
	Hugh Murray, Esq. *Edinburgh.*
1817	Right Honourable Earl of Wemyss and March.
	John Wilson, Esq. *Professor of Moral Philosophy.*
	Hon. Lord Meadowbank.
	Sir James Hamilton Dickson, M.D. *Clifton.*
	William P. Alison, M.D. *Professor of the Theory of Physic.*
	Robert Bald, Esq. *Civil Engineer.*
1818	Robert Richardson, M.D. *Harrowgate.*
	Patrick Miller, M.D. *Exeter.*
	John Craig, Esq. *Edinburgh.*
	John Watson, M.D.
	John Hope, Esq. *Dean of Faculty.*
	William Ferguson, M.D. *Windsor.*
1819	His Grace the Duke of Argyll.
	Patrick Murray, Esq. *of Simprim.*
	James Muttlebury, M.D. *Bath.*
	Thomas Stewart Traill, M.D. *Professor of Medical Jurisprudence.*
	Alexander J. Adie, Esq. *Optician, Edinburgh.*
	William Couper, M.D. *Glasgow.*
	Marshall Hall, M.D. *Nottingham.*
	John Borthwick, Esq. *Advocate.*
	Richard Phillips, Esq. F.R.S. Lond.

Date of Election.	
1819	Reverend William Scoresby, *Exeter.*
	George Forbes, Esq. *Edinburgh.*
1820	James Hunter, Esq. *of Thurston.*
	Right Honourable David Boyle, *Lord Justice-Clerk.*
	James Keith, Esq. *Surgeon, Edinburgh.*
	James Nairne, W.S. *Edinburgh.*
	Charles Babbage, Esq. F.R.S. Lond.
	Thomas Guthrie Wright, Esq. *Auditor of the Court of Session.*
	Sir John F. W. Herschel, Bart., F.R.S. Lond.
	Adam Anderson, A.M., LL.D. *Prof. Nat. Phil. St Andrews.*
	John Shank More, Esq. *Advocate.*
	George Augustus Borthwick, M.D. *Edinburgh.*
	Samuel Hibbert Ware, M.D.
	Robert Haldane, D.D. *Principal of St Mary's College, St Andrews.*
	Sir John Meade, M.D. *Weymouth.*
	Dr William Macdonald *of Ballyshear.*
	Sir John Hall, Baronet, *of Dunglass.*
	Sir George Ballingall, M.D. *Professor of Military Surgery.*
1821	Robert Graham, M.D. *Professor of Botany.*
	Sir James M. Riddell, Baronet, *of Ardnamurchan.*
	Archibald Bell, Esq. *Advocate.*
	John Clerk Maxwell, Esq. *Advocate.*
	John Lizars, Esq. *Surgeon.*
	John Cay, Esq. *Advocate.*
	Robert Kaye Greville, LL.D. *Edinburgh.*
	Robert Hamilton, M.D. *Edinburgh.*
	Sir Archibald Campbell, Baronet, *of Garscube.*
	Sir David Milne, K.C.B.
	A. R. Carson, Esq. LL.D. *Rector of the High School.*
1822	Sir Francis Chantrey, F.R.S. Lond.
	James Smith, Esq. *of Jordanhill*, F.R.S. Lond.
	William Bonar, Esq. *Edinburgh.*
	Captain J. D. Boswall, R.N. *of Wardie.*
	George A. Walker-Arnott, Esq. *Advocate.*
	Very Reverend John Lee, D.D. *Principal of the University.*
	Sir James South, F.R.S. Lond.
	Lieutenant-General Martin White, *Edinburgh.*
	Walter Frederick Campbell, Esq. *of Shawfield*, M.P.
	W. C. Trevelyan, Esq. *Wallington.*
	Sir Robert Abercromby, Baronet, *of Birkenbog.*
	Dr Wallich, *Calcutta.*
	The Right Honourable Sir George Warrender, Baronet, *of Lochend.*
	John Russell, Esq. W.S. *Edinburgh.*
	John Dewar, Esq. *Advocate.*

Date of Election.

1823 Sir Edward Ffrench Bromhead, Baronet, A.M., F.R.S. Lond. *Thurlsby Hall.*
Captain Thomas David Stuart, *of the Hon. East India Company's Service.*
Andrew Fyfe, M.D. *Lecturer on Chemistry, Edinburgh.*
Robert Bell, Esq. *Advocate, Procurator for the Church of Scotland.*
Captain Norwich Duff, R.N.
Warren Hastings Anderson, Esq.
Alexander Thomson, Esq. *of Banchory, Advocate.*
Liscombe John Curtis, Esq. *Ingsdon House, Devonshire.*
Robert Knox, M.D. *Lecturer on Anatomy, Edinburgh.*
Robert Christison, M.D. *Professor of Materia Medica.*
John Gordon, Esq. *of Cairnbulg.*
1824 Dr Lawson Whalley, *Lancaster.*
William Bell, Esq. W.S. *Edinburgh.*
Alexander Wilson Philip, M.D. *London.*
Sir Charles Adam, R.N.
Robert E. Grant, M.D. *Professor of Comparative Anatomy Univ. Coll. London.*
Claud Russell, Esq. *Accountant, Edinburgh.*
Rev. Dr William Muir, *one of the Ministers of Edinburgh.*
W. H. Playfair, Esq. *Architect, Edinburgh.*
John Argyle Robertson, Esq. *Surgeon, Edinburgh.*
James Pillans, Esq. *Edinburgh.*
James Walker, Esq. *Civil Engineer.*
Sir William Newbigging, *Surgeon, Edinburgh.*
William Wood, Esq. *Surgeon, Edinburgh.*
1825 The Venerable Archdeacon John Williams, *Rector of the Edinburgh Academy.*
W. Preston Lauder, M.D. *London.*
Right Honourable Lord Ruthven.
Dr Reid Clanny, *Sunderland.*
Sir William Jardine, Baronet, *of Applegarth.*
Alexander Wood, Esq. *Advocate.*
1826 Sir George Macpherson Grant, Baronet, *of Ballindalloch.*
William Renny, Esq. W.S. *Solicitor of Stamps.*
Rev. George Coventry.
Sir David Hunter Blair, Bart.
John Stark, Esq. *Edinburgh.*
Dr Macwhirter, *Edinburgh.*
1827 John Gardiner Kinnear, Esq. *Edinburgh.*
William Burn, Esq. *Edinburgh.*
James Russell, M.D. *Edinburgh.*
Henry Thornton Maire Witham, Esq. *of Lartington.*
Rev. Dr Robert Gordon, *one of the Ministers of Edinburgh.*
James Wilson, Esq. *Edinburgh.*
Rev. Edward Bannerman Ramsay, A.B. *of St John's College, Cambridge.*
Right Rev. Bishop James Walker, D.D. *Edinburgh.*

Date of Election.

1827 George Swinton, Esq. *Edinburgh.*

1828 Sir Francis Walker Drummond, Baronet, *of Hawthornden.*
Erskine Douglas Sandford, Esq. *Advocate.*
David Maclagan, M.D. *Edinburgh.*
Sir William Maxwell, Bart.
John Forster, Esq. *Architect, Liverpool.*
Thomas Graham, A.M. *Professor of Chemistry, London University.*
Thomas Hamilton, Esq. *Edinburgh.*
David Milne, Esq. *Advocate.*
Dr Manson, *Nottingham.*
William Burn Callender, Esq.

1829 A. Colyar, Esq.
William Gibson-Craig, Esq. *Advocate.*
James Ewing, LL.D. *Glasgow.*
Sir Charles Ferguson, Bart. *Advocate.*
Duncan Macneill, Esq. *Sheriff-depute of Perthshire.*
Rev. John Sinclair, A.M. *Pembroke College, Oxford.*
Arthur Connell, Esq. *Professor of Chemistry, St Andrews.*
James Hope Vere, Esq. *of Craigiehall.*
Bindon Blood, M.R.I.A.
James Walker, Esq. W.S.
William Bald, Esq. M.R.I.A.

1830 J. T. Gibson-Craig, Esq. W.S.
Archibald Alison, Esq. *Advocate, Sheriff-depute of Lanarkshire.*
Hon. Mountstuart Elphinstone.
James Syme, Esq. *Professor of Clinical Surgery.*
Thomas Brown, Esq. *of Langfine.*
James L'Amy, Esq. *Advocate, Sheriff-depute of Forfarshire.*
Thomas Barnes, M.D. *Carlisle.*

1831 James D. Forbes, Esq. F.R.S. Lond. *Professor of Natural Philosophy.*
Right Honourable Lord Dunfermline.
John Abercrombie, M.D. *Edinburgh, First Physician to her Majesty in Scotland.*
Donald Smith, Esq.
Captain Sir Samuel Brown, R.N.
O. Tyndal Bruce, Esq. *of Falkland.*
David Boswell Reid, M.D. *London.*
T. S. Davies, Esq. A.M. *Woolwich.*

1832 John Sligo, Esq. *of Carmyle.*
James Dunlop, Esq. *Astronomer, New South Wales.*
James F. W. Johnston, A.M. *Professor of Chemistry in the University of Durham.*
William Gregory, M.D. *Professor of Medicine, King's College, Aberdeen.*
Robert Allan, Esq. *Advocate.*
Robert Morrieson, Esq. *Hon. E.I.C. Civil Service.*
Montgomery Robertson, M.D.

Date of Election.

1832 William Dyce, Esq. A.M.
1833 Captain Milne, R.N.
Alexander Earle Monteith, Esq. *Advocate.*
His Grace the Duke of Buccleuch.
A. T. J. Gwynne, Esq.
David Craigie, M.D. *Edinburgh.*
George Buchanan, Esq. *Civil-Engineer.*
Sir John Stuart Forbes, Baronet, *of Pitsligo.*
Alexander Hamilton, Esq. LL.B., W.S.
Right Honourable Lord Greenock.
1834 Mungo Ponton, Esq. W.S.
Isaac Wilson, M.D., F.R.S. Lond.
David Low, Esq. *Professor of Agriculture.*
Thomas Henderson, Esq. *Professor of Astronomy.*
Rev. Dr Chalmers, *Professor of Divinity.*
Alexander Kinnear, Esq.
Patrick Boyle Mure Macredie, Esq. *Advocate.*
John Davie Morries Stirling, Esq.
Thomas Jameson Torrie, Esq.
William Copland, Esq. *of Colliston.*
John Steuart Newbigging, Esq. W.S.
Rev. Dr Welsh, *Professor of Church History.*
John Haldane, Esq. *Haddington.*
1835 John Hutton Balfour, M.D.
William Sharpey, M.D. *Professor of Anatomy, University College, London.*
Sir John Campbell, M.P. *Attorney-General.*
William Brown, Esq. *Surgeon, Edinburgh.*
Thomas Edington, F.G.S.
Reverend Edward Craig.
R. Mayne, Esq.
1836 William Paul, Esq. *Accountant.*
Robert Paul, Esq. *Secretary to Commercial Bank.*
David Rhind, Esq. *Architect.*
James Anderson, Esq. *Civil-Engineer.*
Martin Barry, M.D.
Archibald Robertson, M.D., F.R.S. Lond.
J. Macpherson Grant, Esq. *younger of Ballindalloch.*
Alexander Gibson Carmichael, Esq.
Rev. J. P. Nichol, *Professor of Practical Astronomy, University of Glasgow.*
1837 John Archibald Campbell, Esq. W.S.
John Scott Russell, Esq. A.M.
Charles Maclaren, Esq,
A. Smith, Esq. B.A., F.T.C. *Cambridge.*
Richard Parnell, M.D.

Date of Election.	
1837	Peter D. Handyside, M. D.
	John Clark, M. D., K. H.
1838	William Nicol, Esq.
	William Scot, Esq., *H. E. I. C. Service.*
	Thomas Mansfield, Esq. *Accountant.*
	Alan Stevenson, Esq. *Civil Engineer.*
1839	James Auchinleck Cheyne, Esq. *of Kilmaron.*
	David Smith, Esq. W. S.
	Adam Hunter, M. D.
	Rev. Philip Kelland, A. M. *Professor of Mathematics.*
	Henry Marshall, Esq. *Dep. Insp. Gen. of Army Hospitals.*
	William Ferguson, Esq. *Professor of Surgery, King's College, London.*
	William Alexander, Esq. W. S.
	F. Brown Douglas, Esq. *Advocate.*
	Lieutenant-Colonel Swinburne.
1840	Alan A. Maconochie, Esq. *Advocate.*
	Martyn J. Roberts, Esq.
	Robert Daun, M. D. *Dep. Inspector Gen. of Army Hospitals.*
	Robert Chambers, Esq.
	James Forsyth, Esq.
	Sir John MacNeill, G. C. B.
	John Cockburn, Esq.
	Sir William Scott, Bart.
	The Rev. C. H. Terrot.
	The Rev. R. Traill, D. D.
	Robert Bryson, Esq.
	Edward J. Jackson, Esq.
	John Shedden Patrick, Esq.
	John Learmonth, Esq.
	G. A. Stuart, Esq.
	Right Hon. T. B. Macaulay, M. P.
	Gilbert Laurie Finlay, Esq.
	John Mackenzie, Esq.
	John Thomson, Esq.

LIST OF NON-RESIDENT AND FOREIGN MEMBERS.

ELECTED UNDER THE OLD LAWS.

NON-RESIDENT.

Right Honourable Lord Wallace.
Charles Hatchett, Esq. F. R. S. Lond.
Thomas Blizzard, Esq.
Sir James Macgrigor, Baronet, M. D.
Richard Griffiths, Esq. *Civil Engineer.*

FOREIGN.

Dr S. L. Mitchell, *New York.*
M. P. Prevost, *Geneva.*

LIST OF HONORARY FELLOWS.

His Majesty the King of the Belgians.
His Royal Highness the Duke of Sussex.
His Imperial Highness the Archduke John of Austria.
His Royal Highness the Archduke Maximilian.

BRITISH SUBJECTS (LIMITED TO TWENTY BY LAW X.)

Robert Brown, Esq. F. R. S.
Sir John F. W. Herschel, Bart. F.R.S.
Dr Dalton, F.R.S.
Dr Faraday, F.R.S.
James Ivory, Esq. K.H., F.R.S.
G. B. Airy, Esq. F.R.S. *Astronomer-Royal.*
Sir W. R. Hamilton, M.A., M.R.I.A. *Astronomer-Royal, Ireland.*

THE FOLLOWING EIGHT NAMES WERE INCLUDED WITH THE AROVE PRIOR TO THE CHANGE IN THE LAW, 18TH JANUARY 1836.

Baron Humboldt,	*Berlin.*
M. Gay Lussac.	
M. Biot,	*Paris.*
M. Arago,	*Do.*
Chevalier Hammer.	
M. Berzelius,	*Stockholm.*

FOREIGNERS (LIMITED TO THIRTY-SIX.)

	M. Brochant,	*Paris.*
	Le Baron Von Buch,	*Berlin.*
	M. Gauss,	*Göttingen.*
	M. Simond de Sismondi,	*Geneva.*
5	Le Baron Degerando,	*Paris.*
	Le Baron Krusenstern,	*St Petersburgh.*

	M. Oersted,	*Copenhagen.*
	M. Schumacher,	*Altona.*
	Le Baron Larrey,	*Paris.*
10	Sir Henry Bernstein,	*Berlin.*
	M. De Candolle,	*Geneva.*
	Bishop Munter,	*Zealand.*
	Baron Charles Dupin,	*Paris.*
	M. Brongniart,	*Do.*
15	Chevalier Bürg,	*Vienna.*
	M. Bessel,	*Konigsberg.*
	M. Thenard,	*Paris.*
	M. Haidinger,	*Vienna.*
	M. Mitscherlich,	*Berlin.*
20	M. G. Rose.	*Berlin.*
	M. Hausmann,	*Göttingen.*
	J. J. Audubon, Esq.	*United States.*
	Chevalier Bouvard,	*Paris.*
	M. L. A. Necker,	*Geneva.*
25	M. Agassiz,	*Neuchatel.*
	Le Baron Cousin,	*Paris.*
	M. Plana.	*Turin.*
	M. Quetelet,	*Brussels.*
	M. Struve,	*Dorpat.*
30	M. Dulong,	*Paris.*
	Professor Tiedemann,	*Heidelberg.*

LIST OF FELLOWS DECEASED, RESIGNED, AND CANCELLED.

FROM 1837 TO 1840.

HONORARY FELLOWS.

Le Baron Poisson, *Paris.*
Le Baron de Prony, *Paris.*
M. Blumenbach, *Göttingen.*
N. Bowdich, LL. D. *United States.*
Dr Olbers, *Bremen.*
M. G. Moll, *Utrecht.*
M. Dulong, *Paris.*
Professor Müller, *Göttingen.*
Sir William Blizzard, Bart. M. D.
Sir William Ouseley, Bart.
M. Chevallier, *Paris.*
Rev. Bishop Gleig, *Stirling.*
M. Mohs, *Vienna.*
Davies Gilbert, Esq.

ORDINARY FELLOWS DECEASED OR RESIGNED.

Honourable Baron Hume.
Rev. Archibald Alison, LL. B. *Edinburgh.*
The Very Reverend Dr George Husband Baird, *Principal of the University.*
The Honourable Baron Sir Patrick Murray, Bart.
Rev. Dr John Jamieson.
Rev. John Thomson, *Duddingston.*
Sir James Montgomery, Bart. *of Stanhope.*
Rev. Andrew Stewart, M.D.
Right Honourable William Adam, *Lord Chief Commissioner.*
Thomas Jackson, LL.D. *Prof. of Nat. Phil. St Andrews.*
Harry William Carter, M.D. *Oxford.*
Lieutenant-General Sir Joseph Straton, K. C.B.
Right Hon. Sir Samuel Shepherd.
Robert Dundas, Esq. *of Arniston.*

Sir John Hay, Bart. *of Smithfield and Hayston.*
Dr William Dyce, *Aberdeen.*
John Shaw Stewart, Esq. *Advocate.*
Alexander Hamilton, M.D.
William Cadell, Esq. *of Cockenzie.*
Sir Andrew Halliday, M.D.
James Hamilton, M.D. *Professor of Midwifery.*
Edward Turner, M.D. *Professor of Chemistry, London.*
Sir John Archibald Stewart, Bart. *of Grandtully.*
Andrew Clephane, Esq. *Advocate.*
Alexander Copland Hutchison, Esq. *Surgeon, London.*
Sir Whitelaw Ainslie, M.D.
George Meikle, Esq. *Surgeon Hon. E.I.C. Service.*
Thomas Balfour, Esq.
John Mackean, Esq. *Accountant.*

RESIGNATIONS.

James Skene, Esq. *of Rubislaw.*
John Colquhoun, Esq. *Advocate.*
Rev. H. Parr Hamilton.
Right Honourable Sir William Rae, Bart.
Rev. Robert Muirhead, *Northumberland.*
Major Sir E. Leith Hay *of Rannes.*
Rev. Dionysius Lardner.
Elias Cathcart, Esq. *Advocate.*
George Moir, Esq. *Advocate.*
Prideaux John Selby, Esq.
Colonel Pitman, *Hon. E. I. C. Service.*
Major Alexander Anderson.
John Dunlop, Esq. *Advocate.*
Lieutenant-Colonel Sir W. Keith Murray, Bart. *of Ochtertyre.*
Charles Forbes, Esq.
John Stewart Wood, Esq.

ELECTIONS CANCELLED.

Lieutenant-Colonel M. Stewart.
George Joseph Bell, Esq.
Thomas Shortt, M.D.
William MacGillivray, Esq.
Edward Sang, Esq.

LIST OF DONATIONS.

(Continued from Vol. XIII. p. 590.)

December 5. 1836.

DONATIONS.	DONORS.
Bulletin de la Société Géologique de France. Tome vi. Feuilles 21–23, et Tome vii. 3–16.	The Society.
Chemical Tables; exhibiting the present state of our knowledge in regard to the Chemical and Physical Properties of Simple and Compound Bodies. By James F. W. Johnston, A.M., F.R.S.E.	The Author.
Transactions of the American Philosophical Society, held at Philadelphia, for promoting Useful Knowledge. (New Series.) Vol. v. part. 2.	The Society.
Memorie della Reale Accademia della Scienze di Torino. Tome xxxviii.	The Society.
Memoires présentés par divers Savans à l'Académie Royale des Sciences de l'Institut de France. Tome vi.	Royal Academy.
Notices of Communications to the British Association for the Advancement of Science, at Dublin, in August 1835.	British Association.
Philosophical Transactions of the Royal Society of London. 1835, Part 2; and 1836, Part 1.	Royal Society.
Proceedings of the Royal Society. Nos. 19. to 25.	Ditto.
Abhandlungen der Königlichen Akademie der Wissenschaften zu Berlin. Aus dem Jahre 1832.	The Academy.
Beschreibung und Abbildung von 24 Arten kurzschwanzigen Krabben. Von Dr Eduard Ruppell.	The Author.
Flora Batava. Nos. 104, 105, 106, and 107.	King of Holland.
Nouvelles Annales du Muséum d'Histoire Naturelle de France. Tome iv. Livr. 4.	The Editors.
Some Account of Halley's Astronomiæ Cometicæ Synopsis, which contains his investigation of the Orbits of Comets. By Professor Rigaud.	The Author.
Mémoire sur les Courants de la Manche, de la Mer d'Allemagne, et du Canal de Saint George. Par P. Monnier, Ingénieur Hydrographe de la Marine.	Ditto.
Gregorii Barhebræi Scholia in Psalmum quintum et decimum octavum, e Codicis Bibliothecæ Bodleianæ Apographo Bernsteniano. Edita a J. T. G. H. Rhode.	The Editor.
Georgii Gulielmi Kirschii Chrestomathia Syriaca cum Lexico denuo edidit G. H. Bernstein, Theologiæ Philosophiæ et Literarum Humaniorum Doctor in Univer. Liter. Vratislav. Professor. Parts 1, 2.	Ditto.
Discours sur quelques Progrès des Sciences Mathématiques en France, depuis 1830. Par le Baron Charles Dupin.	The Author.
Recherches relative à l'influence du Prix des Grains sur la Population Francaise. Par le Baron Charles Dupin, President de l'Académie des Sciences.	Ditto.
Notice sur le Baron de Prony, Pair de France.	Ditto.
Journal of the Bahama Society for the Diffusion of Knowledge. Nos. 11. to 14.	The Society.

DONATIONS.	DONORS.
The Quarterly Journal of Agriculture; and the Prize Essays and Transactions of the Highland and Agricultural Society of Scotland. Nos. 34. and 35.	Highland and Agricultural Society.
Journal of the Asiatic Society. January to December 1835.	The Society.
Do. do. January, February, March, April, and May 1836.	Ditto.
Journal of the Royal Asiatic Society of Great Britain and Ireland. No. 5. for March 1836.	Ditto.
An Essay on the Primitive Universal Standard of Weights and Measures. By Captain Thomas Best Jervis, Bombay Engineers.	The Author.
Arsberattelser om Vetenskapernas Framsteg, afgifne af Kongl. Vetenskaps Academiens Embetsman, d. 31 Mars 1834.	Academy of Sweden.
Kongl. Vetenskaps-Academiens Handlingar, for ar 1834.	Ditto.
Address of Earl Stanhope, President of the Medico-Botanical Society, for the Anniversary Meeting, January 16. 1836.	The Society.
Memorias da Academia R. das Sciencias de Lisboa. Tome xi. parte 2.	The Academy.
Bulletin de la Société de Géographie. 20 Tomes.	The Society.
Do. do. do. (2de Serie) Tomes iii. iv. v.	Ditto.
A Catalogue of 7385 Stars, chiefly in the Southern Hemisphere, prepared from Observations made in the years 1822, 1823, 1824, 1825, and 1826, at the Observatory at Paramatta, New South Wales, founded by Lieutenant-General Sir T. M. Brisbane, K.C.B. The Computations made, and the Catalogue constructed, by Mr William Richardson, of the Royal Observatory at Greenwich.	The Admiralty.
Abhandlungen der Königlichen Akademie der Wissenschaften zu Berlin. Aus dem Jahre 1834.	The Academy.
A Treatise on Isometrical Drawing. By T. Sopwith, Esq., Land and Mine Surveyor.	The Author.
Analyse d'une partie du Traité sur la Chaleur de M. Poisson. Par A. de la Rive.	Ditto.
Descriptive and Illustrated Catalogue of the Physiological series of Comparative Anatomy contained in the Museum of the Royal College of Surgeons in London. Vol. iii. part. 2.	Royal College of Surgeons.
Contribution to a Natural and Economical History of the Cocoa-Nut Tree. By Henry Marshall, Deputy Inspector-General of Army Hospitals.	The Author.
Verhandelingen van Het Bataafsch Genootschap der Proefordervindelijke Wijsbegeerte te Rotterdam. 12 vols.	The Society.
Nieuwse Verhandelingen, &c. 8 vols.	Ditto.
Bulletin de l'Academie Royale des Sciences et Belles Lettres de Bruxelles. 1836. Nos. 2, 3, 4, 5, 6, 7.	The Academy.
Neue Wirbelthiere zu der Fauna von Abyssinien gehörig, entdeckt und beschrieben, von Dr Eduard Ruppell. Lieferungen 5 and 6.	The Author.
Six Miscellaneous Pamphlets by Monsieur Virlet, Secretary of the Geological Society of France.	Ditto.
Eight Miscellaneous Pamphlets, by M. J. Girardin, Professor of Chemistry at Rouen.	Ditto.
Proceedings of the Geological Society of London. 1836. Nos. 45 and 46.	The Society.
Bridgewater Treatise on Geology and Mineralogy, considered with reference to Natural Theology. By the Rev. William Buckland, D.D. 2 vols.	The Author.
American Journal of Science and Arts. Conducted by Benjamin Silliman, M.D., LL.D. For April.	The Editor.

DONATIONS.	DONORS.
Annalen der Physik und Chemie, Herausgegeben zu Berlin. Von J. C. Poggendorf. 1836. Nos. 3, 4, 5, 6.	The Editor.
Astronomische Nachrichten. Nos. 295 to 311.	Prof. Schumacher.
Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences de Paris. 1835.	The Academy.
Do do. do. 1836. Nos. 1–26. Second Semestre, Nos. 1–16.	Ditto.
Report to a Committee of the Commissioners of the Northern Lighthouses, appointed to take into consideration the subject of illuminating Lighthouses by means of Lenses, on the new Dioptric Light of the Isle of May. By Alan Stevenson, M.A.	The Author.
The Articles America, Greece, and Physical Geography (from the Encyclopædia Britannica). By Charles Maclaren, Esq.	Ditto.
Catalogue Raisonné; or Classified Arrangement of the Books in the Library of the Medical Society of Edinburgh.	The Society.
A Treatise on Naval Tactics; by P. Paul Hoste. Translated by Captain J. D. Boswall, R.N., F.R.S.E.	Captain Boswall.
Mémoires de l'Académie Impériale des Sciences de Saint Petersbourg. (Sciences Politiques, &c.) Tome iii. livrs. 2 and 3; and Tome iv. liv. 1.	Imperial Academy.
Do. do. (Sciences Mathématiques, &c.) Tome i. livr. 3.	Ditto.
Do. do. (Sciences Naturelles.) Tome ii. livrs. 1, 2.	Ditto.
Do. do. (Mémoires preséntés par divers Savans.) Tome iii. livrs. 1, 2.	Ditto.
Recueil des Actes de la Séance Publique de l'Académie Impériale des Sciences de Saint Petersbourg, tenue le 29. Decembre 1835.	Ditto.
Annalium Societatis Eruditæ Hungaricœ, Volumen Secundum.	Hungarian Literary Society.
Maps of the Ordnance Survey of Great Britain. Published by the Board of Ordnance. Nos. 51 and 60.	Board of Ordnance.
Twenty Charts, forming part of the Pilote Français.	Marine Depot of France.
December 19.	
History of the Extinct Volcanoes of the Basin of Neuweid, on the Lower Rhine. By Samuel Hibbert, M.D., F.R.S.E.	The Author.
Proceedings of the Berwickshire Naturalists' Club, No. 4.	The Club.
Description Sommaire des Phare et Fanaux allumés sur les Côtes de France, au 1er Sept. 1836.	The Academy.
Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences de Paris (2d Semestre 1836), Nos. 17, 18, 19, 20, 21, 22, 23, 24.	Ditto.
Flora Batava, No. 108.	King of Holland.
January 2. 1837.	
The American Almanac and Repository of Useful Knowledge for the year 1837.	American Philosophical Society.
The Nervous System of the Human Body; as explained in a series of Papers read before the Royal Society of London. By Sir Charles Bell, K.G.H., F.R.SS.L. & E.	The Author.
Tijdschrift voor Natuurlijke Geschiedenis en Physiologie door J. Van der Hoeven, M.D., en W. H. De Vriese, M.D. Vol. iii. part 1.	The Editors.
The Article Mammalia, or a Treatise on Quadrupeds (from the Encyclopædia Britannica). By James Wilson, Esq. F.R.S.E.	The Author.
Report by a Committee of the Royal Society regarding the New Dioptric Light of the Isle of May.	Professor Forbes.

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January 16.	
Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences de Paris (2d Semestre 1836), Nos. 25, 26.	The Academy.
On the Unity of Structure in the Animal Kingdom. By Martin Barry, M.D., F.R.S.E.	The Author.
Transactions of the Institution of Civil Engineers. Vol. i.	The Institution.
Nouveaux Mémoires de la Société Impériale des Naturalistes de Moscow. Tome iv.	Imperial Society.
Bulletin de la Société Impériale des Naturalistes des Moscow. Tome ix.	Ditto.
Safety Apparatus for Steam Boilers. By A. D. Bache, Professor of Natural Philosophy and Chemistry, University of Pennsylvania.	The Author.
Historical Notes. By A. D. Bache, Esq. &c.	Ditto.
Experimental Illustrations of the Radiating and Absorbing Powers of Surfaces for Heat, &c. By A. D. Bache, Esq.	Ditto.
Experiments on the alleged Influence of Colour on Radiation. By A. D. Bache, Esq.	Ditto.
Replies to a Circular in relation to the occurrence of an unusual Meteoric Display on the 13th November 1834, addressed by the Secretary of War to the Military Posts of the United States. By A. D. Bache, Esq.	Ditto.
Notes and Diagrams illustrative of the New Brunswick Tornado. By A. D. Bache, Esq.	Ditto.
On the Magnetic Dip at various places in the United States. By A. D. Bache, Esq.	Ditto.
On the relative Horizontal Intensities of Terrestrial Magnetism at several places in the United States. By A. D. Bache, Esq.	Ditto.
Report on the Geological Survey of the State of New Jersey. By Henry D. Rogers, Professor of Geology and Mineralogy, &c.	Ditto.
Report of the Managers of the Franklin Institute of the State of Pennsylvania, for the promotion of Mechanic Arts in relation to Weights and Measures.	Franklin Institution.
Report of the Committee of the Franklin Institute of Pennsylvania, on the Explosion of Steam Boilers.	Ditto.
General Report on the Explosion of Steam Boilers, by a Committee of the Franklin Institute of Pennsylvania.	Ditto.
Report of the Geological Reconnaissance of the State of Virginia, made under the appointment of the Board of Public Works. By William B. Rogers, Professor of Natural Philosophy.	The Author.
February 6.	
Bulletin de la Société Géologique de France. Tome vii. Feuilles 17–19.	The Society.
Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences de Paris (1er Semestre 1837). Nos. 1, 2, 3. 4.	The Academy.
The American Journal of Science and Arts. Conducted by Benjamin Silliman, M.D., LL.D. Vol. xxxi. No. 1. for October 1836.	The Editor.
Carte de la Côte Septentrionale d'Afrique entre Alger et les Iles Zafarines.	Le Ministre de la Marine.
Plan de la Baie de Coquimbo—Plan de la rade d'Iquique.	Ditto.
Carte des Côtes de France, partie comprise entre le cap Fréhel et Caucale.	Ditto.
Carte des Côtes de France, (anse de Vauville, cap de la Hague), &c.	Ditto.
Carte des Côtes de France, (partie comprise le fort de Querqueville et le fort La Hague).	Ditto.

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Carte des Côtes de France, (partie comprise entre la pointe de Barfleur et Grandchamp), &c.	Le Ministre de la Marine.
Plan de Barfleur et de ses environs.	Ditto.
Plan de la rade de la Hongue.	Ditto.
Plan particulier de Mouillage d'Alger.	Ditto.
February 20.	
Asiatic Researches; or Transactions of the Society instituted in Bengal for inquiring into the History, the Antiquities, the Arts and Sciences, and Literature of Asia. Vol. xx. part 1.	The Society.
The Journal of the Asiatic Society of Bengal for June and July 1836.	Ditto.
Mémoires de la Société de Physique et d'Histoire Naturelle de Genève. Tome vii. partie 2.	Ditto.
Recherches sur la Cause de l'Electricité Voltaique. Par M. le Professeur Auguste De la Rive.	The Author.
Report on the New Standard Scale of the Royal Astronomical Society. By Francis Baily, Esq. F. R. S. &c.	Ditto.
Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences de France, (1er Semestre 1837). Nos. 5 and 6.	The Academy.
Brief Outlines illustrative of the Alterations in the House of Commons, in reference to the Acoustic and Ventilating Arrangements. By D. B. Reid, M.D., F.R.S.E.	The Author.
March 6.	
The Journal of the Asiatic Society of Bengal for August 1836.	The Society.
Transactions of the Zoological Society of London. Vol. ii. part 1.	Ditto.
Statuti dell' Accademia di Palermo.	The Academy.
De Redigendis ad unicam seriem comparabilem Meteorologicis ubique factis observationibus conventio proposita, et Tabulæ supputatæ, ab equite Nicolao Cacciatore, Regii Observatorii Panormitani Directore.	The Author.
Considerations regarding the Edinburgh, Leith, and Newhaven Railway. By Patrick Neill, LL.D., F. R. S. E.	Ditto.
Specimen of a Treatise on the Differential Calculus or Fluxions; founded on an original principle derived from the Ancient Geometry. By the Rev. John Forbes, D.D., Minister of St Paul's, Glasgow.	Ditto.
The Quarterly Journal of Agriculture; and the Prize Essays and Transactions of the Highland and Agricultural Society of Scotland. No. 36, March 1837.	The Society.
Transactions of the Society instituted at London for the Encouragement of Arts, Manufactures, and Commerce. Vol. ii. part 1.	Ditto.
Two Essays on the Geography of Ancient Asia; intended partly to illustrate the Campaigns of Alexander, and the Anabasis of Xenophon. By the Rev. John Williams, Vicar of Lampeter, and Rector of the Edinburgh Academy.	The Author.
March 20.	
Mémoire sur l'Instruction secondaire dans le Royaume de Prusse. Par M. V. Cousin, Directeur de l'Ecole Normale.	Ditto.
The Journal of the Royal Asiatic Society of Great Britain and Ireland. No. 2, (November 1834).	The Society.
Transactions of the Royal Asiatic Society of Great Britain and Ireland. Vol. ii. part 1.	Ditto.
Transactions of the Horticultural Society of London (Second Series). Vol. i., and Vol. ii. parts 1, 2.	Ditto.

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Table des Positions Géographiques des Principaux Lieux du Globe. Par M. Daussy.	The Author.
Sur l'Influence de la Pression Atmosphérique sur le Niveau moyen de la Mer. Par M. Daussy.	Ditto.
Novorum Actorum Academiæ Cæsareæ Leopoldino-Carolinæ Naturæ Curiosorum Voluminis Septimi Decimi Supplementum.	The Academy.
Transactions of the Institute of British Architects of London, Session 1835–6. Vol. i. part 1.	The Institute.
The Anatomy, Particular and Surgical, of the Human Body, illustrated by a series of Engravings; by A. Fyfe, F.R.C.S.E. Corrected and arranged, and with an Explanatory Letterpress, by P. D. Handyside, M.D., &c. Part 1.	Dr Handyside.
An Elementary Introduction to Mineralogy; comprising a Notice of the Characters and Elements of Minerals. By William Phillips, F.L.S., M.G.S.L. and C. Considerably augmented by Robert Allan, F.R.S.E., M.G.S.L., &c.	Robert Allan, Esq.
April 3.	
Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences de Paris (1837, 1er Semestre), Nos. 7, 8, 9, 10, 11.	The Academy.
Annalen der Physik und Chemie. Herausgegeben zu Berlin, von J. C. Poggendorff. 1836. Nos. 10, 11, 12.	The Editor.
On the Arenarius of Archimedes. By S. P. Rigaud, M.A., Savilian Professor of Astronomy.	The Author.
A Catalogue of the Collection of British Quadrupeds in the Museum of the Cambridge Philosophical Society.	The Society.
Transactions of the Cambridge Philosophical Society. Vol. vi. part 1.	Ditto.
Supplement to the Account of the Rev. John Flamsteed, the first Astronomer-Royal. By Francis Baily, Esq. F.R.S., &c.	The Author.
On the Theory of the Moon, and on the Perturbations of the Planets. By J. W. Lubbock, Esq. F.R.R.A. and L.S.S.	Ditto.
Report of the Fifth Meeting of the British Association for the Advancement of Science, held at Dublin in 1835.	British Association.
Discussion of the Magnetical Observations made by Captain Back, R.N., during his late Arctic Expedition. By S. Hunter Christie, Esq., M.A., F.R.S., &c.	The Author.
List of the Fellows of the Royal Society (1836).	The Society.
Addresses delivered at the Anniversary Meetings of the Royal Society on Saturday, November 30. 1833, and on Wednesday, November 30. 1836, by His Royal Highness the Duke of Sussex, K.G., &c. &c. &c., the President.	Ditto.
Proceedings of the Royal Society. Nos. 19 to 27.	Ditto.
Philosophical Transactions of the Royal Society of London for the year 1836. Part 2.	Ditto.
Astronomical Observations made at the Royal Observatory at Greenwich, 1834, Parts 4 and 5, and 1835, Parts 1, 2, 3, 4, 5, under the direction of John Pond, Esq. Astronomer-Royal.	Ditto.
April 17.	
Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences de Paris (1837, 1er Semestre), Nos. 12 and 13.	The Academy.

DONATIONS.	DONORS.
Bulletin de l'Académie Royale des Sciences et Belles Lettres de Bruxelles. Années 1832, 1834, 1835, Nos. 4, 5, 6, 7; and 1836, Nos. 8 and 12.	Royal Academy.
Essai Historique sur la Vie et la Doctrine d'Ammonius-Saccas, chef d'une des plus célèbres Ecoles Philosophiques de l'Alexandrie. Par L. J. Dehaut.	Ditto.
Mémoire sur les Propriétés et l'Analyse de la Phloridzine. Par L. de Koninck.	Ditto
Annuaire de l'Observatoire de Bruxelles pour l'an 1837, par le Directeur A. Quetelet.	Ditto.
Annuaire de l'Académie Royale des Sciences et Belles Lettres de Bruxelles.	Ditto.
Sur la Latitude de l'Observatoire de Bruxelles. Par A. Quetelet, Directeur de cet Etablissement, &c.	The Author.
Bulletin de la Société de Géographie (Deuxieme Serie). Tome vi.	The Society.
Astronomische Nachrichten. Nos. 312–22.	Prof. Schumacher.
Ueber die Kessels'schen Chronometer von Hansen.	The Author.
Astronomische Beobachtungen auf der Königlichen Universitäts Sternwarte in Königsberg. Von F. W. Bessel.	Ditto.
Arsberättelser om Vetenskapernas Framsteg, afgifne af Kongl. Vetenskaps-Academiens Embetsmän, d' 31 Mars 1835.	The Academy.
Kongl. Vetenskaps-Academiens Handlingar, för är 1835.	Ditto.
Account of some Experiments made in different parts of Europe on Terrestrial Magnetic Intensity. By James D. Forbes, Esq. F.R.SS.L. & E., &c.	The Author.

December 4.

Bulletin de la Société Géologique de France. Tome vii. Feuilles 20–23; and Tome viii. Feuilles 1–20.	La Société.
Mémoires de la Société Géologique de France. Tome ii. parts 1, 2.	Ditto.
Flora Batava, Nos. 80, 95, 96, 97, 98, 99, 109, and 111.	King of Holland.
On the Results of Experiments made on the Weight, Height, and Strength, of above 800 Individuals. By Professor Forbes.	The Author.
On the Muscular Effort required to ascend Planes of different Inclinations. By Professor Forbes.	Ditto.
Note relative to the supposed Origin of the deficient Rays in the Solar Spectrum; being an account of an Experiment made at Edinburgh during the Annular Eclipse of 15th May 1836. By Professor Forbes.	Ditto.
On the Temperatures and Geological Relations of certain Hot Springs, particularly those of the Pyrenees; and on the Verification of Thermometers. By Professor Forbes.	Ditto.
Biographical Sketch of the late Edward Turner, M.D., Professor of Chemistry in University College, London. By Robert Christison, M.D., Professor of Materia Medica in the University of Edinburgh, &c.	Ditto.
The American Journal of Science and Arts. Conducted by Benjamin Silliman, M.D., LL.D. For October 1833, January 1835, and January and July 1837.	The Editor.
The Journal of the Asiatic Society of Bengal for November and December 1836, January, February, April, and May, 1837.	The Society.
First Part of the Nineteenth Volume of Asiatic Researches, or Transactions of the Society instituted in Bengal for inquiring into the History, the Antiquities, the Arts and Sciences, and Literature of Asia.	Ditto.
The Anatomy, Particular and Surgical, of the Human Body, illustrated by a series of Engravings; by A. Fyfe, F.R.C.S.E. Corrected and arranged, and with explanatory letterpress, by P. D. Handyside, M.D., F.R.S.E. Part 2.	Dr Handyside.

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The Quarterly Journal of Agriculture; and the Prize Essays and Transactions of the Highland and Agricultural Society of Scotland. Nos. 37, 38, and 39.	Highland and Agricultural Society.
Investigation of the Equation to Fresnel's Wave Surface. By Archibald Smith, Esq. Trinity College, Cambridge.	The Author.
Account of the Andersonian Museum, Glasgow.	James Smith, Esq. of Jordanhill.
Memoirs, chiefly Anatomical and Physiological. By Robert Knox, M.D., F.R.S.E.	The Author.
Astronomische Nachrichten, Nos. 323 to 336.	Prof. Schumacher.
The Fourth Annual Report of the Royal Cornwall Polytechnic Society, 1836.	The Society.
Nieuwe Verhandelingen der Eerste Klasse van het Koninklijk-Nederlandsche Institut van Wetenschappen, Letterkunde en Schoone Kunsten, te Amsterdam. Vols. 1, 2, 3, 4, 5.	The Institute.
Naturkuundige Verhandelingen van de Hollandsche Maatschappij der Wetenschappen te Haarlem. Deels 13 to 23.	Dutch Society of Scien. at Haarlem.
Histoire des Maladies observées à la Grande Armée Francaise, pendant les Campagnes de Russie en 1812, et d'Allemagne en 1813. Par le Chevalier J. R. L. De Kerckhove dit De Kirckhoff.	The Author.
Transactions of the Philosophical and Literary Society of Leeds. Vol. i. part 1.	The Society.
Annalen der Physik und Chemie. Herausgegeben zu Berlin, von J. C. Poggendorff. 1836, Nos. 2, 7, 8, 9. 1837, Nos. 1, 2, 3, 4, 5.	The Editor.
Natuur-en Scheikundig Archief, uitgegeven door G. J. Mulder en W. Wenckebach. 1836. Stuk 4.	The Editors.
Tijdschrift voor Natuurlijke Gescheidenis en Physiologie. Uitgegeven door J. Van Der Hoeven, M.D., en W. H. De Vriese, M.D. Vol. iii. part 3.	Ditto.
Transactions of the American Philosophical Society held at Philadelphia for promoting Useful Knowledge. Vol. v., and Vol. vi. part 1; and of the New Series, Vol. i., and Vol. iii. parts 1 and 3.	The Society.
Recueil de Voyages et de Mémoires, publié par la Société de Geographie. Tome v.	Ditto.
Neue Wirbelthiere zu der Fauna von Abyssinien gehorig, entdeckt und beschrieben von Dr Eduard Rüppell. Lieferungs 7, 8, 9.	The Author.
The Ancient Kalendars and Inventories of the Treasury of his Majesty's Exchequer, together with other documents illustrating the History of that Repository. Collected and edited by Sir Francis Palgrave, K.H. 3 vols.	Commissioners on the Public Records.
Proceedings and Ordinances of the Privy Council of England. Edited by Sir Harris Nicolas. Vols. vi. and vii.	Ditto.
Excerpta è Rotulis Finium in Turri Londinensi asservatis, Henrico Tertio Rege. A. D. 1246–1272, curâ Caroli Robert. Vol. ii.	Ditto.
Nova Acta Physico-Medica Academiæ Cæsareæ Leopoldino-Carolinæ Naturæ Curiosorum. Vol. xvi., et Vol. xvii. part 2.	The Academy.
Uebersicht der Säugthiere, Vogel, Amphibien und Fische Schlesiens von Dr C. L. Gloger.	The Author.
Das Abandern der Vogel durch Einfluss des Klima's von Dr C. L. Gloger.	Ditto.
Disquisitionum de Avibus ab Aristotele commemoratis Specimen I. Scripsit C. L. Gloger.	Ditto.
Bulletin de la Société Impériale des Naturalistes de Moscow. 1837, Nos. 1, 2, 3, 4.	The Society.
Library Catalogue and Regulations of the Telford Premiums of the Institution of Civil Engineers.	The Institution.

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Charter and Bye-Laws of the Institute of British Architects of London.	The Institute.
Transactions of the Statistical Society of London. Vol. i. part 1.	The Society.
Notice of two Roman Inscriptions relative to the Conquest of Britain by the Emperor Claudius Cæsar. By John Hogg, Esq. A.M., Fellow of St Peter's College, Cambridge.	The Author.
Etudes des Gîtes Houillers et Métallifères du Bocage Vendéen faite en 1834 et 1835. Par Henri Fournel, Ingenieur des Mines.	M. le Duc de Cases.
Atlas au Même.	Ditto.
Memorias da Academia R. das Sciencias de Lisboa. Tomo xii. parta 1.	The Society.
Elements of Chemistry; by the late Edward Turner, M.D. Sixth Edition. Enlarged and revised by Professor Liebig and Wilton G. Turner. Part 1.	The Editors.
The Madras Journal of Literature and Science for October 1836 and January 1837.	Madras Literary Society.
Observations upon a "Report by the Select Committee on Salmon Fisheries, Scotland: together with the Minutes of Evidence, Appendix and Index." 30th June 1836. By Robert Knox, F.R.S.E.	The Author.
Proceedings of the Geological Society of London. Nos. 48, 49, 50, 51.	The Society.
Stellarum Duplicium et Multiplicium Mensuræ Micrometricæ per magnum Fraunhoferi Tubum annis a 1824 ad 1837 in Specula Dorpatensi institutæ, adjecta est Synopsis observationum de Stellis compositis Dorpati annis 1814 ad 1824 per minora instrumenta perfectarum, Auctore F. G. W. Struve.	The Author.
Mesures Micrométriques obtenues à l'Observatoire de Dorpat avec la Grande Lunette de Fraunhofer de 1824 à 1837. Par F. G. G. Struve.	Ditto.
Mémoires de l'Académie Impériale des Sciences de Saint Petersbourg. (Sciences Mathématiques, &c.) Tome i. livr. 4.	L'Academie Imperiale.
Do. do. (Sciences Naturelles.) Tome ii. livr. 3.	Ditto.
Do. do. (Sciences Politiques, &c.) Tome iii. livr. 6, and Tome iv. livr. 2.	Ditto.
Recueil des Actes de la Séance publique de l'Académie Impériale des Sciences de Saint Petersbourg, tenue le 30 Decembre 1836.	Ditto.
Systematic Treatise on Zoology. By Professor Jarotski of Warsaw. 5 vols.	Prof. Johnston of Durham.
Observations on the Influence of Climate on Health and Mortality. By Arthur Saunder Thomson, M.D.	The Author.
Principal Documents relating to the Survey of the Coast of the United States; and the construction of uniform Standards of Weights and Measures for the Custom-Houses and States. By F. R. Hassler. 3 parts.	Ditto.
Commentatio de Definienda Quantitate Vaporis Aquei in Atmosphæra vel Aëre quocunque. Auctore A. C. G. Suerman.	Dr Moll.
Dissertatio Physica Inauguralis de Calore Fluidorum Elasticorum specifico. Auctore A. C. G. Suerman.	Ditto.
Specimen Inaugurale de Fractionibus Continuis. Auctore P. O. C. Vorsselman De Heer.	Professor Moll.
Memoirs on the Nervous System. By Marshall Hall, M.D., F.R.S.L. & E.	The Author.
Recherches Expérimentales sur les Propriétés et les Fonctions du Systeme Nerveux dans les Animaux Vertébrés. Par P. Flourens.	Ditto.
Expériences sur le Systeme Nerveux. Par P. Flourens.	Ditto.
Nouvelles Expériences sur le Systeme Nerveux. Par P. Flourens.	Ditto.
Eloge Historique de G. Cuvier. Par M. Flourens.	Ditto.

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Eloge Historique de Jean Antoine Chaptal. Par M. Flourens.	The Author.
Eloges Historiques de R. Louiche Desfontaines et de J. Jul. De Labillardiere. Par M. Flourens.	Ditto.
Report of the Sixth Meeting of the British Association for the Advancement of Science, held at Bristol in August 1836.	British Association.
The American Almanac and Repository of Useful Knowledge for the Year 1838.	American Philosophical Society.
Report by a Committee of the Society of Arts in Scotland, on the best Alphabet and Method of Printing for the use of the Blind, 1837.	Society of Arts for Scotland.
On the Elements of the Orbit of Halley's Comet at its appearance in the years 1835 and 1836. By Lieutenant W. S. Stratford, R.N.	The Author.
Taylor's Calendar of the Meetings of the Scientific Bodies of London for 1837–8.	Lieut. W. S. Stratford, R. N.
Bulletin de la Société d'Encouragement pour l'Industrie Nationale. Pour Oct., Nov., Dec. 1835, et Jan. au Decembre 1836.	La Société.
A Dissertation on the Causes and Effects of Disease considered in reference to the Moral Constitution of Man. By Henry Clark Barlow, M.D.	The Author.
Constitution and Regulations of the Glasgow and Clydesdale Statistical Society, instituted April 1836.	The Society.
Maps of the Ordnance Survey of Great Britain, No. 59.	Board of Ordnance.
December 18.	
Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences de Paris, 1837, 2me Semestre. Nos. 20, 21, 22.	The Academy.
Journal of the Asiatic Society of Bengal for March 1837.	The Society.
Scientific Memoirs, selected from the Transactions of Foreign Academies of Science and Learned Societies, and from Foreign Journals. Edited by Richard Taylor, F.S.A., &c. Vol. i.	The Editor.
A Synopsis of Chronology from the era of Creation, according to the Septuagint, to the year 1837. By William Cuninghame, Esq.	The Author.
January 1. 1838.	
Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences de Paris. 1837, 2me Semestre. Nos. 23 and 24.	The Academy.
Essays on Unexplained Phenomena. By Graham Hutchison.	The Author.
Observations Météorologiques et Magnétiques faites dans l'étendue de l'Empire de Russie. Redigées et publiées par A. T. Kupffer. Tome i. No. 1.	Le Ministre des Finances de Russie.
January 15.	
Natuur-en Scheikundig Archief, uitgegeven door G. J. Mulder en W. Wenckebach. 1837. St. 1, 2.	The Editors.
Tijdschrift voor Natuurlijke Gescheidenis en Physiologie. Uitgegeven door J. Van der Hoeven, M.D., en W. H. De Vriese, M.D. Vol. iii. part 4.	Ditto.
Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences de Paris. 1837, 2me Semestre. Nos. 25, 26.	The Academy.
Account of a Case of Hermaphrodism. By P. D. Handyside, M. D.	The Author.
Account of a remarkable Case of Suicide, with Observations on the fatal issue of the rapid introduction of Air in large quantity into the Circulation during Surgical Operations. By P. D. Handyside, M.D., F.R.S.E.	Ditto.

DONATIONS.	DONORS.
February 5.	
Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences (1837, 2me Semestre), No. 26, et (1838, 1er Semestre), Nos. 1, 2.	The Academy.
Bulletin de la Société de Géographie, 2me Serie, Tome vii.	The Society.
Bulletin de la Société Géologique de France. Tome viii. Feuilles 21–25.	Ditto.
Journal of the Asiatic Society of Bengal for June 1837.	Ditto.
February 19.	
Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences de Paris. 1838, 1er Semestre. Nos. 3, 4, 5.	The Academy.
Bulletin des Séances de l'Académie Royale de Bruxelles (1837), Nos. 1, 2, 3, 4, 5, 6, 7, 8.	Ditto.
Nouveaux Mémoires de l'Académie Royale des Sciences et Belles Lettres de Bruxelles. Tome x.	Ditto.
Mémoires Couronnés par l'Académie Royale des Sciences et Belles Lettres de Bruxelles. Tome xi.	Ditto.
Annales de l'Observatoire de Bruxelles, publiés, aux frais de l'etat, par le Directeur A. Quetelet. Tome i. partie 2.	The Author.
Mémoires sur Trois Intégrales Définies, par M. J. Plana, Directeur de l'Observatoire de Turin.	Ditto.
Transactions of the Cambridge Philosophical Society. Vol. vi. part 2.	The Society.
Bulletin de la Société de Géographie, 2me Serie, Tome viii.	Ditto.
March 5.	
Elements of Chemistry; by the late Edward Turner, M.D. Sixth edition, enlarged and revised by Professor Liebig and Wilton G. Turner. Part 11.	The Editors.
Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences. No. 6, 1er Semestre 1838.	The Academy.
Flora Batava. Nos. 112 and 113.	King of Holland.
On the Nature and Treatment of the Diseases of the Heart; with some views on the Physiology of the Circulation. By James Wardrop, M.D., Surgeon to his late Majesty George IV. &c. &c.	The Author.
Notice sur les Marbres; par M. Theodore Virlet.	Ditto.
The Quarterly Journal of Agriculture, and the Prize Essays and Transactions of the Highland and Agricultural Society of Scotland. No. 40, March 1838.	The Society.
Bulletin de la Société Geologique de France. Tome ix. Feuilles 1–5.	Ditto.
Recherches Historiques et Statistiques sur la Population de Génève. Par Edouard Mallet.	The Author.
Researches into the Cause of Voltaic Electricity. By Mons. Auguste de la Rive.	Ditto.
De l'Influence qu'exerce la Chaleur sur la Facilité que le Courant Electrique possede a passer d'une liquide dans un Metal. Par M. la Professeur A. de la Rive.	Ditto.
Proceedings of the Royal Society. Nos. 28, 29, 30.	Royal Society.
Address of his Royal Highness the Duke of Sussex, K.G., the President, read at the Anniversary Meeting of the Royal Society on November 30. 1837.	Ditto.
Address to her Majesty, referred to in the Address of his Royal Highness the President of the Royal Society.	Ditto.
Philosophical Transactions of the Royal Society of London, for the year 1837. Parts 1 and 2.	Ditto.

DONATIONS.	DONORS.
List of the Fellows of the Royal Society for the year 1837.	Royal Society.
Astronomical Observations made at the Royal Observatory, Greenwich, in the year 1836, under the direction of George Biddell Airy, Esq. M.A., Astronomer-Royal.	Ditto.
The Cambridge Mathematical Journal. Nos. 1 and 2.	The Editor.
Malacologia Monensis. A Catalogue of the Mollusca inhabiting the Isle of Man and the neighbouring Sea. By Edward Forbes, President of the Royal Physical Society of Edinburgh, &c. &c.	The Author.

March 19.

Annals of Natural History, or Magazine of Zoology, Botany, and Geology. Conducted by Sir William Jardine, Bart.; P. J. Selby, Esq.; Sir W. J. Hooker; and Richard Taylor, Esq. (No. 1, New Series).	The Editors.
Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences de Paris. 1er Semestre 1838. Nos. 7, 8, 9.	The Academy.
Proceedings of the Geological Society of London. Nos. 52, 53.	The Society.
Transactions of the Geological Society of London. (Second Series.) Vol. v. part 1.	Ditto.
Theory of Heat. By Philip Kelland, M.A., Fellow and Tutor of Queens' College, Cambridge.	The Author.
Hortus Mauritianus, ou Enumeration des Plantes Exotiques et Indigenes, qui croissent à l'Ile Maurice, disposées d'apres la Methode Naturelle. Par W. Bojer.	Lord Glenelg, H.M. Sec. of State for the Colonies.
Journal of the Asiatic Society of Bengal, for June and August 1837.	The Society.

April 2.

Journal of the Royal Asiatic Society of Great Britain and Ireland. No. 8.	Ditto.
Comptes Rendus Hebdomadaires des Séances de l'Academie des Sciences de Paris. 1838, 1er Semestre. No. 10.	The Academy.
Madras Journal of Literature and Science. Published under the auspices of the Madras Literary Society, and Auxiliary Royal Asiatic Society. Nos. 15, 16, and 17.	The Society.

April 9.

Report on the Physical Condition of the Assam Tea Plant, with reference to Geological Structure, Soils, and Climate. By John M'Clelland, Esq. Assistant-Surgeon, Bengal Establishment.	The Author.
Seventeenth Report of the Council of the Leeds Philosophical and Literary Society at the close of the Session 1836–7.	The Society.
Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences de Paris. 1838, 1er Semestre. Nos. 11, 12.	Ditto.
Proceedings of the Geological Society of London, Nos. 54 and 55.	Ditto.
Nieuwe Verhandelingen der Eerste Klasse van het Konninklijk-Nederlandsche Instituut van Wetenschappen, Letterkunde en Schoone Kunsten te Amsterdam. Vol. vi.	The Institute.
Abhandlungen der Königlichen Akademie der Wissenschaften zu Berlin. 1835.	The Academy.
Bericht über die zur Bekanntmachung geeigneten Verhandlungen der Königl. Preuss. Akademie der Wissenschaften zu Berlin vom Mai 1836 bis Juni 1837.	Ditto.
Bestimmung der Länge des einfachen Secundenpendels fur Berlin von F. W. Bessel.	The Author.

DONATIONS.	DONORS.
Astronomische Beobachtungen auf der Königlichen Universitäts-Sternwarte in Konigsberg von F. W. Bessel. 18th part.	The Author.
The Glasgow Mortality Bill for the year ending 31st December 1837. By Henry Paul, Accountant.	Ditto.
Ordnance Survey of the County of Londonderry. Colonel Colby, R.E., F.R.S. L. & E. &c., Superintendent. Vol. i.	Lord Lieutenant of Ireland.
Views of the Architecture of the Heavens, in a Series of Letters to a Lady. By J. P. Nichol, LL. D., F.R.S.E.	The Author.
April 16.	
Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences de Paris. 1838, 1er Semestre. No. 13.	The Academy.
A Systematic and Stratigraphical Catalogue of the Fossil Fish in the Cabinets of Lord Cole and Sir Philip Grey Egerton. By Sir Philip Grey Egerton, Bart.	The Author.
May 7.	
A Catalogue of Circumpolar Stars, deduced from the Observations of Stephen Groombridge, Esq. F.R.S. Edited by George Biddell Airy, A.M., Astronomer-Royal.	Lords Commissioners of the Admiralty.
Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences, 1838, 1er Semestre. Nos. 14, 15.	The Academy.
Memorias da Academia R. das Sciencias de Lisboa. Vol. xi. part 2, and Vol. xii. part 1.	Ditto.
Proceedings of the Royal Irish Academy for the year 1836–37. Part 1.	Ditto.
Transactions of the Royal Irish Academy. Vol. xvii. parts 2 and 3.	Ditto.
Journal of the Asiatic Society of Bengal for September and October 1837.	The Society.
Inquisitionum in Officio Rotulorum Cancellariæ Hiberniæ Asservatarum, Repertorium. Vols. i. ii.	Commissioners on the Public Records of the Kingdom.
Rotulorum Patentium et Clausorum Cancellariæ Hiberniæ Calendarium. Vol. i. part 1. Hen. II.—Hen. VII.	Ditto.
Rotuli Chartarum in Turri Londinensi Asservati. Accurante Thoma Duffus Hardy, S. S. A. è Soc. Int. Templ. Lond. Vol. i. part 1. Ab anno MCXCIX. ad annum MCCXVI.	Ditto.
Registrum vulgariter nuncupatum, "The Record of Caernarvon;" è Codice m.sto. Harleiano 696 Descriptum.	Ditto.
General Report to the King in Council from the Honourable Board of Commissioners on the Public Records, appointed by His Majesty King William IV., by a Commission dated the 12th March in the first year of his reign; with an Appendix and Index.	Ditto.
December 3.	
The Journal of the Royal Geographical Society of London. Vols. i. to vii., and Vol. viii. part 1.	The Society.
Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences 1838. 1er Semestre, Nos. 17 to 26. 2me Semestre, Nos. 1 to 20.	The Academy.
Annalen der Physik und Chemie. Herausgegeben zu Berlin, von J. C. Poggendorff. 1837, Nos. 10, 11, 12.	The Editor.
Researches on Heat. 3d Series. By James D. Forbes, F.R.SS. L. & E.	The Author.
Nova Acta Physico-Medica Academiæ Cæsareæ Leopoldino-Carolinæ Naturæ Curiosorum. Vol. xviii. part 1.	The Academy.

DONATIONS.	DONORS.
Statistique de la Grande-Bretagne et de l'Ireland, par Alex. Moreau de Jonnés. 2 tomes.	The Author.
The American Journal of Science and Arts. Conducted by Benjamin Silliman, M.D., LL.D. For January and April 1838.	The Editor.
Bulletin de la Société Géologique de France. Tome ix. Feuilles 6–19.	The Society.
Journal of the Statistical Society of London, for June, July, August, September, October, and November.	Ditto.
Mémoires de la Société de Physique et d'Histoire Naturelle de Geneve. Tome viii. part 1.	Ditto.
The Fifth Report of the Royal Cornwall Polytechnic Society, 1837.	Ditto.
A Dissertation on the Nature and Character of the Chinese System of Writing, in a Letter to John Vaughan, Esq. By Peter S. Du Ponceau, LL.D., President of the American Philosophical Society, &c.	American Philosophical Society.
Journal of the Asiatic Society of Bengal for December 1837 and February 1838.	The Society.
Proceedings of the Geological Society of London. No. 56.	Ditto.
Revue Zoologique par la Société Cuverienne. No. 1.	Ditto.
Astronomische Nachrichten. Nos. 337 to 348.	Prof. Schumacher.
Astronomical Observations made at the Royal Observatory, Edinburgh. By Thomas Henderson, F.R.S.E. and R.A.S., &c.	Royal Society of London.
Maps of the Ordnance Survey of Great Britain. Nos. 49, 50, 66, 67, and 68.	Board of Ordnance.
The Malacological and Conchological Magazine. Conducted by G. A. Sowerby, F.L.S., &c. &c. &c. Part 1.	The Editor.
Nieuwe Verhandelingen der Eerste Klasse van het Konninklijk-Nederlandsche Instituut van Wetenschappen, Letterkunde en Schoone Kunsten te Amsterdam. Deel 7, Stucks 1, 2, 3.	The Institute.
Elements of Chemistry, including the Applications of the Sciences to the Arts. By Thomas Graham, F.R.SS.L. & E., &c. Part 2.	The Author.
Fifteenth Report of the Whitby Literary and Philosophical Society, presented at the Annual Meeting, October 31. 1837.	The Society.
Statistical Report of the Sickness, Mortality, and Invaliding, among the Troops in the West Indies. By Captain Alexander Tulloch, and Henry Marshall, Esq.	The Authors.
Description Nautique des Côtes de l'Algerie, par M. A. Berard, Capitaine de Corvette; suivies de Notes par M. de Tessau Ingenieur-Hydrographe, 1837. 8vo, avec Carte.	Le Directeur General du Dépot des Cartes de la Marine.
Bulletin de la Société de Géographie. Deuxieme Serie. Tome ix.	The Society.
Transactions of the Society for the Encouragement of Arts, &c. Vol. ii. part 2.	Ditto.
Mémoire sur la Chaleur Solaire, sur les pouvoirs rayonnants et absorbants de l'air Atmospherique, et sur la Temperature de l'espace. Par M. Pouillet.	The Author.
Flora Batava. Nos. 114 and 115.	King of Holland.
Tijdschrift voor Natuurlijke Geschiedenis en Physiologie. Uitgegeven door J. Van Der Hoeven, M.D., en W.H. De Vriese, M.D. Deel iv. Stucks 3, 4.	The Editors.
Minutes of Proceedings of the Institution of Civil Engineers.	The Institution.
An Experimental Essay on the Physiology of the Blood. By Charles Maitland, M.D.	The Author.
Christiani Hugenii aliorumque seculi xvii. Virorum celebrium Exercitationes Mathematicæ et Philosophicæ ex Manuscriptis in Bibliotheca Academiæ Lugduno-Batavæ servatis. Edidit Petrus Joannes Uylenbroeck in eadem Academia Physices et Astronomiæ Prof. Extraord. Fascic. 1, 2.	The Academy.

DONATIONS.	DONORS.
Transactions of the American Philosophical Society held at Philadelphia for promoting Useful Knowledge. New Series. Vol. vi. part 1.	The Society.
Transactions of the Royal Irish Academy. Vol. xviii. part 1.	The Academy.
Proceedings of the Royal Irish Academy for the Year 1837–88. Part 2.	Ditto.
Second Report of the Commissioners appointed to consider and recommend a General System of Railways for Ireland, with Atlas, Plans, and Sections of the several Lines.	Irish Railway Commissioners.
On the present state of our knowledge in regard to Dimorphous Bodies. By J. F. W. Johnston, M.A.	The Author.
The Economy of a Coal-Field. By J. F. W. Johnston, M.A.	Ditto.
Bericht über die zur Bekanntmachung geeigneten Verhandlungen der Konigl. Preuss. Akademie der Wissenschaften zu Berlin vom Juli 1837 bis Juni 1838.	The Academy.
Abhandlungen der Koniglichen Akademie der Wissenschaften zu Berlin, 1836.	Ditto.
The Seventh Report of the British Association for the Advancement of Science, held at Liverpool in 1837.	The Association.
Eloge Historique d'Antoine Laurent de Jussieu, par M. Flourens.	The Author.
Catalogue of the Chinese Library of the Royal Asiatic Society.	The Society.
Journal of the Royal Asiatic Society of Great Britain and Ireland. No. 9, for August 1838.	Ditto.
Mémoires de la Société Geologique de France. Tome iii. part 1.	Ditto.
Eighteenth Report of the Council of the Leeds Philosophical and Literary Society, at the close of the Session 1837–38.	Ditto.
The American Almanac and Repository of Useful Knowledge for the Year 1839.	Ditto.
Proceedings of the American Philosophical Society, January to August 1838.	Ditto.
Recueil d'Observations Magnétiques faites à St Petersbourg et sur d'autres points de l'Empire de Russie. Par A. T. Kupffer.	The Author.
Bulletin de la Société Impériale des Naturalistes de Moscow. 1837, Nos. 5, 6, 7, 8; and 1838, Nos. 1, 2, 3.	The Society.
Recherches sur le Mouvement et l'Anatomie du Stylidium Graminifolium. Par Ch. Morren, Professeur de Botanique à l'Universitie de Liege.	The Author.
Ten other Botanical Tracts.	The same Author.
On a New Correction in the Construction of the Double Achromatic Object Glass. By Richard Potter, Esq.	The Author.
Sketch of the Geology of Exeter and the Neighbourhood. By T. Shapter, M.D.	Ditto.
Documents and Records illustrating the History of Scotland, and the Transactions between the Crowns of Scotland and England, preserved in the Treasury of Her Majesty's Exchequer. Collected and edited by Sir Francis Palgrave, K.H. Vol. i.	Commissioners on the Public Records.
Annalium Societatis Eruditæ Hungaricæ Volumen Tertium.	The Society.
Memoirs of the Royal Academy of Stockholm for 1836.	The Academy.
Annual Reports on the Progress of the Sciences, presented to the Royal Academy of Stockholm in 1836.	Ditto.
Maps of the Ordnance Survey of Ireland.	Lord Lieutenant.
The Quarterly Journal of Agriculture; and the Prize Essays and Transactions of the Highland and Agricultural Society of Scotland. For June, September, and December 1838.	The Society.

December 17.

Mémoires Couronnés par l'Académie Royale des Sciences et Belles Lettres de Bruxelles. Tome xii. and xiii.	The Academy.

DONATIONS.	DONORS.
Bulletin des Séances de l'Académie Royale des Sciences et Belles Lettres de Bruxelles. 1837, Nos. 10, 11, 12. 1838, Nos. 1, 2, 3, 4, 5, 6, 7, 8.	The Academy.
Annuaire de l'Académie Royale des Sciences et Belles Lettres de Bruxelles pour l'an 1838.	Ditto.
Annuaire de l'Observatoire de Bruxelles pour l'an 1838, par le Directeur A. Quetelet.	The Author.
De l'Influence des Saisons sur la Mortalité aux differens Ages dans la Belgique, par A. Quetelet.	Ditto.
Résumé des Observations Météorologiques faites en 1837 à l'Observatoire de Bruxelles, et communiquées par A. Quetelet.	Ditto.
Recherches sur les Propriétés des Courants Magnéto-Electriques, par M. le Prof. Aug. de la Rive.	Ditto.
Examen Critique d'un Mémoire de M. P. Leroux, intitulé du Bonheur, par L. A. Gruyer.	Ditto.
Discurso lido em 15 de Maio de 1838 na Sessas publica da Academia Real das Sciencias de Lisboa, par Joaquim José Da Costa de Macedo.	The Academy.
Astronomische Nachrichten, Nos. 349 to 354.	Prof. Schumacher.
Mémoires de l'Académie Impériale des Sciences de St Petersbourg. (Sciences Mathématiques, &c.) Tome i. livrs. 5, 6 ; et Tome ii. livrs. 1, 2.	The Academy.
Do. do. (Sciences Naturelles.) Tome ii. livrs. 4, 5, 6.	Ditto.
Do. do. (Mémoires présentés par divers Savans.) Tome iii. livrs. 3, 4, 5, 6 ; et Tome iv. livrs. 1, 2.	Ditto.
Recueil des Actes de la Séance Publique de l'Académie Impériale des Sciences de St Petersbourg tenue le 29 Décembre 1837.	Ditto.
Comptes Rendus Hebdomadaires des Séances de l'Academie des Sciences de Paris. 1838, (2me Semestre.) Nos. 21, 22.	Ditto.
Natuur-en-Scheikundig Archief, uitgegeven door G. J. Mulder en W. Wenckebach. Jaargang 1837. 3d Stuk.	The Editors.

January 7. 1839.

Eulogy on Nathaniel Bowdich, LL.D., President of the American Academy of Arts and Sciences. By John Pickering.	American Academy.
Eulogy on the Life and Character of Nathaniel Bowdich, LL.D., F.R.S. By Daniel Appleton White.	Ditto.
A Discourse on the Life and Character of the Hon. Nathaniel Bowdich, LL.D., F.R.S. By Alexander Young.	Ditto.
Journal of the Statistical Society of London for December 1838.	The Society.
Memoirs of the Wernerian Natural History Society for the Years 1831–37. Vol. vii.	Ditto.
The Journal of the Royal Geographical Society of London. Vol. viii. parts 2–3.	Ditto.
A Sketch of the Geology of Fife and the Lothians, including detailed Descriptions of Arthur's Seat and Pentland Hills. By Charles Maclaren, Esq. F.R.S.E.	The Author.
The American Journal of Science and the Arts. Conducted by Benjamin Silliman jun., A.B.	The Editor.
Memorie della Reale Accademia della Scienze di Torino. Tome xl.	The Academy.
Transactions of the Institution of Civil Engineers. Vol. ii.	The Institution.
Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences de Paris. (1838, 2me Semestre.) Nos. 23, 24, 25.	The Academy.
Journal of the Asiatic Society of Bengal. April, May, June, 1838.	The Society.

DONATIONS.	DONORS.
January 21.	
The Natural History of the Fishes of the Firth of Forth and Tributaries. By Richard Parnell, M. D., F. R. S. E.	The Author.
The Laws of Harmonious Colouring, adapted to Interior Decorations, Manufactures, and other useful purposes. By R. D. Hay, house-painter.	Ditto.
The Silurian System, founded on Geological Researches in the Counties of Salop, Hereford, Radnor, Montgomery, Caermarthen, Brecon, Pembroke, Monmouth, Gloucester, Worcester, and Stafford ; with Descriptions of the Coal-Fields and Overlying Formations; with a large separate Map. By Roderick Impey Murchison, F. R. S., F. L. S.	Ditto.
Comptes Rendus Hebdomadaires des Séances de l'Academie des Sciences (1838, 2d Semestre). Nos. 26, 27.	The Academy.
Flora Batava, No. 116.	King of Holland.
February 4.	
Tijdschrift voor Natuurlijke Gescheidenis en Physiologie. Uitgegeven door J. Van Der Hoeven, M. D., en W. H. De Vriese, M. D.	The Editors.
First and Second Annual Reports, Laws, and Transactions of the Royal Botanical Society of Edinburgh.	The Society.
Address of His Royal Highness the Duke of Sussex, K. G., &c. &c., President of the Royal Society, read at the Anniversary Meeting on Friday the 30th November 1838.	Ditto.
Proceedings of the Geological Society of London, 1838. No. 59.	Ditto.
Journal of the Statistical Society of London for January 1839.	Ditto.
Comptes Rendus Hebdomadaires des Séances de l'Academie des Sciences de Paris. 1839, 1er Semestre. Nos. 1, 2.	The Academy.
Natuur-en-Scheikundig Archief, uitgegeven door G. J. Mulder en W. Wenkebach. 1837. St. 4.	The Editors.
February 18.	
Journal of the Asiatic Society of Bengal, No. 49, for January 1836.	The Society.
Researches in Embryology. First Series. By Martin Barry, M. D., F.R.S.E.	The Author.
Elements of Chemistry. By the late Edward Turner, M.D. 6th edition. Revised by Justice Liebig, M.D., and W. G. Turner, Ph. D. Part 3.	The Editors.
Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences, 1839. 1er Semestre. Nos. 3 and 4.	The Academy.
Philosophical Transactions of the Royal Society of London for the year 1838. Part 1.	Royal Society.
Proceedings of the Royal Society of London, 1837–38. Nos. 31, 32, 33, 34, and 35.	Ditto.
Astronomical Observations made at the Royal Observatory, Greenwich, in the year 1837, under the direction of George Biddell Airy, Esq. M. A., Astronomer-Royal.	Ditto.
Transactions of the Cambridge Philosophical Society. Volume vi. Part 3.	The Society.
Astronomische Nachrichten, Nos. 355 to 364.	Prof. Schumacher.
Annual Report of the Institution of Civil Engineers. Session 1839.	The Institution.
Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences. 1839. 1er Semestre. Nos. 5, 6.	The Academy.
Journal of the Statistical Society of London, No. 10, for February.	The Society.
The Quarterly Journal of Agriculture ; and the Prize Essays and Transactions of the Highland and Agricultural Society of Scotland, No. 44, for March 1839.	Ditto.

DONATIONS.	DONORS.
A Geometrical Practical Treatise named Pantometria, Longimetra, Planimetra, and Stereometria; by Thomas Digges, Esq.	P. D. Handyside. M.D.
Abhandlungen der Königlichen Akademie der Wissenschaften zu Berlin. Aus dem Jahre, 1833.	The Academy.
Uber die Länderverwaltung unter dem Chalifate. Von Joseph von Hammer.	Royal Academy at Berlin.
Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences de Paris. 1839, 1er Semestre. Nos. 7, 8.	The Academy.
The American Journal of Arts and Science. Conducted by Benjamin Silliman, M.D., LL.D. Vol. xxxv. No. 2, for January 1839.	The Editor.
Journal of the Asiatic Society of Bengal for July 1838.	The Society.
Natuur-en-Scheikundig Archief, uitgegeven door G. J. Mulder en W. Wenckebach. 1838. St. 1, 2.	The Editors.
Tijdschrift voor Natuurlijke Geschiedenis en Physiologie. Uitgegeven door J. Van Der Hoeven, M.D., en W. H. De Vriese, M.D. Deel v. St. 3.	Ditto.
Recherches sur l'Histoire Naturelle et l'Anatomie des Limules. Par J. Van Der Hoeven.	The Author.
Tables of Logarithms; published by Taylor and Walton, booksellers, London.	The Publishers.
Journal of the Statistical Society of London for March 1839.	The Society.

April 1.

Memoirs of the Royal Astronomical Society. Vols. ix and x.	The Society.
Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences de Paris. 1839, 1er Semestre. No. 9.	The Academy.
Transactions of the Society for the Encouragement of Arts, Manufactures, and Commerce. Vol. lii. Part 1.	The Society.
Annuaire de l'Académie Royale des Sciences et Belles Lettres de Bruxelles, pour 1839.	The Academy.
Bulletin de l'Academie Royale des Sciences et Belles Lettres de Bruxelles. 1838. Nos. 9, 10, 11, 12.	Ditto.
Nouveaux Mémoires de l'Academie Royale des Sciences et Belles Lettres de Bruxelles. Tome xi.	Ditto.
Mémoires Couronnés par l'Académie Royale des Sciences et Belles Lettres de Bruxelles. Tome xiv. Premiere partie.	Ditto.
Annuaire de l'Observatoire de Bruxelles pour l'an 1839, par le Directeur A. Quetelet, Directeur de cet Etablissement.	The Author.
Résumé des Observations Météorologiques faites en 1838 à l'Observatoire de Bruxelles, par A. Quetelet.	Ditto.
Journal of the Asiatic Society of Bengal for August 1838.	The Society.
Astronomical Observations made at the Royal Observatory, Edinburgh. By Thomas Henderson, F.R.S.E. and R.A.S. Vol. ii. for the year 1836.	Royal Society of London.
Journal of the Royal Geographical Society. Vol. ix. Part 1.	The Society.
Geometrical Theorems and Analytical Formulæ, with their application to the Solution of certain Geodetical Problems. By William Wallace, LL.D. &c.	The Author.

April 15.

Bulletin de la Sociétié de Géographie. Deuxieme serie. Tome x.	The Society.
Comptes Rendus Hebdomadaires des Séances de l'Academie des Sciences de Paris. 1839, 1er Semestre. Nos. 10, 11, 12.	The Academy.
Report on the Geology of Cornwall, Devon, and West Somerset. By Henry T. De La Beche, F.R.S. &c.	Lords of the Treasury.

DONATIONS.	DONORS.
Flora Batava, No. 117.	King of Holland.
Journal of the Statistical Society of London, No. 12, for April.	The Society.
Herniarum Corporis Humani Tabulæ Anatomico-Pathologicæ ac Chirurgicæ qua edidere Professores Imperatoriæ Medico-Chirurgicæ Academiæ Petropolitanæ Christianus Salomon Petrus Savenko.	Imperial Medico-Chirurgical Acad. of St Petersburgh.
Eliæ Buialsky Tabulæ Anatomico-Chirurgicæ, Operationes Ligandarum Arteriarum Majorum exponentes.	Ditto.
December 2.	
Outlines of Human Physiology. By William Pulteney Alison, M.D., F.R.S.E.	The Author.
Two Essays. I. On the Nature of the Boetian Numerical Contractions. II. Notes on Early Calendars. By James Orchard Halliwell, Esq. of Jesus College, Cambridge.	Ditto.
Remarks on the Classification of the different branches of Human Knowledge. By J. W. Lubbock, Esq. F.R.S.	Ditto.
An Elementary Treatise on the Tides. By J. W. Lubbock, Esq. F.R.S.	Ditto.
Philosophical Transactions of the Royal Society of London for the year 1838. Part 2.	Royal Society.
Comptes Rendus Hebdomadaires des Séances de l'Académie Royale des Sciences, 1839. 1er Semestre, Nos. 13–25. 2d Semestre, Nos. 1–19.	The Academy.
Tijdschrift voor Natuurlijke Geschiedenis en Physiologie. Uitgegeven door J. Van Der Hoeven, en W. H. De Vriese, M.D. Deel vi. St. 1, 2, 3.	The Editors.
Proceedings of the Geological Society of London. Vol. iii. Nos. 60, 61, 62.	The Society.
Transactions of the Institution of Civil Engineers. Vol. iii. part 1.	The Institution.
Proceedings of the American Philosophical Society. Nos. 5, 6, 7.	The Society.
Bulletin de la Société de Géologique de France. Tome ix. Feuilles 28–32. Tome x. Feuilles 5–15.	Ditto.
Journal of the Asiatic Society of Bengal, for September, October, November, December 1838.	Ditto.
The Quarterly Journal of Agriculture, and the Prize Essays and Transactions of the Highland and Agricultural Society of Scotland, for June, September, and December 1839.	Ditto.
Nova Acta Physico-Medica Academiæ Cæsareæ Leopoldino-Carolinæ Naturæ Curiosorum. Vol. xviii. part 2.	The Academy.
Eloge Historique d'Antoine-Laurent de Jussieu. Par M. Flourens.	The Author.
Archives du Museum d'Histoire Naturelle, publiées par les Professeurs-Administrateurs de cet Etablissement. Tome i. livn. 1.	The Editors.
Recherches sur les Ossemens Fossiles de la Russie. Par G. Fischer de Waldheim.	The Author.
Memoir of William Vaughan, Esq.	Ditto.
Transactions of the American Philosophical Society, held at Philadelphia, for promoting Useful Knowledge. Vol. vi. part 2. (New Series.)	The Society.
Memoires de la Société Géologique de France. Tome iii. part 2.	Ditto.
Transactions of the Royal Society of Literature of the United Kingdom. Vol. i. part 1; and Vol. iii. parts 1, 2.	Ditto.
On the Enlisting, Discharging, and Pensioning of Soldiers, with the official documents on these branches of Military Duty. By Henry Marshall, F.R.S.E., Deputy Inspector-General of Army Hospitals.	The Author.
The Transactions of the Linnean Society of London. Vol. xvii. part 2.	The Society.
Proceedings of the Linnean Society of London. Nos. 1, 2.	Ditto.
Quarterly Journal of the Statistical Society of London. July and October.	Ditto.
Memoires sur le Canada, depuis 1749 jusqu'a 1760.	Lit. and Hist. Soc. of Quebec.

DONATIONS.	DONORS.
An Essay on the State of Literature and Learning under the Anglo-Saxons. By Thomas Wright, Esq.	Royal Society of Literature.
Madras Journal of Literature and Science. 1838, April to December. 1839, January to March.	Madras Literary Society.
Transactions of the Royal Irish Academy. Vol. xviii. part 2.	The Academy.
Memoir of the late Honourable Nathaniel Bowdich, LL.D., of Boston. By N. J. Bowdich.	The Author.
The Narrative of Captain David Woodward and Four Seamen, who lost their ship while in a boat at sea, and surrendered themselves up to the Malays in the Island of Celebes.	Wm. Vaughan, Esq.
Recueil de Voyages et de Mémoires, publié par la Société de Geographie. Tome iv.	The Society.
Annuaire des Marées de Côtes de France pour l'an 1839. Par A. M. R. Chazallon.	The Author.
Mémoire sur les divers moyens de se procurer une Base pour la mesure directe, par la vitesse du Son, par des Observations Astronomiques, &c. Par A. M. R. Chazallon.	Ditto.
Instructions pour Naviguer sur la Côte Orientale de Terre Neuve, depuis le Cap de Bonavista jusqu'au cap Normand. Par M. Ch. Lavand.	Ditto.
De l'Etablissement des Français dans la Régence d'Alger, et des Moyens d'en assurer la prospérité. Par M. P. Genty de Bussey. 2 tomes.	Ditto. Ditto.
Rara Mathematica; or a Collection of Treatises on the Mathematics and subjects connected with them, from ancient unedited manuscripts. Edited by James Orchard Halliwell, Esq. F.R.S., &c.	The Editor.
Mécanique Céleste. By the Marquis De La Place. Translated, with a Commentary, by Nathaniel Bowditch, LL. D., and with a Memoir of the Translator, by his Son. Vol. iv.	Representatives of the Translator.
Astronomische Nachrichten. Nos. 365 to 379.	Prof. Schumacher.
Astronomische Beobachtungen auf der Königlichen Universitäts-Sternwarte in Königsberg. Von F. W. Bessel. 9th part.	The Author.
Neue Wirbelthiere zu der Fauna von Abyssinien gehorig, entdecht und beschrieben von Dr Eduard Rüppell. Nos. 10, 11, 12.	Ditto.
Report upon the Military and Hydrographical Chart of the Extremity of Cape Cod, including the Townships of Province Town and Truro, with their Seacoasts and Ship Harbour, projected from Surveys executed under the direction of James D. Graham.	Prof. A. D. Bache.
Manuel Complet du Micrographie. Par Charles Chevalier.	The Author.
The Journal of the Royal Geographical Society of London. Vol. ix. part 2.	The Society.
Address at the Anniversary Meeting of the Royal Geographical Society, 27th May 1839. By William R. Hamilton, Esq., F.R.S., President.	Ditto.
Narrative of the Discoveries of Sir Charles Bell in the Nervous System. By Alex. Shaw, assistant-surgeon to the Middlesex Hospital.	The Author.
Report of the Eighth Meeting of the British Association for the advancement of Science, held at Newcastle in August 1838. Vol. vii.	British Association.
Memoir on the Mid-Lothian and East-Lothian Coal-Fields. By David Milne, Esq. F.R.S.E. and F.G.S.	The Author.
Natuur-en Scheikundig Archief, uitgegeven door G. J. Mulder en W. Wenckebach. 1838. St. 3.	The Editors.
Abhandlungen der Königlichen Akademie der Wissenschaften zu Berlin. Aus dem Jahre 1837.	The Academy.

DONATIONS.	DONORS.
Bericht uber die zur Bekanntmachung geeigneten Verhandlungen der Konigl. Preuss. Akademie der Wissenschaften zu Berlin. Monats Juli 1838 bis Juni 1839.	The Academy.
Remarkable Case of Extrophy of the Urinary Bladder, with Remarks. By P. D. Handyside, M.D.	The Author.
History of the Sternoptixinæ, a family of the Osseous Fishes, and their anatomical peculiarities, with a description of the Sternoptix Celibes, a species not hitherto noticed. By P. D. Handyside, M.D.	Ditto.
Elements of Chemistry, including the applications of the Science to the Arts. By Thomas Graham, F. R. S. L. & E. Part 1, 2, 3.	Ditto.
The Journal of the Royal Asiatic Society of Great Britain and Ireland. No. 10.	The Society.
Nineteenth Report of the Council of the Leeds Philosophical and Literary Society at the close of the Session 1838–9.	Ditto.
Annuaire Magnetique et Meteorologique du Corps des Ingenieurs des Mines de Russie. Par A. T. Kupfer.	Le Ministre des Finances.
The American Almanac and Repository of Useful Knowledge for the year 1840.	Amer. Phil. Soc.
Voyage dans la Russie Meridionale, sous la direction de M. A. Demidoff. Livraisons 1–23. Atlas au meme, folio.	Count Demidoff.
Pilote Français, Quatrieme Partie, comprenant les Côtes Septentrionales de France depuis l'Ile Brebat jusqu'à Barfleurs.	Le Ministre de la Marine.
Maps of the Ordnance Survey of England and Wales. Nos. 71 and 74.	Board of Ordnance.
Maps of the Ordnance Survey of Ireland (County Kildare), 45 sheets.	Ditto.

December 16.

Journal of the Asiatic Society of Bengal, No. 85, for January 1839.	The Society.
Transactions of the Meteorological Society. Vol. i.	Ditto.
Comptes Rendus Hebdomadaires des Séances de l'Academie des Sciences (1839, 2d Semestre). Nos. 20, 21, 22.	Ditto.

January 6. 1840.

The Sixth Annual Report of the Royal Cornwall Polytechnic Society, 1838.	Ditto.
Abstract of the Returns of the Overseers of the Poor in Massachusetts for 1837 and 1838, prepared by the Secretary of the Commonwealth.	Family of the late Dr Bowdich.
Abstract of the Returns of Insurance Companies, incorporated with Specific Capital in 1838.	Ditto.
Abstract exhibiting the condition of the Banks in Massachusetts, in February and October 1838.	Ditto.
Abstract of the Massachusetts School Returns for 1837.	Ditto.
First and Second Annual Reports of the Board of Education.	Ditto.
Report of the Secretary of the Board of Education on the subject of School-Houses.	Ditto.
Report of the Committee on Education relative to the School-Fund.	Ditto.
Report and Resolves in relation to the North-Eastern Boundary in March 1838.	Ditto.
Report of the Bank Commissioners.	Ditto.
Schedule exhibiting the condition of the Banks in Massachusetts for every year from 1803 to 1837.	Ditto.
Report of the Commissioners on Codification of the Common Law.	Ditto.
Preliminary Report of the Commissioners on Criminal Law.	Ditto.
Annual Reports of the Railroad Corporations in the State of Massachusetts for 1838.	Ditto.

DONATIONS.	DONORS.
First Report on the Geology of the Public Lands in the State of Maine.	Family of the late Dr Bowdich.
Report on a Re-examination of the Economical Geology of Massachusetts.	Ditto.
Statistical Tables exhibiting the condition and products of certain Branches of Industry in Massachusetts, for the year 1837.	Ditto.
Second Report of the Agriculture of Massachusetts.	Ditto.
Reports of the Commissioners of the Zoological Survey of the State of Massachusetts.	Ditto.
Reports on the Fishes, Reptiles, and Birds of Massachusetts.	Ditto.
Second Part of the twentieth volume of the Asiatic Researches.	The Society.
Journal of the Asiatic Society of Bengal, for February and March 1839.	Ditto.
Transits as observed, and calculation of the Apparent Right Ascension at the Cape of Good Hope, 1834.	Lords Commissioners of Admiralty.
Zenith Distances observed with the Mural Circle, at the Royal Observatory, Cape of Good Hope, and the Calculation of the Geocentric South Polar Distances, 1836 and 1837.	Ditto.
Bessel's Refraction Tables.	Ditto.
Observations of Halley's Comet, made at the Royal Observatory, Cape of Good Hope, in the years 1835 and 1836. By Thomas Maclear, Esq.	Ditto.
On the Declinations of the Principal Fixed Stars, deduced from Observations made at the Observatory, Cape of Good Hope, in 1832 and 1833. By Thomas Henderson, Esq.	Ditto.
Prospectus and Illustrations of the Natural History of the Scottish Salmonidæ. By Sir William Jardine, Bart.	The Author.
February 17.	
Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences, 1839. 2d Semestre, Nos. 23, 24, 25, 26, and 27. 1840. 1er Semestre, Nos. 1, 2, 3.	The Academy.
Quarterly Journal of the Statistical Society of London. Vol. ii. Part 6. January 1840.	The Society.
Bulletin de la Société de Geographie. 2me Series. Tome xi.	Ditto.
Flora Batava. Part 118.	King of Holland.
March 2.	
Transactions of the Geological Society of London. Second Series. Vol. iv. Part 2 ; and Vol. v. Part 2.	The Society.
Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences. 1840. 1er Semestre. Nos. 4, 5.	The Academy.
Transactions of the American Philosophical Society held at Philadelphia for promoting Useful Knowledge. Vol. vi. New Series, Part 3.	The Society.
Proceedings of the American Philosophical Society. No. 8.	Ditto.
Journal of the Society of Bengal for April and May 1839.	Ditto.
Philosophical Transactions of the Royal Society of London for the year 1839. Parts 1, 2.	Ditto.
Proceedings of the Royal Society of London. Nos. 37, 38, 39, 40.	Ditto.
Voyage dans la Russie Méridionale et la Crimée, par M. de Demidoff (Partie Scientifique). Livs. 3 et 4 en 8vo, et Planches en fol.	The Author.
The Journal of the Royal Geographical Society of London. Vol. ix. Part 3.	The Society.
Ordnance Survey of the County Mayo in Ireland, in 125 sheets.	Lord Lieutenant of Ireland.

DONATIONS.	DONORS.
The Quarterly Journal of Agriculture; and the Prize Essays and Transactions of the Highland and Agricultural Society of Scotland. No. 48, for March 1840.	The Society.
April 6.	
Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences, 1840. 1er Semestre. Nos. 6, 7, 8, 9.	The Academy.
Astronomische Nachrichten. Nos. 380 to 386.	Prof. Schumacher.
Premier Memoire sur les Kaolins ou Argiles à Porcelaine, sur la Nature, le Gisement, l'origine et le emploi de cette sorte d'Argile. Par M. Alexandre Brongniart, Professeur de Mineralogie au Museum d'Histoire Naturelle.	The Author.
Memoires de l'Académie Royale des Sciences et Belles Lettres de Bruxelles. Tome xii.	The Academy.
Bulletins de l'Académie Royale des Sciences et Belles Lettres de Bruxelles. Tome vi.	Ditto.
Annuaire de l'Académie Royale des Sciences et Belles Lettres de Bruxelles. Sixieme Année. 1840.	Ditto.
Annuaire de l'Observatoire de Bruxelles, pour l'an 1840; par le Directeur A. Quetelet.	The Author.
De la Liberté Physique et Morale; par L. A. Gruyer.	Ditto.
Voyage dans la Russie Meridionale et la Crimée; par M. de Demidoff (Partie Scientifique). Livraison 5 en 8vo, et Planches en fol.	Ditto.
Nova Acta Physico-Medica Academiæ Cesareæ Leopoldino-Carolinæ Naturæ Curiosorum. Tome xix. Part 1.	The Academy.
The Dedication of the Sanctuary; a Poem. By James Kennedy Bailie, M.D., M.R.I.A.	The Author.
Observations on the Application of the Catadioptric Zones to Lights of the First Order in the System of Fresnel; with Tables of the Elements of Zones adapted to these Lights. By Alan Stevenson, LL.B., F.R.S.E.	Ditto.
The Journal of the Royal Geographical Society of London. Vol. x. Part 1.	The Society.
Collection de Memoires et de Relations sur l'Histoire Ancienne du Canada.	Lit. and Hist. Soc. of Quebec.
The Quarterly Journal of Agriculture; and the Prize Essays and Transactions of the Highland and Agricultural Society of Scotland. No. 48, for March 1840.	The Society.
April 20.	
Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences 1840. 1er Semestre. Nos. 10, 11, 12.	The Academy.
Third Annual Report and Proceedings of the Botanical Society. Session 1838–39.	The Society.
Specimens of Printing Types in the office of Neill & Co. Printers and Type-Founders.	Neill & Co.
A collection of Fossil Organic Remains from Touraine was presented by	Sir G. Mackenzie, Bart.
Specimens of Fossil Vegetables and Shells from Shetland and Skye, by Professor Necker of Geneva, Hon. F.R.S. Ed.	Professor Necker.

LAWS

OF THE

ROYAL SOCIETY OF EDINBURGH.

JANUARY 18. 1836.

LAWS.

I.

THE ROYAL SOCIETY OF EDINBURGH shall consist of Ordinary and Honorary Fellows.

II.

Every Ordinary Fellow, within three months after his election, shall pay Five Guineas as fees of admission, and Three Guineas as his contribution for the Session in which he has been elected; and annually at the commencement of every Session, Three Guineas into the hands of the Treasurer.*

III.

All Fellows who shall have paid Twenty-five years' annual contributions shall be exempt from further payment.

IV.

Ordinary Fellows, not residing in Scotland, shall compound for the annual contribution at the rate of fifteen years' purchase.

V.

Members failing to pay their contribution for three successive years (due application having been made to them by the Treasurer), shall be reported to the Council, and, if they see fit, shall be declared from that period to be no longer Fellows, and the legal means for recovering such arrears shall be employed.

* A modification of this rule, in certain cases, was agreed to 3d January 1831.

VI.

None but Ordinary Fellows shall bear any office in the Society, or vote in the choice of Fellows or Office-bearers, or interfere in the patrimonial interests of the Society.

VII.

The number of Ordinary Fellows shall be unlimited.

VIII.

The Ordinary Fellows, upon producing an order from the TREASURER, shall be entitled to receive from the Publisher, gratis, the Parts of the Society's Transactions which shall be published subsequent to their admission.

IX.

No person shall be proposed as an Ordinary Fellow, without a recommendation subscribed by *One* Ordinary Fellow, to the purport below.* This recommendation shall be delivered to the Secretary, and by him laid before the Council, and shall afterwards be printed in the circulars for three ordinary meetings of the Society, previous to the day of the election, and shall lie upon the table during that time.

X.

Honorary Fellows shall not be subject to any Contribution. This class shall consist of persons eminently distinguished for science or literature. Its number shall not exceed Fifty-six, of whom twenty may be British subjects, and thirty-six may be subjects of foreign states.

* " A. B., a gentleman well skilled in several branches of Science *(or Polite Literature " as the case may be)*, being to my knowledge desirous of becoming a Fellow of the Royal " Society of Edinburgh, I hereby recommend him as deserving of that honour, and as likely " to prove an useful and valuable Member."

This recommendation to be accompanied by a request of admission signed by the Candidate.

XI.

Personages of Royal Blood may be elected Honorary Fellows, without regard to the limitation of numbers specified in Law X.

XII.

Honorary Fellows may be proposed by the Council, or by a recommendation (in the form given below)* subscribed by three Ordinary Fellows; and in case the Council shall decline to bring this recommendation before the Society, it shall be competent for the proposers to bring the same before a General Meeting. The election shall be by ballot, after the proposal has been communicated *viva voce* from the Chair at one meeting, and printed in the circular for the meeting at which the ballot is to take place.

XIII.

The election of Ordinary Fellows shall take place at the ordinary meetings of the Society. The election shall be by ballot, and shall be determined by a majority of at least two-thirds of the votes, provided Twenty-four Fellows be present and vote.

XIV.

The Ordinary Meetings shall be held on the first and third Mondays of every month, from November to June inclusive. Regular minutes shall be kept of the proceedings, and the Secretaries shall do the duty alternately, or according to such agreement as they may find it convenient to make.

* We hereby recommend ______________________________
for the distinction of being made an Honorary Fellow of this Society, declaring that each of us from our own knowledge of his services to *(Literature or Science as the case may be)* believe him to be worthy of that honour.

(To be signed by three Ordinary Fellows.)

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To the President and Council of the Royal Society of Edinburgh.

XV.

The Society shall from time to time publish its Transactions and Proceedings. For this purpose the Council shall select and arrange the papers which they shall deem it expedient to publish in the *Transactions* of the Society, and shall superintend the printing of the same.

XVI.

The Transactions shall be published in Parts or *Fasciculi* at the close of each session, and the expense shall be defrayed by the Society.

There shall be elected annually for conducting the publications and regulating the private business of the Society, a Council, consisting of a President; Six Vice-Presidents, two at least of whom shall be resident; Twelve Counsellors, a General Secretary, Two Secretaries to the Ordinary Meetings, a Treasurer, and a Curator, and an Assistant-Curator of the Museum and Library.

XVII.

Four Counsellors shall go out annually, to be taken according to the order in which they stand on the list of the Council.

XVIII.

An Extraordinary Meeting for the Election of Office-Bearers shall be held on the 4th Monday of November annually.

XIX.

Special Meetings of the Society may be called by the Secretary, by direction of the Council; or on a requisition signed by six or more Ordinary Fellows. Notice of not less than two days must be given of such meetings.

XX.

The Treasurer shall receive and disburse the money belonging to the Society, granting the necessary receipts, and collecting the money when due.

He shall keep regular accounts of all the cash received and expended, which shall be made up and balanced annually; and at the last Ordinary Meeting in January, he shall present the accounts for the preceding year, duly audited. At this Meeting, the Treasurer shall also lay before the Council a list of all arrears due above two years, and the Council shall thereupon give such directions as they may deem necessary for recovery thereof.

XXI.

At the Extraordinary Meeting in November, a Committee of Three Fellows shall be chosen to audit the Treasurer's accounts, and give the necessary discharge of his intromissions.

The report of the examination and discharge shall be laid before the Society at the last Ordinary Meeting in January, and inserted in the records.

XXII.

The General Secretary shall keep Minutes of the Extraordinary Meetings of the Society, and of the Meetings of the Council, in two distinct books. He shall, under the direction of the Council, conduct the correspondence of the Society, and superintend its publications. For these purposes, he shall, when necessary, employ a clerk, to be paid by the Society.

The Secretaries to the ordinary Meetings shall keep a regular Minute-book, in which a full account of the proceedings of these Meetings shall be entered: they shall specify all the Donations received, and furnish a list of them, and of the donors' names, to the Curator of the Library and Museum: they shall likewise furnish the Treasurer with notes of all admissions of Ordinary Fellows. They shall assist the General Secretary in superintending the publications, and in his absence shall take his duty.

XXIII.

The Curator of the Museum and Library shall have the custody and charge of all the Books, Manuscripts, objects of Natural History, Scientific Productions, and other articles of a similar description belonging to the Society; he shall take an account of these when received, and keep a regular catalogue of the whole, which shall lie in the Hall, for the inspection of the Fellows.

XXIV.

All articles of the above description shall be open to the inspection of the Fellows, at the Hall of the Society, at such times, and under such regulations, as the Council from time to time shall appoint.

XXV.

A Register shall be kept, in which the names of the Fellows shall be enrolled at their admission, with the date.

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